L-functions and Converse Theorems for GL_n

James W. Cogdell

Contents

IAS/Park City Mathematics Series
Volume 00, 0000

L-functions and Converse Theorems for GL_n

James W. Cogdell

Introduction

The purpose of these notes is to develop the analytic theory of L-functions for cuspidal automorphic representations of GL_n over a global field. There are two approaches to L-functions of GL_n: via integral representations or through analysis of Fourier coefficients of Eisenstein series. In these notes we develop the theory via integral representations.

The theory of L-functions of automorphic forms (or modular forms) via integral representations has its origin in the paper of Riemann on the ζ-function [72]. However the theory was really developed in the classical context of L-functions of modular forms for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ by Hecke and his school [34]. Much of our current theory is a direct outgrowth of Hecke's. L-functions of automorphic representations were first developed by Jacquet and Langlands for GL_2 [30, 37, 39]. Their approach followed Hecke combined with the local-global techniques of Tate's thesis [91]. The theory for GL_n was then developed along the same lines in a long series of papers by various combinations of Jacquet, Piatetski-Shapiro, and Shalika [40–47, 64, 66, 85]. In addition to associating an L-function to an automorphic form, Hecke also gave a criterion for a Dirichlet series to come from a modular form, the so called Converse Theorem of Hecke [35]. In the context of automorphic representations, the Converse Theorem for GL_2 was developed by Jacquet and Langlands [39], extended and significantly strengthened to GL_3 by Jacquet, Piatetski-Shapiro, and Shalika [40], and then extended to GL_n [9, 12].

What we have attempted to present here is a synopsis of this work and in doing so present the paradigm for the analysis of automorphic L-functions via integral representations. Lecture 1 deals with the Fourier expansion of automorphic forms on GL_n and the related Multiplicity One and Strong Multiplicity One Theorems. Lecture 2 then develops the theory of Eulerian integrals for GL_n. In Lecture 3 we turn to the local theory of L-functions for GL_n, in both the archimedean and

[1]Department of Mathematics Oklahoma State University, Stillwater, OK 74078, USA.
E-mail address: `cogdell@math.okstate.edu`.
1991 *Mathematics Subject Classification.* 11F70 (22E55).
The author was supported in part by the NSA and the Clay Mathematics Institute.

non-archimedean local contexts, which comes out of the Euler factors of the global integrals. In Lecture 4 we finally combine the global Eulerian integrals with the definition and analysis of the local L-functions to define the global L-function of an automorphic representation and derive their major analytic properties. In Lecture 5 we turn to the various Converse Theorems for GL_n. Lecture 6 is devoted to the application of the Converse Theorem to questions of Functoriality, that is, the lifting or transfer of automorphic representations from a group H to GL_n.

We have tried to keep the tone of the notes informal for the most part. We have tried to provide complete proofs where feasible, at least sketches of most major results, and references for technical facts.

There is another body of work on integral representations of L-functions for GL_n which developed out of the classical work on zeta functions of algebras. This is the theory of principal L-functions for GL_n as developed by Godement and Jacquet [31,37]. This approach is related to the one pursued here, but we have not attempted to present it here.

The other approach to these L-functions is via the Fourier coefficients of Eisenstein series. In the context of automorphic representations, and in a broader context than GL_n, this approach was originally laid out by Langlands [60] but then most fruitfully pursued by Shahidi. Some of the major papers of Shahidi on this subject are [74–84]. In particular, in [77] he shows that the two approaches give the same L-functions for GL_n. We will not pursue this approach in these notes, but the interested reader should consult Shahidi's lectures in this volume [84].

For a balanced presentation of all three methods we recommend the book of Gelbart and Shahidi [24]. They treat not only L-functions for GL_n but L-functions of automorphic representations of other groups as well.

We have not discussed the arithmetic theory of automorphic representations and L-functions. For the connections with motives, we recommend the surveys of Clozel [5] and Ramakrishnan [68].

The original version of these notes was prepared for and distributed at the School on Automorphic Forms on $GL(n)$ held at The Abdus Salam International Centre for Theoretical Physics (ICTP) in Trieste, Italy, 31 July – 18 August 2000. That version, entitled "Notes on L-functions for GL_n", is available on the ICTP web site. Since then I have used the ICTP notes in conjunction with lectures given in the Programme on Lie Groups 2001 at the Institute of Mathematical Research of Hong Kong University, 20 May – 26 June 2001, and most recently the IAS/PCMI Graduate Summer School on Automorphic Forms held in Park City, Utah, 30 June – 20 July 2002. In the version of these notes presented here Lectures 1–4 are essentially the same as in the ICTP notes, with some corrections and updates. Lecture 5 has been rewritten to conform with the presentation at the PCMI school. Lecture 6 is new and was added to give an exposition of the application of the Converse Theorem to the question of Functoriality, which was one of the points of emphasis for the PCMI school. The Lectures as presented in these notes, of which there are 6, do not coincide with the actual lectures I gave at Park City, where I gave only 4 lectures. Lectures 1 and 2 here were covered in one lecture at Park City, Lectures 3 and 4 were covered in one lecture, and Lectures 5 and 6 were alloted one lecture each.

