Contents

How to Read This Note

This note were written mostly to serve as a secondary reference with worked examples. I suspect most people will only get something out of the worked examples, if anything.

There are three parts to this note.

I The first part of these notes attempts to clarify a common point of confusion I have noticed.
II The second part contains a worked example and a useful illustration about concave and convex functions.
III The third part is just a reference for unusual indeterminate forms. It contains worked examples too.

In the index for these notes above, I have put a star $*$ by anything that I feel is less useful for a student that just wants to figure out what the techniques are and how to deploy them.

Part I: Unpacking MVT & IVT

Checking When a Function Satisfies the Conditions of the MVT

Let us recall the statement of the Mean Value Theorem (abbreviated as MVT).

Theorem (MVT). *If f is a function continuous on $[a, b]$ and differentiable on (a, b), then there exists a point c in (a, b) such that*

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Conditions to Check

First, for concreteness, let's suppose our interval $[a, b]$ is $[1, 5]$, so that $a = 1$ and $b = 5$. Then to check that the conditions of the MVT theorem hold on for a function f on the interval $[1, 5]$, we must check two things:

MVT 1. f is continuous on $[a, b]$ and hence also *defined* on the interval $[a, b]$.
MVT 2. f is differentiable on (a, b).

When these two things hold, the MVT *guarantees* the existence of a point c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Now let's consider the case that $[a, b]$ is any interval, where $a < b$ so that this makes sense. Then to check that the conditions of the MVT for a function f on the interval $[a, b]$, we must check that

MVT 1. f is continuous on $[a, b]$ and hence also *defined* on the interval $[a, b]$.
MVT 2. f is differentiable on (a, b).

When these two things hold, the MVT *guarantees* the existence of a point c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Remark. When the two conditions **MVT 1** and **MVT 2** fail to be true, we *are not guaranteed* the existence of a point c satisfying

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

This *does not* mean that for every function $f(x)$ which does not satisfy both **MVT 1** and **MVT 2** on $[a, b]$, there cannot exist a point c in (a, b) satisfying

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

The MVT gives us two conditions for a function to satisfy on an interval $[a, b]$ which are enough to conclude that the point c as above exists.

Checking When a Function Satisfies the Conditions of the IVT

Let us recall the statement of the Intermediate Value Theorem (abbreviated IVT).

Theorem (IVT). *If f is a function which is continuous on $[a, b]$, then for each value d between $f(a)$ and $f(b)$, there is a number c in (a, b) such that $f(c) = d$.*

Based on our discussion above on the MVT, you can probably answer the following question.

Question. *What is the single condition we need to check for a function f to satisfy the conditions of the IVT?*

If you're curious about this question, feel free to chat with me about it.

* What Can Happen When MVT 1 or MVT 2 Are Not True

Let us look at some examples clarifying what we've just discussed. For these examples, we adopt the following convention.

Convention. In the following examples we consider the interval $[a, b] = [1, 5]$ and we consider whether or not the given function satisfies the conditions (and/or conclusion) of the MVT on $[1, 5]$.

First, let's see an example where a function does not satisfy the conditions of the MVT, but the conclusion of the MVT still holds. In figure 1 below, the function $f(x)$ is not continuous on $[a, b] = [1, 5]$, so item **MVT 1** in our list above fails to be true, but the conclusion of the MVT still holds for the function f! This is because, for instance, $f'(2) = 0$ and $\dfrac{f(5) - f(1)}{5 - 1} = \dfrac{0}{4} = 0$.

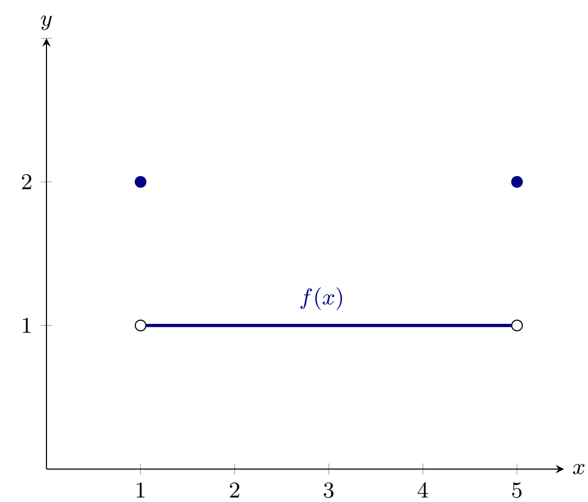

Fig. 1: Conclusion of MVT holds on $[1, 5]$ but $f(x)$ does not satisfy **MVT 1** on $[1, 5]$

Now let's see an example where a function does not satisfy the conditions of the MVT, and the conclusion of the MVT does not hold.

