Definitions, Theorems and Examples

Definition (Power Series). A function of the form $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ is called a ***power series centered at*** c.

Definition (Absolute Convergence). A power series $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$ is said to ***absolutely converge*** on an interval $[a, b]$ if for all $a \leq x \leq b$, $\sum_{n=0}^{\infty} |a_n(x-c)^n|$ is a convergent power series.

Remark. This definition specializes to series in the following way—a series $\sum_{n=0}^{\infty} a_n$ is said to absolutely converge if $\sum_{n=0}^{\infty} |a_n| < \infty$. If a series $\sum_{n=0}^{\infty} a_n$ converges absolutely, then it doesn't matter in what order we sum the terms a_n. This is not true in general—the terms in the series $\sum_{n=0}^{\infty}(-1)^n$ can be rearranged to sum to either ∞ or $-\infty$.

It turns out that absolute convergence implies convergence in the usual sense.

Lemma 1. *If a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ converges absolutely, then it converges in the usual sense. If a series $\sum_{n=0}^{\infty} a_n$ converges absolutely, it converges in the usual sense.*

Theorem 1. *Let $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ and suppose $f(x_0) = \sum_{n=0}^{\infty} a_n(x_0-a)^n$ converges. Then for any x with $|x-a| < |x_0-a|$, $f(x) = \sum_{n=0}^{\infty} a_n(x-a)^n$ converges. In other words, the radius of convergence of the power series $f(x)$ is at least $|x_0-a|$.*

Example 1. Suppose $f(x) = \sum_{n=0}^{\infty} a_n(x-3)^n$ is a power series centered at 3 and that $f(1)$ converges. Does $f(2) = \sum_{n=0}^{\infty} a_n(2-3)^n$ converge?

Solution. $f(2) = \sum_{n=0}^{\infty} a_n(-1)^n$ converges by the **Theorem 1**. Explicitly, since $1 = |2-3| < |1-3| = 2$, **Theorem 1** tells us that $f(2) = \sum_{n=0}^{\infty} a_n(-1)^n$ converges. //

Theorem 2. *The radius of convergence r of a power series $\sum_{n=0}^{\infty} a_n(x-a)^n$ is given by*

$$\frac{1}{r} = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|,$$

should the limit exist.

Remark. If the limit $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$ fails to exist, a more subtle analysis may be required.

Warning. While this theorem provides a general means of computing the radius of convergence of a power series, it may not be the best approach for doing so for a particular series, since it can easily lead to mistakes. To illustrate the possible dangers, consider the geometric series

$$\sum_{n=0}^{\infty} \frac{1}{4^n} x^{2n}.$$

It is easy to *misinterpret* **Theorem 2** by taking $\frac{1}{r} = \lim_{n\to\infty}\left|4^{-(n+1)}/4^{-n}\right| = \frac{1}{4}$ and concluding that $r = 4$ is the radius of convergence of this geometric series. *This is wrong.*

Theorem 2 only applies when a_{n+1} is the coefficient of the term x^{n+1} and a_n is the coefficient of the term x^n in the power series. The geometric series $\sum_{n=0}^{\infty} \frac{1}{4^n} x^{2n}$ can be rewritten as $\sum_{n=0}^{\infty} a_n x^n$ where

$$a_n = \begin{cases} \dfrac{1}{4^k} & \text{if } n = 2k \text{ is even,} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Thus, $\lim_{n\to\infty} |a_{n+1}/a_n|$ does not exist, so we cannot apply **Theorem 2**.

To remedy this, we forget about power series and apply the ratio test *directly* to the series in question. Fixing x_0, we compute

$$\lim_{n\to\infty} \left| \frac{\frac{1}{4^{n+1}} x_0^{2(n+1)}}{\frac{1}{4^n} x_0^{2n}} \right| = \lim_{n\to\infty} \left| \frac{x_0^2}{4} \right| = \frac{x_0^2}{4}.$$

To have $\frac{x_0^2}{4} < 1$, we must have $x_0^2 < 4$ or $|x_0| < 2$. Thus, we have convergence for $|x| < r$. Thus, the *real* radius of convergence is $r = 2$. This is different than the *incorrect* answer of $r = 4$ we found above!

Example 2. Find the radius of convergence of the power series $\sum_{n=0}^{\infty} \frac{x^n}{n!}$.

Solution. Applying **Theorem 2**, we see that $\frac{1}{r} = \lim_{n\to\infty} \left| \frac{((n+1)!)^{-1}}{(n!)^{-1}} \right| = \lim_{n\to\infty} |1/(n+1)| = 0$. Thus, the radius of convergence $r = \infty$. //

Proofs of Lemma and Theorems

Obvious the proofs are not important in this course. Since I don't want to introduce more definitions, I will only indicate what is needed to prove the lemma and give a proof for the theorems.

Lemma 1

Proof Sketch (Lem. 1). The idea is to use what is called the *Cauchy Criterion* for convergence of series. //

Theorem 1

Proof (Thm. 1). Since $\sum_{n=0}^{\infty} a_n(x_0 - a)^n$ converges, $\lim_{n\to\infty} a_n(x_0 - a)^n = 0$. Hence, the sequence $\{a_n(x_0 - a)^n\}_{n=0}^{\infty}$ is bounded, say $|a_n(x_0 - a)^n| \leq M$. Thus if $|x - a| < |x_0 - a|$, then

$$|a_n(x - a)^n| \leq |a_n| \, |x_0 - a|^n = |a_n| \, |x_0 - a|^n \left| \frac{x - a}{x_0 - a} \right|^n \leq M \left| \frac{x - a}{x_0 - a} \right|^n.$$

Since $\frac{x - a}{x_0 - a} < 1$, the geometric series

$$\sum_{n=0}^{\infty} M \left| \frac{x - a}{x_0 - a} \right|^n$$

converges. But for each n,

$$|a_n(x - a)^n| \leq M \left| \frac{x - a}{x_0 - a} \right|^n$$

so

$$0 \leq \sum_{n=0}^{\infty} |a_n(x - a)^n| \leq \sum_{n=0}^{\infty} M \left| \frac{x - a}{x_0 - a} \right|^n < \infty$$

so the series $\sum_{n=0}^{\infty} |a_n(x - a)^n|$ converges by the monotone convergence theorem. ■

Theorem 2

Proof (Thm. 2). Fix x_0. First suppose $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < \infty$. By the ratio test, $\sum_{n=0}^{\infty} a_n(x_0-a)^n$ converges if

$$\lim_{n\to\infty}\left|\frac{a_{n+1}(x_0-a)^{n+1}}{a_n(x_0-a)^n}\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}(x_0-a)\right| < 1.$$

But,

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}(x_0-a)\right| = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right||x_0-a| = |x_0-a|\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x_0-a|}{r}.$$

Thus, $\frac{|x_0-a|}{r} < 1$ when $|x_0-a| < r$ which is precisely the statement that r is the radius of convergence of the power series $\sum_{n=0}^{\infty} a_n(x-a)^n$.

When $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \infty$, the only way we could have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}(x-a)\right| < 1$$

is if $x = a$, so that the radius of convergence is said to be $r = 0$. ■