Most of what I know about L-functions for GL_n I have learned through my years of work with Piatetski-Shapiro. I owe him a great debt of gratitude for all

that he has taught me. For several years Piatetski-Shapiro and I have envisioned writing a book on L-functions for GL_n [17]. The contents of these notes essentially follows our outline for that book. In particular, the exposition in Lectures 1, 2, and parts of 3 and 4 is drawn from drafts for this project. The exposition in Lecture 5 is drawn from the survey of our work on Converse Theorems in [13]. I would like to thank Piatetski-Shapiro for graciously allowing me to present part of our joint efforts in these notes. I would also like to thank Jacquet for many enlightening conversations over the years on his work on L-functions for GL_n. Finally I would like all those who provided me with comments on and corrections to the ICTP notes.

LECTURE 1
Fourier expansions and multiplicity one

In this section we let k denote a global field, $\mathbb{A}$, its ring of adeles, and ψ will denote a continuous additive character of $\mathbb{A}$ which is trivial on k. For the basics on adeles, characters, etc. we refer the reader to Weil [96] or the book of Gelfand, Graev, and Piatetski-Shapiro [26].

We begin with a cuspidal automorphic representation (π, V_π) of $\mathrm{GL}_n(\mathbb{A})$. For us, automorphic forms are assumed to be smooth (of uniform moderate growth) but not necessarily K_∞–finite at the archimedean places. This is most suitable for the analytic theory. For simplicity, we assume the central character ω_π of π is unitary. Then V_π is the space of smooth vectors in an irreducible unitary representation of $\mathrm{GL}_n(\mathbb{A})$. We will always use cuspidal in this sense: the smooth vectors in an irreducible unitary cuspidal automorphic representation. (Any other smooth cuspidal representation π of $\mathrm{GL}_n(\mathbb{A})$ is necessarily of the form $\pi = \pi^\circ \otimes |\det|^t$ with π° unitary and t real, so there is really no loss of generality in the unitarity assumption. It merely provides us with a convenient normalization.) By a cusp form on $\mathrm{GL}_n(\mathbb{A})$ we will mean a function lying in a cuspidal representation. By a cuspidal function we will simply mean a smooth function φ on $\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A})$ satisfying $\int_{\mathrm{U}(k)\backslash \mathrm{U}(\mathbb{A})} \varphi(ug)\, du \equiv 0$ for every unipotent radical U of standard parabolic subgroups of GL_n.

The basic references for this section are the papers of Piatetski-Shapiro [64, 66] and Shalika [85].

1.1. Fourier Expansions

Let $\varphi(g) \in V_\pi$ be a cusp form in the space of π. For arithmetic applications, and particularly for the theory of L-functions, we will need the *Fourier expansion* of $\varphi(g)$.

If $f(\tau)$ is a holomorphic cusp form on the upper half plane $\mathfrak{H}$, say with respect to $\mathrm{SL}_2(\mathbb{Z})$, then f is invariant under integral translations, $f(\tau+1) = f(\tau)$ and thus has a Fourier expansion of the form

$$f(\tau) = \sum_{n=1}^{\infty} a_n e^{2\pi i n\tau}.$$

If $\varphi(g)$ is a smooth cusp form on $\mathrm{GL}_2(\mathbb{A})$ then the translations correspond to the maximal unipotent subgroup $\mathrm{N}_2 = \left\{ n = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \right\}$ and $\varphi(ng) = \varphi(g)$ for $n \in \mathrm{N}_2(k)$. So, if ψ is any continuous character of $k\backslash\mathbb{A}$ we can define the ψ-Fourier coefficient or ψ-Whittaker function by

$$W_{\varphi,\psi}(g) = \int_{k\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) \psi^{-1}(x)\ dx.$$

We have the corresponding Fourier expansion

$$\varphi(g) = \sum_{\psi} W_{\varphi,\psi}(g).$$

(Actually from abelian Fourier theory, one has

$$\varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right) = \sum_{\psi} W_{\varphi,\psi}(g)\psi(x)$$

as a periodic function of $x \in \mathbb{A}$. Now set $x = 0$.)

If we fix a single non-trivial character ψ of $k\backslash\mathbb{A}$, then by standard duality theory [26, 96] the additive characters of the compact group $k\backslash\mathbb{A}$ are isomorphic to k via the map $\gamma \in k \mapsto \psi_\gamma$ where ψ_γ is the character $\psi_\gamma(x) = \psi(\gamma x)$. Now, an elementary calculation shows that $W_{\varphi,\psi_\gamma}(g) = W_{\varphi,\psi}\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g \right)$ if $\gamma \neq 0$. If we set $W_\varphi = W_{\varphi,\psi}$ for our fixed ψ, then the Fourier expansion of φ becomes

$$\varphi(g) = W_{\varphi,\psi_0}(g) + \sum_{\gamma \in k^\times} W_\varphi\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g \right).$$