In the figure below, on the next page, we have another example of a function for which the conclusion of the MVT does not hold and for which **MVT 1** fails to be true—this time becuase the function f was not defined on the whole interval $[1, 5]$ to begin with.

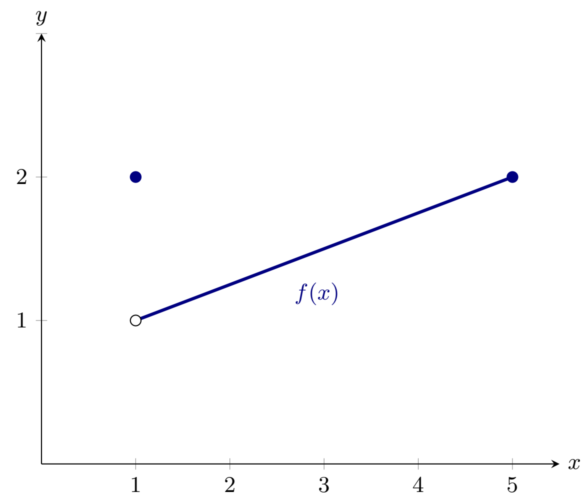

Fig. 2: Conclusion of MVT on $[1, 5]$ does not hold and $f(x)$ does not satisfy **MVT 1** on $[1, 5]$

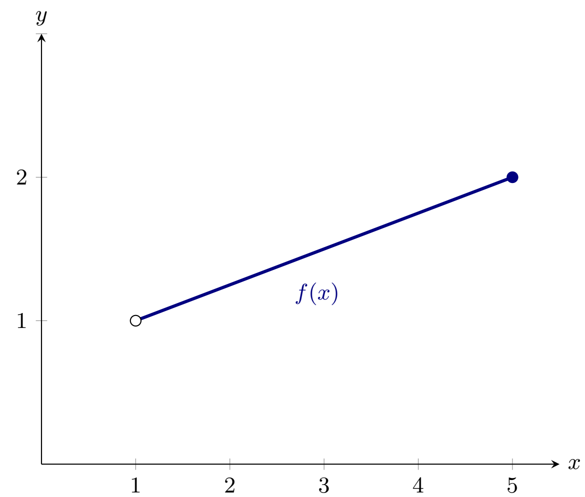

Fig. 3: Conclusion of MVT does not hold on $[1, 5]$ and $f(x)$ does not satisfy **MVT 1** on $[1, 5]$

Finally, the figure below is an example where the function $f(x)$ is continuous on $[1, 5]$ but not everywhere differentiable on $(1, 5)$ and for which the conclusion of the MVT on $[1, 5]$ does not hold.

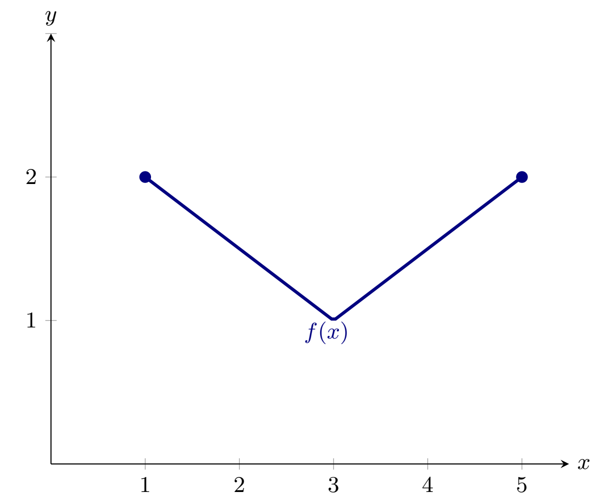

Fig. 4: Conclusion of MVT does not hold on $[1, 5]$ and $f(x)$ does not satisfy **MVT 2** on $[1, 5]$

Part II: More on Linear Approximations

Using Linear Approximations to Approximate Values of Functions

There's not too much to says here beyond the following.

