Differentials & Linear Approximation

Reminder. The tangent line to the graph of a function f at a point a is given by

$$y = f'(a)(x-a) + f(a).$$

Definition (Differential). If f is a differentiable function. Its ***differential*** at a point a is defined by

$$\boxed{df \overset{\text{def}}{=} f'(a)(x-a) \text{ or } df \overset{\text{def}}{=} f'(a)dx \text{ where } dx = (x-a)}$$

We generally prefer the more concise formulation of $df = f'(a)dx$.

Definition (Linear Approximation). The ***linear approximation*** of a function differentiable function f at a point a is the function

$$L(x) = f'(a)(x-a) + f(a).$$

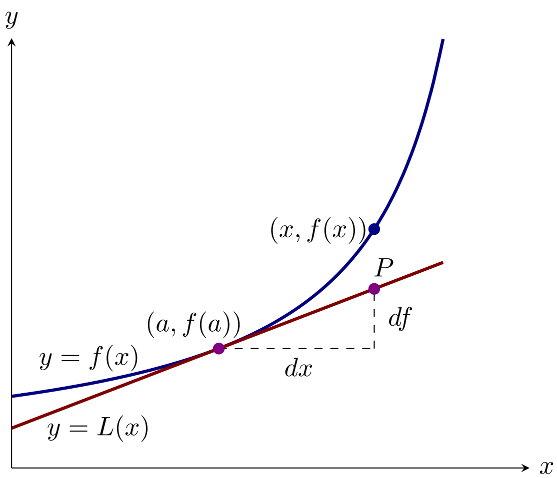
That is, it is the function which is simply the tangent line to the graph at the point a.

Remarks. One basic idea of calculus is that the tangent line to the graph of a function f at a point a represents the best linear approximation to the function at that point. But the tangent line carries the extra information of $f(a)$. The differential best represents the *change* in the value of a function as we move from a to $a + dx$. In fact,

$$f(a+dx) \approx f(a) + df \text{ equivalently, } df \approx f(a+dx) - f(a).$$

Notice, however, that $df + f(a) = f'(a)(x-a) + f(a) = L(x)$. The upshot is this: the differential attempts to quantify the change in the value of f as we move from a to $a + dx$.

The differential (and also the tangent line) help us approximate the value of a function, as is illustrated in the figure below (lightly modified from the Ximera textbook).



Here is a modified example from your homework.

Example. Suppose we measure the radius of a disk at 17 m and that our ruler only allows us to measure up to an accuracy of ± 1 cm. Estimate the potential error in the area which results from this potential error in measurement.

Solution. The area of the disk is given by $A(r) = \pi r^2$. If we measured $r = 17$ m, then $dA = A'(17)dr$ where dr is our error. In this case, since we either measured over by 1 cm or under by 1 cm, $dr = 1$ cm. That is, $dr = \frac{1}{100}$ m. Since $A'(r) = 2\pi r$, $dA = A'(17)dr = \frac{2 \cdot 17\pi}{100}$.

Mean Value Theorem

Theorem 1 (Mean Value Theorem). *If f is a function continuous on $[a,b]$ and differentiable on (a,b), then there exists a point c in (a,b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Remark. Commit this theorem's statement to memory. Conceptually, what the Mean Value Theorem (MVT) tells us is that, in any given road trip, your average velocity must be attained at some point during that trip.