Since φ is cuspidal

$$W_{\varphi,\psi_0}(g) = \int_{k\backslash\mathbb{A}} \varphi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g \right)\ dx \equiv 0$$

and the Fourier expansion for a cusp form φ becomes simply

$$\varphi(g) = \sum_{\gamma \in k^\times} W_\varphi\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g \right).$$

We will need a similar expansion for cusp forms φ on $\mathrm{GL}_n(\mathbb{A})$. The translations still correspond to the maximal unipotent subgroup

$$\mathrm{N}_n = \left\{ n = \begin{pmatrix} 1 & x_{1,2} & & & * \\ & 1 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 1 & x_{n-1,n} \\ 0 & & & & 1 \end{pmatrix} \right\},$$

but now this is non-abelian. This difficulty was solved independently by Piatetski-Shapiro [64] and Shalika [85]. We fix our non-trivial continuous character ψ of $k\backslash\mathbb{A}$ as above. Extend it to a character of N_n by setting $\psi(n) = \psi(x_{1,2} + \cdots + x_{n-1,n})$ and define the associated Fourier coefficient or Whittaker function by

$$W_\varphi(g) = W_{\varphi,\psi}(g) = \int_{\mathrm{N}_n(k)\backslash \mathrm{N}_n(\mathbb{A})} \varphi(ng)\psi^{-1}(n)\ dn.$$

Since φ is continuous and the integration is over a compact set this integral is absolutely convergent, uniformly on compact sets. The Fourier expansion takes the following form.

Theorem 1.1. Let $\varphi \in V_\pi$ be a cusp form on $\mathrm{GL}_n(\mathbb{A})$ and W_φ its associated ψ-Whittaker function. Then

$$\varphi(g) = \sum_{\gamma \in \mathrm{N}_{n-1}(k)\backslash GL_{n-1}(k)} W_\varphi\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} g\right)$$

with convergence absolute and uniform on compact subsets.

The proof of this fact is an induction. It utilizes the *mirabolic subgroup* P_n of GL_n which seems to be ubiquitous in the study of automorphic forms on GL_n. Abstractly, a mirabolic subgroup of GL_n is simply the stabilizer of a non-zero vector in (either) standard representation of GL_n on k^n. We denote by P_n the stabilizer of the row vector $e_n = (0, \ldots, 0, 1) \in k^n$. So

$$\mathrm{P}_n = \left\{ p = \begin{pmatrix} h & y \\ & 1 \end{pmatrix} \middle| h \in \mathrm{GL}_{n-1}, y \in k^{n-1} \right\} \simeq \mathrm{GL}_{n-1} \ltimes \mathrm{Y}_n$$

where

$$\mathrm{Y}_n = \left\{ y = \begin{pmatrix} I_{n-1} & y \\ & 1 \end{pmatrix} \middle| y \in k^{n-1} \right\} \simeq k^{n-1}.$$

Simply by restriction of functions, a cusp form on $\mathrm{GL}_n(\mathbb{A})$ restricts to a smooth cuspidal function on $\mathrm{P}_n(\mathbb{A})$ which remains left invariant under $\mathrm{P}_n(k)$. (A smooth function φ on $\mathrm{P}_n(\mathbb{A})$ which is left invariant under $\mathrm{P}_n(k)$ is called cuspidal if

$$\int_{\mathrm{U}(k)\backslash \mathrm{U}(\mathbb{A})} \varphi(up)\, du \equiv 0$$

for every standard unipotent subgroup $\mathrm{U} \subset \mathrm{P}_n$.) Since $\mathrm{P}_n \supset \mathrm{N}_n$ we may define a Whittaker function attached to a cuspidal function φ on $\mathrm{P}_n(\mathbb{A})$ by the same integral as on $\mathrm{GL}_n(\mathbb{A})$, namely

$$W_\varphi(p) = \int_{\mathrm{N}_n(k)\backslash \mathrm{N}_n(\mathbb{A})} \varphi(np)\psi^{-1}(n)\, dn.$$

We will prove by induction that for a cuspidal function φ on $\mathrm{P}_n(\mathbb{A})$ we have

$$\varphi(p) = \sum_{\gamma \in \mathrm{N}_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k)} W_\varphi\left(\begin{pmatrix}\gamma & 0 \\ 0 & 1\end{pmatrix} p\right)$$

with convergence absolute and uniform on compact subsets.

The function on $\mathrm{Y}_n(\mathbb{A})$ given by $y \mapsto \varphi(yp)$ is invariant under $\mathrm{Y}_n(k)$ since $\mathrm{Y}_n(k) \subset \mathrm{P}_n(k)$ and φ is automorphic on $\mathrm{P}_n(\mathbb{A})$. Hence by standard abelian Fourier analysis for $\mathrm{Y}_n \simeq k^{n-1}$ we have as before

$$\varphi(p) = \sum_{\lambda \in (\widehat{k^{n-1}\backslash \mathbb{A}^{n-1}})} \varphi_\lambda(p)$$

where

$$\varphi_\lambda(p) = \int_{\mathrm{Y}_n(k)\backslash \mathrm{Y}_n(\mathbb{A})} \varphi(yp)\lambda^{-1}(y)\, dy.$$