Upshot (Approximation). Suppose f is a differentiable function. We can use the linear approximation $L(x)$ to f at a (see the definition in the document here) to approximate nearby values of f. Thus, if b is a point sufficiently close a, then $L(b)$ furnishes an approximation to $f(b)$

$$\boxed{f(b) \approx L(b).}$$

In general, the further away from a we go, the worse the approximation.

Example. Find the linear approximation L to the function $f(x) = \ln(2x-5)$ at $a = 4$ and use this to estimate the value of $\ln(3.1)$.

Solution. From the definition,

$$L(x) = f'(4)(x-4) + f(4).$$

We can easily compute that $f(4) = \ln(3)$, so it is left to find $f'(4)$. By the chain rule, $f'(x) = \dfrac{2}{2x-5}$ and so, $f'(4) = \dfrac{2}{3}$. Putting this together, we find that

$$L(x) = \frac{2}{3}(x-4) + \ln(3).$$

Now, we wish to estimate $\ln(3.1)$ with this. If we can find a value of b for which $f(b) = \ln(3.1)$, it will follow that $\ln(3.1) \approx L(b)$, so let us find such a value for b. The only way we could have $\ln(2b-5) = \ln(3.1)$ is if $2b - 5 = 3.1$ since the function f is a one-to-one function. Thus, we have $b = \dfrac{81}{20} = 4.05$. It follows that $f(4.05) = \ln(3.1)$ and since $f(4.05) \approx L(4.05)$, that $\ln(3.1) \approx L(4.05)$, where $L(81/20) = \ln(3) + \frac{1}{30}$, after converting everything into fractions.

Concavity & Over/Under Approximations

Theorem 1. *If $f(x)$ is everywhere concave up, then each tangent line to f lies below the graph of f.*

To understand what this theorem says, let's look at an example. Recall that $f(x) = e^x$ is everywhere concave up—that is, its second derivative is everywhere positive. The figure below illustrates a few examples of tangent lines to the graph of e^x. Note that they all lie *below* the graph of e^x.

Remark. If we wanted to be picky, we might say that each tangent line to an everywhere concave up function $f(x)$ lies below the graph of $f(x)$ everywhere *except* at the point it meets the graph of $f(x)$. For our purposes, it is basically enough to remember figure 5 above and what it tells us (i.e., **Theorem 1**).

There is a similar statement for concave down functions.

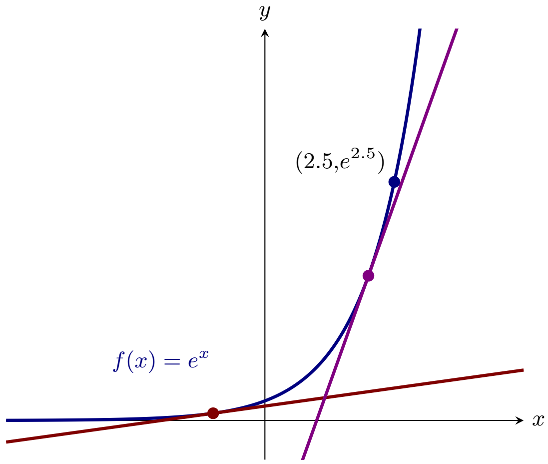

Fig. 5: The tangent lines to e^x lie ***below*** its graph.

Theorem 2. *If $g(x)$ is everywhere concave down, then each tangent line to g lies above the graph of g.*

To understand what this theorem says, let's look at an example. Recall that $g(x) = -e^x$ is a concave down function—that is, its second derivative is everywhere negative. The figure below illustrates a few examples of tangent lines to the graph of $-e^x$. Note that they all lie *above* the graph of $-e^x$.

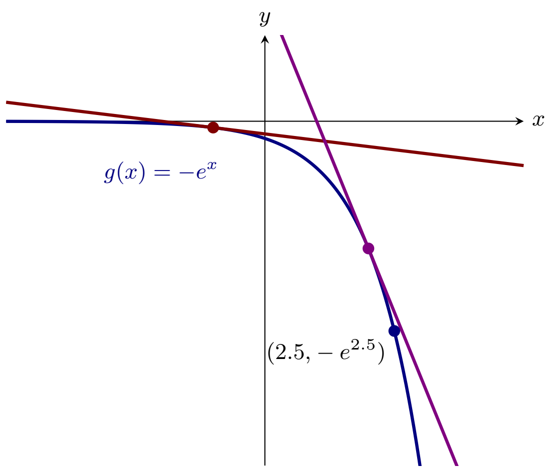

Fig. 6: The tangent lines to $-e^x$ lie ***above*** its graph.