Now, by duality theory [96], $(\widehat{k^{n-1}\backslash\mathbb{A}^{n-1}}) \simeq k^{n-1}$. In fact, if we let $\langle\ ,\ \rangle$ denote the pairing $k^{n-1}\times k^{n-1}\to k$ by $\langle x,y\rangle=\sum x_iy_i$ we have

$$\varphi(p)=\sum_{x\in k^{n-1}}\varphi_x(p)$$

where now we write

$$\varphi_x(p)=\int_{k^{n-1}\backslash\mathbb{A}^{n-1}}\varphi(yp)\psi^{-1}(\langle x,y\rangle)\ dy.$$

$\mathrm{GL}_{n-1}(k)$ acts on k^{n-1} with two orbits: $\{0\}$ and $k^{n-1}-\{0\}=\mathrm{GL}_{n-1}(k)\cdot{}^te_{n-1}$ where $e_{n-1}=(0,\dots,0,1)$. The stabilizer of ${}^te_{n-1}$ in $\mathrm{GL}_{n-1}(k)$ is ${}^t\mathrm{P}_{n-1}$. Therefore, we may write

$$\varphi(p)=\varphi_0(p)+\sum_{\gamma\in\mathrm{GL}_{n-1}(k)/{}^t\mathrm{P}_{n-1}(k)}\varphi_{\gamma\cdot{}^te_{n-1}}(p).$$

Since $\varphi(p)$ is cuspidal and Y_n is a standard unipotent subgroup of GL_n, we see that

$$\varphi_0(p)=\int_{\mathrm{Y}_n(k)\backslash\mathrm{Y}_n(\mathbb{A})}\varphi(yp)\ dy\equiv 0.$$

On the other hand an elementary calculation as before gives

$$\varphi_{\gamma\cdot{}^te_{n-1}}(p)=\varphi_{{}^te_{n-1}}\left(\begin{pmatrix}{}^t\gamma & 0\\ 0 & 1\end{pmatrix}p\right).$$

Hence we have

$$\varphi(p)=\sum_{\gamma\in\mathrm{P}_{n-1}(k)\backslash\mathrm{GL}_{n-1}(k)}\varphi_{{}^te_{n-1}}\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}p\right)$$

and the convergence is still absolute and uniform on compact subsets.

Note that if $n=2$ this is exactly the fact we used previously for GL_2. This then begins our induction.

Next, let us write the above in a form more suitable for induction. Structurally, we have $\mathrm{P}_n=\mathrm{GL}_{n-1}\ltimes\mathrm{Y}_n$ and $\mathrm{N}_n=\mathrm{N}_{n-1}\ltimes\mathrm{Y}_n$. Therefore, $\mathrm{N}_{n-1}\backslash\mathrm{GL}_{n-1}\simeq\mathrm{N}_n\backslash\mathrm{P}_n$. Furthermore, if we let $\widetilde{\mathrm{P}}_{n-1}=\mathrm{P}_{n-1}\ltimes\mathrm{Y}_n\subset\mathrm{P}_n$, then $\mathrm{P}_{n-1}\backslash\mathrm{GL}_{n-1}\simeq\widetilde{\mathrm{P}}_{n-1}\backslash\mathrm{P}_n$. Next, note that the function $\varphi_{{}^te_{n-1}}(p)$ satisfies, for $y\in\mathrm{Y}_n(\mathbb{A})\simeq\mathbb{A}^{n-1}$,

$$\varphi_{{}^te_{n-1}}(yp)=\psi(y_{n-1})\varphi_{{}^te_{n-1}}(p).$$

Since ψ is trivial on k we see that $\varphi_{{}^te_{n-1}}(p)$ is left invariant under $\mathrm{Y}_n(k)$. Hence

$$\varphi(p)=\sum_{\gamma\in\mathrm{P}_{n-1}(k)\backslash\mathrm{GL}_{n-1}(k)}\varphi_{{}^te_{n-1}}\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}p\right)=\sum_{\delta\in\widetilde{\mathrm{P}}_{n-1}(k)\backslash\mathrm{P}_n(k)}\varphi_{{}^te_{n-1}}(\delta p).$$

To proceed with the induction, fix $p\in\mathrm{P}_n(\mathbb{A})$ and consider the function $\varphi'(p')=\varphi'_p(p')$ on $\mathrm{P}_{n-1}(\mathbb{A})$ given by

$$\varphi'(p')=\varphi_{{}^te_{n-1}}\left(\begin{pmatrix}p' & 0\\ 0 & 1\end{pmatrix}p\right).$$

φ' is a smooth function on $\mathrm{P}_{n-1}(\mathbb{A})$ since φ was smooth. One checks that φ' is left invariant by $\mathrm{P}_{n-1}(k)$ and cuspidal on $\mathrm{P}_{n-1}(\mathbb{A})$. Then we may apply our inductive assumption to conclude that

$$\begin{aligned}\varphi'(p') &= \sum_{\gamma'\in \mathrm{N}_{n-2}(k)\backslash \mathrm{GL}_{n-2}(k)} W_{\varphi'}\left(\begin{pmatrix}\gamma' & 0\\ 0 & 1\end{pmatrix} p'\right)\\ &= \sum_{\delta'\in \mathrm{N}_{n-1}(k)\backslash \mathrm{P}_{n-1}} (k) W_{\varphi'}(\delta' p').\end{aligned}$$

If we substitute this into the expansion for $\varphi(p)$ we see

$$\begin{aligned}\varphi(p) &= \sum_{\delta\in \widetilde{\mathrm{P}}_{n-1}(k)\backslash \mathrm{P}_n(k)} \varphi_{{}^te_{n-1}}(\delta p)\\ &= \sum_{\delta\in \widetilde{\mathrm{P}}_{n-1}(k)\backslash \mathrm{P}_n(k)} \varphi'_{\delta p}(1)\\ &= \sum_{\delta\in \widetilde{\mathrm{P}}_{n-1}(k)\backslash \mathrm{P}_n(k)} \sum_{\delta'\in \mathrm{N}_{n-1}(k)\backslash \mathrm{P}_{n-1}(k)} W_{\varphi'_{\delta p}}(\delta').\end{aligned}$$

Now, as before, $\mathrm{N}_{n-1}\backslash \mathrm{P}_{n-1} \simeq \mathrm{N}_n\backslash\widetilde{\mathrm{P}}_{n-1}$ and $\mathrm{N}_n \simeq \mathrm{N}_{n-1} \ltimes \mathrm{Y}_{n-1}$. Thus

$$\begin{aligned}W_{\varphi'_{\delta p}}(\delta') &= \int_{\mathrm{N}_{n-1}(k)\backslash \mathrm{N}_{n-1}(\mathbb{A})} \varphi'_{\delta p}(n'\delta')\psi^{-1}(n')\, dn'\\ &= \int_{\mathrm{N}_{n-1}(k)\backslash \mathrm{N}_{n-1}(\mathbb{A})}\int_{\mathrm{Y}_n(k)\backslash \mathrm{Y}_n(\mathbb{A})} \varphi(yn'\delta'\delta p)\psi^{-1}(y_{n-1})\psi^{-1}(n')\, dy\, dn'\\ &= \int_{\mathrm{N}_n(k)\backslash \mathrm{N}_n(\mathbb{A})} \varphi(n\delta'\delta p)\psi^{-1}(n)\, dn\\ &= W_\varphi(\delta'\delta p)\end{aligned}$$

and so

$$\begin{aligned}\varphi(p) &= \sum_{\delta\in \widetilde{\mathrm{P}}_{n-1}(k)\backslash \mathrm{P}_n(k)} \sum_{\delta'\in \mathrm{N}_n(k)\backslash\widetilde{\mathrm{P}}_{n-1}(k)} W_\varphi(\delta'\delta p)\\ &= \sum_{\delta\in \mathrm{N}_n(k)\backslash \mathrm{P}_n(k)} W_\varphi(\delta p)\\ &= \sum_{\gamma\in \mathrm{N}_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k)} W_\varphi\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} p\right)\end{aligned}$$

which was what we wanted.

To obtain the Fourier expansion on GL_n from this, if φ is a cusp form on $\mathrm{GL}_n(\mathbb{A})$, then for $g \in \Omega$ a compact subset the functions $\varphi_g(p) = \varphi(pg)$ form a compact family of cuspidal functions on $\mathrm{P}_n(\mathbb{A})$. So we have

$$\varphi_g(1) = \sum_{\gamma\in \mathrm{N}_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k)} W_{\varphi_g}\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix}\right)$$

with convergence absolute and uniform. Hence

$$\varphi(g) = \sum_{\gamma\in \mathrm{N}_{n-1}(k)\backslash \mathrm{GL}_{n-1}(k)} W_\varphi\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} g\right)$$

again with absolute convergence, uniform for $g \in \Omega$.

1.2. Whittaker Models and the Multiplicity One Theorem

Consider now the functions W_φ appearing in the Fourier expansion of a cusp form φ. These are all smooth functions $W(g)$ on $\mathrm{GL}_n(\mathbb{A})$ which satisfy $W(ng) = \psi(n)W(g)$ for $n \in \mathrm{N}_n(\mathbb{A})$. If we let $\mathcal{W}(\pi, \psi) = \{W_\varphi \mid \varphi \in V_\pi\}$ then $\mathrm{GL}_n(\mathbb{A})$ acts on this space by right translation and the map $\varphi \mapsto W_\varphi$ intertwines V_π with $\mathcal{W}(\pi, \psi)$. $\mathcal{W}(\pi, \psi)$ is called the *Whittaker model* of π.