Remark. If we wanted to be picky, we might say that each tangent line to an everywhere concave down function $g(x)$ lies above the graph of $g(x)$ everywhere *except* at the point it meets the graph of $g(x)$. For our purposes, it is basically enough to remember figure 6 above and what it tells us (i.e., **Theorem 2**).

In both figure 5 and figure 6, the magenta colored tangent lines are tangent the graphs of $f(x)$ and $g(x)$ at $x = 2$. Let's call this tangent line $L(x)$ and treat it as a linear approximation. In the case of $L(x)$ being a linear approximation to $f(x)$, we see that this is a *under approximation* and in the case that $L(x)$ is a linear approximation to $g(x)$, we see that this is a *over approximation.*

Part III: L'Hôpital's Rule

The Statement of L'Hôpital's Rule

Theorem (L'Hôpital's Rule). *If $f(x)$ and $g(x)$ are functions differentiable near a*[1] *and*

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0 \qquad \text{or} \qquad \lim_{x \to a} f(x) = \pm\infty \text{ and } \lim_{x \to a} g(x) = \pm\infty,$$

and $\lim_{x \to a} \dfrac{f'(x)}{g'(x)}$ *exists, and $g'(x) \neq 0$ for all x near a, then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}.$$

Remark. The statement of L'Hôpital's Rule makes sense for one-sided limits under very similar hypotheses! For example, using a trick below, we will consider $\lim_{x \to 0^+} x^x$.

Remark. In practice, you will *almost never* run into a function g for which $g'(x) = 0$ for all x sufficiently near a. If you are always cautious about dividing by zero, you should be OK.

Warning. Be cognizant of the fact that L'Hôpital only applies to limits having the indeterminate form $\dfrac{0}{0}$ or $\dfrac{\pm\infty}{\pm\infty}$. Sometimes you will have to manipulate the thing you're taking the limit of into an expression of this form. We will go over some common tricks for doing this below.

Some L'Hôpital Tricks For Unusual Indeterminate Forms

There are some standard tricks for the indeterminate forms

1. $\pm\infty \cdot 0$
2. $\infty - \infty$
3. 0^0
4. ∞^0
5. 1^∞

Warning (FORM: 0^∞). We consider a limit of the form 0^∞ to be *determinate* with limit 0. The reason why is that if $\lim_{x \to a} f(x)^{g(x)}$ has this form, then for all x sufficiently close to a, $f(x) < 1$ and $g(x) > 1$, so $f(x)^{g(x)} < f(x) < 1$ (e.g., if you have half of a whole pie and take half of that half, you have one-fourth of a pie).

[1] Since the limit is insensitive to any single point, we do not need to require that they be differentiable at a, in the case that $a \neq \pm\infty$.

The Forms $\pm\infty \cdot 0$ *and* $\infty - \infty$

Trick 1 (FORM: $\pm\infty \cdot 0$). If you want to find a limit $\lim_{x\to a} f(x)g(x)$ and $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = 0$, then the standard trick to convert this into a L'Hôpital problem is to attack the problem as

$$\lim_{x\to a} f(x)g(x) = \lim_{x\to a} \frac{f(x)}{1/g(x)} \qquad \text{or} \qquad \lim_{x\to a} f(x)g(x) = \lim_{x\to a} \frac{g(x)}{1/f(x)}.$$

The form of the limit $\lim_{x\to a} \dfrac{f(x)}{1/g(x)}$ is $\pm\infty/\pm\infty$ and the form of the limit $\lim_{x\to a} \dfrac{g(x)}{1/f(x)}$ is $0/0$, so we can apply L'Hôpital to either of these.

Remark. Sometimes, it is easier to apply L'Hôpital to $\lim_{x\to a} \dfrac{f(x)}{1/g(x)}$ than to $\lim_{x\to a} \dfrac{g(x)}{1/f(x)}$ and sometimes it is the other way around. I would say that, as a rule of thumb, shoot for making your life as easy as possible. For example, if $1/g(x)$ is *harder* to differentiate than $1/f(x)$, you might consider applying L'Hôpital to $\lim_{x\to a} \dfrac{g(x)}{1/f(x)}$.