The notion of a Whittaker model of a representation makes perfect sense over a local field or even a finite field. Much insight can be gained by pursuing these ideas over a finite field [28,67], but that would take us too far afield. So let k_v be a local field (a completion of k for example [26,96]) and let (π_v, V_{π_v}) be an irreducible admissible smooth representation of $\mathrm{GL}_n(k_v)$. Fix a non-trivial continuous additive character ψ_v of k_v. Let $\mathcal{W}(\psi_v)$ be the space of all smooth functions $W(g)$ on $\mathrm{GL}_n(k_v)$ satisfying $W(ng) = \psi_v(n)W(g)$ for all $n \in \mathrm{N}_n(k_v)$, that is, the space of all smooth Whittaker functions on $\mathrm{GL}_n(k_v)$ with respect to ψ_v. This is also the space of the smooth induced representation $\mathrm{Ind}_{\mathrm{N}_v}^{\mathrm{GL}_n}(\psi_v)$. $\mathrm{GL}_n(k_v)$ acts on this by right translation. If we have a non-trivial continuous intertwining $V_{\pi_v} \to \mathcal{W}(\psi_v)$ we will denote its image by $\mathcal{W}(\pi_v, \psi_v)$ and call it a Whittaker model of π_v.

Whittaker models for a representation (π_v, V_{π_v}) are equivalent to continuous Whittaker functionals on V_{π_v}, that is, continuous functionals Λ_v satisfying $\Lambda_v(\pi_v(n)\xi_v) = \psi_v(n)\Lambda_v(\xi_v)$ for all $n \in \mathrm{N}_n(k_v)$. To obtain a Whittaker functional from a model, set $\Lambda_v(\xi_v) = W_{\xi_v}(e)$, and to obtain a model from a functional, set $W_{\xi_v}(g) = \Lambda_v(\pi_v(g)\xi_v)$. This is a form of Frobenius reciprocity, which in this context is the isomorphism between $\mathrm{Hom}_{\mathrm{N}_n}(V_{\pi_v}, \mathbb{C}_{\psi_v})$ and $\mathrm{Hom}_{\mathrm{GL}_n}(V_{\pi_v}, \mathrm{Ind}_{\mathrm{N}_n}^{\mathrm{GL}_n}(\psi_v))$ constructed above.

The fundamental theorem on the existence and uniqueness of Whittaker functionals and models is the following.

Theorem 1.2. Let (π_v, V_{π_v}) be a smooth irreducible admissible representation of $\mathrm{GL}_n(k_v)$. Let ψ_v be a non-trivial continuous additive character of k_v. Then the space of continuous ψ_v–Whittaker functionals on V_{π_v} is at most one dimensional. That is, Whittaker models, if they exist, are unique.

This was first proven for non-archimedean fields by Gelfand and Kazhdan [27] and their results were later extended to archimedean local fields by Shalika [85]. The method of proof is roughly the following. It is enough to show that $\mathcal{W}(\pi_v) = \mathrm{Ind}_{\mathrm{N}_n}^{\mathrm{GL}_n}(\psi_v)$ is multiplicity free, i.e., any irreducible representation of $\mathrm{GL}_n(k_v)$ occurs in $\mathcal{W}(\psi_v)$ with multiplicity at most one. This in turn is a consequence of the commutativity of the endomorphism algebra $\mathrm{End}(\mathrm{Ind}(\psi_v))$. Any intertwining map from $\mathrm{Ind}(\psi_v)$ to itself is given by convolution with a so-called Bessel distribution, that is, a distribution B on $\mathrm{GL}_n(k_v)$ satisfying $B(n_1gn_2) = \psi_v(n_1)B(g)\psi_v(n_2)$ for $n_1, n_2 \in \mathrm{N}_n(k_v)$. Such quasi-invariant distributions are analyzed via Bruhat theory. By the Bruhat decomposition for GL_n, the double cosets $\mathrm{N}_n \backslash \mathrm{GL}_n / \mathrm{N}_n$ are parameterized by the monomial matrices. The only double cosets that can support Bessel distributions are associated to permutation matrices of the form

$$\begin{pmatrix} & & I_{r_k} \\ & \iddots & \\ I_{r_1} & & \end{pmatrix}$$

and the resulting distributions are then stable under the involution $g \mapsto g^\sigma = w_n\, {}^t g\, w_n$ with $w_n = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ the long Weyl element of GL_n. Thus for the convolution of Bessel distributions we have $B_1 * B_2 = (B_1 * B_2)^\sigma = B_2^\sigma * B_1^\sigma = B_2 * B_1$. Hence the algebra of intertwining Bessel distributions is commutative and hence $\mathcal{W}(\psi_v)$ is multiplicity free.

A smooth irreducible admissible representation (π_v, V_{π_v}) of $\mathrm{GL}_n(k_v)$ which possesses a Whittaker model is called *generic* or *non-degenerate*. Gelfand and Kazhdan in addition show that π_v is generic iff its contragredient $\widetilde{\pi}_v$ is generic, in fact that $\widetilde{\pi} \simeq \pi^\iota$ where ι is the outer automorphism $g^\iota = {}^t g^{-1}$, and in this case the Whittaker model for $\widetilde{\pi}_v$ can be obtained as $\mathcal{W}(\widetilde{\pi}_v, \psi_v^{-1}) = \{\widetilde{W}(g) = W(w_n\, {}^t g^{-1}) \mid W \in \mathcal{W}(\pi, \psi_v)\}$.