Trick 2 (FORM: $\infty - \infty$). If you want to find a limit $\lim_{x\to a} (f(x) - g(x))$ and the form is $\infty - \infty$, a standard trick is to try and divide through by something to make the problem tractable, or put things under a common denominator.

The Forms 0^0, ∞^0 *and* 1^∞

The remaining tricks are for the indeterminate forms 0^0, ∞^0 and 1^∞. These all involve replacing $\lim_{x\to a} f(x)^{g(x)}$ by $\lim_{x\to a} e^{g(x)\ln(f(x))}$ and then using continuity of e^x and the properties of logarithms conclude that

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{\ln(f(x)^{g(x)})} = \lim_{x\to a} e^{g(x)\ln(f(x))} = e^{\lim_{x\to a} g(x)\ln(f(x))}.$$

Trick 3 (FORM: 0^0). If you want to find a limit $\lim_{x\to a} f(x)^{g(x)}$ and the form is 0^0, a standard trick use the fact that $e^{\ln(x)} = x$ and try to compute

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{\ln(f(x)^{g(x)})} = \lim_{x\to a} e^{g(x)\ln(f(x))}.$$

Since e^x is continuous, $\lim_{x\to a} e^{g(x)\ln(f(x))}$ is *totally determined* by $\lim_{x\to a} g(x)\ln(f(x))$. There are three cases to consider.

1. If $\lim_{x\to a} g(x)\ln(f(x)) = c$, then $\lim_{x\to a} f(x)^{g(x)} = e^c$.
2. If $\lim_{x\to a} g(x)\ln(f(x)) = \infty$, then $\lim_{x\to a} f(x)^{g(x)} = \infty$.
3. If $\lim_{x\to a} g(x)\ln(f(x)) = -\infty$, then $\lim_{x\to a} f(x)^{g(x)} = 0$.

Trick 4 (FORM: ∞^0). If you want to find a limit $\lim_{x\to a} f(x)^{g(x)}$ and the form is ∞^0, a standard trick use the fact that $e^{\ln(x)} = x$ and try to compute

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{\ln(f(x)^{g(x)})} = \lim_{x\to a} e^{g(x)\ln(f(x))}.$$

Since e^x is continuous, $\lim_{x\to a} e^{g(x)\ln(f(x))}$ is *totally determined* by $\lim_{x\to a} g(x)\ln(f(x))$. There are three cases to consider.

1. If $\lim_{x\to a} g(x)\ln(f(x)) = c$, then $\lim_{x\to a} f(x)^{g(x)} = e^c$.
2. If $\lim_{x\to a} g(x)\ln(f(x)) = \infty$, then $\lim_{x\to a} f(x)^{g(x)} = \infty$.
3. If $\lim_{x\to a} g(x)\ln(f(x)) = -\infty$, then $\lim_{x\to a} f(x)^{g(x)} = 0$.

Trick 5 (FORM: 1^∞). If you want to find a limit $\lim_{x\to a} f(x)^{g(x)}$ and the form is 1^∞, a standard trick use the fact that $e^{\ln(x)} = x$ and try to compute

$$\lim_{x\to a} f(x)^{g(x)} = \lim_{x\to a} e^{\ln(f(x)^{g(x)})} = \lim_{x\to a} e^{g(x)\ln(f(x))}.$$

Since e^x is continuous, $\lim_{x\to a} e^{g(x)\ln(f(x))}$ is *totally determined* by $\lim_{x\to a} g(x)\ln(f(x))$. There are three cases to consider.

1. If $\lim_{x\to a} g(x)\ln(f(x)) = c$, then $\lim_{x\to a} f(x)^{g(x)} = e^c$.
2. If $\lim_{x\to a} g(x)\ln(f(x)) = \infty$, then $\lim_{x\to a} f(x)^{g(x)} = \infty$.
3. If $\lim_{x\to a} g(x)\ln(f(x)) = -\infty$, then $\lim_{x\to a} f(x)^{g(x)} = 0$.

Remark. One could also do these last few "tricks" by jumping straight to computing $\lim_{x\to a} f(x)\ln(g(x))$ and then use the fact that the logarithm is one-to-one. I would recommend doing it whatever way works best for you—neither is better than the other.

Some L'Hôpital Examples

Standard Examples

Example. Compute $\lim_{x\to\infty} x - \ln(x)$.