As a consequence of the local uniqueness of the Whittaker model we can conclude a global uniqueness. If (π, V_π) is an irreducible smooth admissible representation of $\mathrm{GL}_n(\mathbb{A})$ then π factors as a restricted tensor product of local representations $\pi \simeq \otimes' \pi_v$ taken over all places v of k [19, 26]. Consequently we have a continuous embedding $V_{\pi_v} \hookrightarrow V_\pi$ for each local component. Hence any Whittaker functional Λ on V_π determines a family of local Whittaker functionals Λ_v on each V_{π_v} and conversely such that $\Lambda = \otimes' \Lambda_v$. Hence global uniqueness follows from the local uniqueness. Moreover, once we fix the isomorphism of V_π with $\otimes' V_{\pi_v}$ and define global and local Whittaker functions via Λ and the corresponding family Λ_v we have a factorization of global Whittaker functions

$$W_\xi(g) = \prod_v W_{\xi_v}(g_v)$$

for $\xi \in V_\pi$ which are factorizable in the sense that $\xi = \otimes' \xi_v$ corresponds to a pure tensor. As we will see, this factorization, which is a direct consequence of the uniqueness of the Whittaker model, plays a most important role in the development of Eulerian integrals for GL_n.

Now let us see what this means for our cuspidal representations (π, V_π) of $\mathrm{GL}_n(\mathbb{A})$. We have seen that for any smooth cusp form $\varphi \in V_\pi$ we have the Fourier expansion

$$\varphi(g) = \sum_{\gamma \in \mathrm{N}_{n-1}(k) \backslash GL_{n-1}(k)} W_\varphi\left(\begin{pmatrix} \gamma & \\ & 1 \end{pmatrix} g\right).$$

We can thus conclude that $\mathcal{W}(\pi, \psi) \neq 0$ and that π is (globally) generic with Whittaker functional

$$\Lambda(\varphi) = W_\varphi(e) = \int \varphi(ng)\psi^{-1}(n)\, dn.$$

Thus φ is completely determined by its associated Whittaker function W_φ. From the uniqueness of the global Whittaker model we can derive the Multiplicity One Theorem of Piatetski-Shapiro [66] and Shalika [85].

Theorem (Multiplicity One)**.** Let (π, V_π) be an irreducible smooth admissible representation of $\mathrm{GL}_n(\mathbb{A})$. Then the multiplicity of π in the space of cusp forms on $\mathrm{GL}_n(\mathbb{A})$ is at most one.

Proof: Suppose that π has two realizations (π_1, V_{π_1}) and (π_2, V_{π_2}) in the space of cusp forms on $\mathrm{GL}_n(\mathbb{A})$. Let $\varphi_i \in V_{\pi_i}$ be the two cusp forms associated to the vector

$\xi \in V_\pi$. Then we have two nonzero Whittaker functionals on V_π, namely $\Lambda_i(\xi) = W_{\varphi_i}(e)$. By the uniqueness of Whittaker models, there is a nonzero constant c such that $\Lambda_1 = c\Lambda_2$. But then we have $W_{\varphi_1}(g) = \Lambda_1(\pi(g)\xi) = c\Lambda_2(\pi(g)\xi) = cW_{\varphi_2}(g)$ for all $g \in \mathrm{GL}_n(\mathbb{A})$. Thus

$$\begin{aligned}\varphi_1(g) &= \sum_{\gamma \in \mathrm{N}_{n-1}(k)\backslash GL_{n-1}(k)} W_{\varphi_1}\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} g\right) \\ &= c \sum_{\gamma \in \mathrm{N}_{n-1}(k)\backslash GL_{n-1}(k)} W_{\varphi_2}\left(\begin{pmatrix}\gamma & \\ & 1\end{pmatrix} g\right) = c\varphi_2(g).\end{aligned}$$

But then V_{π_1} and V_{π_2} have a non-trivial intersection. Since they are irreducible representations, they must then coincide. $\square$

1.3. Kirillov models and the Strong Multiplicity One Theorem

The Multiplicity One Theorem can be significantly strengthened. The Strong Multiplicity One Theorem is the following.

Theorem (Strong Multiplicity One)**.** Let (π_1, V_{π_1}) and (π_2, V_{π_2}) be two cuspidal representations of $\mathrm{GL}_n(\mathbb{A})$. Suppose there is a finite set of places S such that for all $v \notin S$ we have $\pi_{1,v} \simeq \pi_{2,v}$. Then $\pi_1 = \pi_2$.

There are two proofs of this theorem. One is based on the theory of L-functions and we will come to it in Lecture 4. The original proof of Piatetski-Shapiro [66] is based on the *Kirillov model* of a local generic representation.