Solution. The form of this limit is $\infty-\infty$, so we should try the first trick. Divide through by x to write this as $\lim_{x\to\infty} x - \ln(x) = \lim_{x\to\infty} x\left(1 - \dfrac{\ln(x)}{x}\right)$. By L'Hôpital, $\lim_{x\to\infty}\left(1 - \dfrac{\ln(x)}{x}\right) = 1$ so the form of the limit $\lim_{x\to\infty} x\left(1 - \dfrac{\ln(x)}{x}\right)$ is $\infty\cdot\#$ and the sign is $+\cdot+$ so the limit $\lim_{x\to\infty} x-\ln(x) = \lim_{x\to\infty} x\left(1 - \dfrac{\ln(x)}{x}\right) = \infty$.

Example. Compute $\lim_{x\to 0^+} x^x$.

Solution. The form of this limit is 0^0, so we'll use the third trick. We write $\lim_{x\to0^+} x^x = \lim_{x\to0^+} e^{\ln(x^x)} = \lim_{x\to0^+} e^{x\ln(x)}$. The form of the limit $x\lim_{x\to0^+} x\ln(x)$ is $0\cdot-\infty$, so, using the first trick for limits of the form $\pm\infty\cdot 0$, we write $\lim_{x\to0^+} x\ln(x) = \lim_{x\to0} \dfrac{\ln(x)}{1/x}$. The form of this limit is now $\dfrac{-\infty}{\infty}$, so we may at last apply L'Hôpital to this. Thus,

$$\lim_{x\to0^+} \frac{\ln(x)}{1/x} \overset{\text{LR}}{=} \lim_{x\to0^+} \frac{1/x}{-1/x^2} = \lim_{x\to0^+} -x = 0.$$

Thus,

$$\lim_{x\to0^+} x^x = \lim_{x\to0^+} e^{\ln(x^x)} = \lim_{x\to0^+} e^{x\ln(x)} = e^{\lim_{x\to0^+} x\ln(x)} = e^0 = 1.$$

Example. Compute $\lim_{x\to0} \dfrac{x}{\tan(x)}$.

Solution. The form of the limit is $0/0$, so we may apply L'Hôpital directly and find that

$$\lim_{x\to0} \frac{x}{\tan(x)} \overset{\text{LR}}{=} \lim_{x\to0} \frac{1}{\sec^2(x)} = \frac{1}{1} = 1.$$

Example. Compute $\lim_{x\to0^+} (1+x+2\sin(x^2))^{1/x}$.

Solution. The form of this limit is 1^∞, so we are in the case of the fifth trick, where $f(x) = 1+x+2\sin(x^2)$ and $g(x) = 1/x$. Thus, we consider $\lim_{x\to0^+} \dfrac{\ln(1+x+2\sin(x^2))}{x}$. The form of this limit is $\dfrac{\ln(1)}{0}$. Recall that $\ln(1) = 0$, so that, in fact, the form of $\lim_{x\to0^+} \dfrac{\ln(1+2\sin(x))}{x}$ is $\frac{0}{0}$. We are again in a position to apply L'Hôpital.

$$\lim_{x\to0^+} \frac{\ln(1+x+2\sin(x^2))}{x} \overset{\text{LR}}{=} \lim_{x\to0^+} \frac{\frac{d}{dx}\ln(1+x+2\sin(x^2))}{1}.$$

We must compute $\frac{d}{dx}\ln(1+x+2\sin(x^2))$. Let $f(x) = \ln(x)$ and $g(x) = 1+x+2\sin(x^2)$. Then

$$\ln(1+x+2\sin(x^2)) = f(g(x)),$$

since $f(g(x)) = \ln(g(x)) = \ln(1+x+2\sin(x^2))$. By the chain rule,

$$\begin{aligned}\frac{d}{dx}\ln(1+x+2\sin(x^2)) &= \frac{d}{dx}f(g(x)) = f'(g(x))g'(x) = \frac{1}{g(x)}g'(x)\\ &= \frac{1}{1+x+2\sin(x^2)}\frac{d}{dx}\left(1+x+2\sin(x^2)\right)\\ &= \frac{1}{1+x+2\sin(x^2)}\left(1+\frac{d}{dx}2\sin(x^2)\right).\end{aligned}$$

To complete this step, we need to find, $\frac{d}{dx}2\sin(x^2)$—this is a simple application of the chain rule, and we get

$$\frac{d}{dx}2\sin(x^2) = 4x\cos(x^2).$$

Putting this together,

$$\frac{d}{dx}\ln(1+x+2\sin(x^2)) = \frac{1}{1+x+2\sin(x^2)}\left(1+4x\cos(x^2)\right) = \frac{1+4x\cos(x^2)}{1+x+2\sin(x^2)}.$$

Finally,

$$\lim_{x\to 0^+}\frac{\ln(1+x+2\sin(x^2))}{x} \overset{\text{LR}}{=} \lim_{x\to 0^+}\frac{1+4x\cos(x^2)}{1+x+2\sin(x^2)} = 1.$$

Hence,

$$\lim_{x\to 0^+}(1+x+2\sin(x^2))^{1/x} = \lim_{x\to 0^+} e^{\frac{\ln(1+x+2\sin(x^2))}{x}} = e^{\lim_{x\to 0^+}\frac{\ln(1+x+2\sin(x^2))}{x}} = e^1 = e.$$

Remark. If we did not know $\ln(1)$, we could try to find it by remembering that $e^{\ln(1)} = 1$, and $e^0 = 1$.

$*$ *Pathological Example: Applying L'Hôpital Gives an Infinite Loop*

Example. Compute $\lim_{x\to-\infty}\frac{\sqrt{x^2+1}}{x}$.

Solution. The form of this limit is $\infty/-\infty$. Applying L'Hôpital, we find that

$$\lim_{x\to-\infty}\frac{\sqrt{x^2+1}}{x} \overset{\text{LR}}{=} \lim_{x\to-\infty}\frac{x/\sqrt{x^2+1}}{1} = \lim_{x\to-\infty}\frac{x}{\sqrt{x^2+1}}.$$

The form of this limit is $-\infty/\infty$. Applying L'Hôpital again gives us

$$\lim_{x\to-\infty}\frac{x}{\sqrt{x^2+1}} \overset{\text{LR}}{=} \lim_{x\to-\infty}\frac{\sqrt{x^2+1}}{x}$$

which is what we wanted to compute in the first place!

We'll have to try something else. Since we are considering $x\to-\infty$, the limit is not affected by values of $x\geq 0$, so we might as well suppose $x<0$ in $\lim_{x\to-\infty}\sqrt{x^2+1}/x$. In that case,

$$\lim_{x\to-\infty}\frac{\sqrt{x^2+1}}{x} = \lim_{x\to-\infty}-\sqrt{\frac{x^2+1}{x^2}} = \lim_{x\to-\infty}-\sqrt{1+\frac{1}{x^2}}.$$

Since we know that $\lim_{x\to-\infty}1/x^2 = 0$,

$$\lim_{x\to-\infty}\frac{\sqrt{x^2+1}}{x} = \lim_{x\to-\infty}-\sqrt{1+\frac{1}{x^2}} = -1.$$

Warning. We needed to put the negative sign out front when we moved x inside the square root because, for the purposes of the limit $x\to-\infty$, we may assume that $x<0$. This means that the entire expression

$\dfrac{\sqrt{x^2+1}}{x}$ was negative in the limit $x \to -\infty$, whereas $\sqrt{\dfrac{x^2+1}{x^2}}$ is positive for all values of $x < 0$. This is resolved by considering the limit of $-\sqrt{\dfrac{x^2+1}{x^2}}$ as $x \to -\infty$.

Remark. Let $f(x) = \dfrac{\sqrt{x^2+1}}{x}$. Applying L'Hôpital told us that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{1}{f(x)}$. This means that if the limit of $\lim_{x\to\infty} f(x) = c$ exists, then it must be that $c = \pm 1$, since we will have $\lim_{x\to\infty} f(x) = c = \lim_{x\to\infty} \dfrac{1}{f(x)} = 1/c$. Since the sign of the limit is $+/-$, it will have to be that $c = -1$. The problem with this attack is in verifying that one of these limits actually exists. The argument we gave in the solution is what, for the purposes of this class, we consider as sufficient for showing this.

Upshot. Sometimes, in applying L'Hôpital, we end up where we started. This is a sign we have to do try something else first. In other words, it can happen that L'Hôpital won't get you anywhere at first—sometimes, like above, the method is to do some purely algebraic maneuvers to make your life easier.