Let k_v be a local field and let (π_v, V_{π_v}) be an irreducible admissible smooth generic representation of $\mathrm{GL}_n(k_v)$, such as a local component of a cuspidal representation π. Since π_v is generic it has its Whittaker model $\mathcal{W}(\pi_v, \psi_v)$. Each Whittaker function $W \in \mathcal{W}(\pi_v, \psi_v)$, since it is a function on $\mathrm{GL}_n(k_v)$, can be restricted to the mirabolic subgroup $\mathrm{P}_n(k_v)$. A fundamental result of Bernstein and Zelevinsky in the non-archimedean case [1] and Jacquet and Shalika in the archimedean case [45] says that the map $\xi_v \mapsto W_{\xi_v}|_{\mathrm{P}_n(k_v)}$ is injective. Hence the representation has a realization on a space of functions on $\mathrm{P}_n(k_v)$. This is the Kirillov model

$$\mathcal{K}(\pi_v, \psi_v) = \{W(p) | W \in \mathcal{W}(\pi_v, \psi_v)\}.$$

$\mathrm{P}_n(k_v)$ acts naturally by right translation on $\mathcal{K}(\pi_v, \psi_v)$ and the action of all of $\mathrm{GL}_n(k_v)$ can be obtained by transport of structure. But for now, it is the structure of $\mathcal{K}(\pi_v, \psi_v)$ as a representation of $\mathrm{P}_n(k_v)$ which is of interest.

For k_v a non-archimedean field, let (τ_v, V_{τ_v}) be the compactly induced representation $\tau_v = \mathrm{ind}_{\mathrm{N}_n(k_v)}^{\mathrm{P}_n(k_v)}(\psi_v)$. Then Bernstein and Zelevinsky have analyzed the representations of $\mathrm{P}_n(k_v)$ and shown that whenever π_v is an irreducible admissible generic representation of $\mathrm{GL}_n(k_v)$ then $\mathcal{K}(\pi_v, \psi_v)$ contains V_{τ_v} as a $\mathrm{P}_n(k_v)$ sub-representation and if π_v is supercuspidal then $\mathcal{K}(\pi_v, \psi_v) = V_{\tau_v}$ [1].

For k_v archimedean, then we let (τ_v, V_{τ_v}) be the smooth vectors in the irreducible smooth unitarily induced representation $\mathrm{Ind}_{\mathrm{N}_n(k_v)}^{\mathrm{P}_n(k_v)}(\psi_v)$. Then Jacquet and Shalika have shown that as long as π_v is an irreducible admissible smooth unitary representation of $\mathrm{GL}_n(k_v)$ then in fact $\mathcal{K}(\pi_v, \psi_v) = V_{\tau_v}$ as representations of $\mathrm{P}_n(k_v)$ [45, Remark 3.15].

Therefore, for a given place v the local Kirillov models of any two irreducible admissible generic smooth unitary representations have a certain $\mathrm{P}_n(k_v)$-submodule in common, namely V_{τ_v}.

Let us now return to Piatetski-Shapiro's proof of the Strong Multiplicity One Theorem [66].

Proof: We begin with our cuspidal representations π_1 and π_2. Since π_1 and π_2 are irreducible, it suffices to find a cusp form $\varphi \in V_{\pi_1} \cap V_{\pi_2}$. Let $\mathrm{P}'_n = \mathrm{P}_n \mathrm{Z}_n$ be the $(n-1,1)$ parabolic subgroup of GL_n. Then $\mathrm{P}'_n(k)\backslash \mathrm{P}'_n(\mathbb{A})$ is dense in $\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A})$. (This follows from the fact that $\mathrm{P}'_n \backslash \mathrm{GL}_n \simeq \mathbb{P}^{n-1}$ and $\mathbb{P}^{n-1}(k)$ is dense in $\mathbb{P}^{n-1}(\mathbb{A})$.) So it suffices to find find two cusp forms $\varphi_i \in V_{\pi_i}$ which agree on $\mathrm{P}'_n(\mathbb{A})$. If we let ω_i be the central character of π_i then by assumption $\omega_{1,v} = \omega_{2,v}$ for $v \notin S$ and the weak approximation theorem then implies $\omega_1 = \omega_2$. So it suffices to find two φ_i which agree on $\mathrm{P}_n(\mathbb{A})$. But as in the proof of the Multiplicity One Theorem, via the Fourier expansion, to show that $\varphi_1(p) = \varphi_2(p)$ for $p \in \mathrm{P}_n(\mathbb{A})$ it suffices to show that $W_{\varphi_1}(p) = W_{\varphi_2}(p)$. Since we can take each W_{φ_i} to be of the form $\prod_v W_{\varphi_{i,v}}$ then this reduces to a question about the local Kirillov models. For $v \notin S$ we have by assumption that $\mathcal{K}(\pi_{1,v}, \psi_v) = \mathcal{K}(\pi_{2,v}, \psi_v)$ and for $v \in S$ we have seen that $V_{\tau_v} \subset \mathcal{K}(\pi_{1,v}, \psi_v) \cap \mathcal{K}(\pi_{2,v}, \psi_v)$. So we can construct a common Whittaker function in the restriction of $\mathcal{W}(\pi_i, \psi)$ to $\mathrm{P}_n(\mathbb{A})$. This completes the proof. $\square$

LECTURE 2
Eulerian integrals for GL_n

Let $f(\tau)$ again be a holomorphic cusp form of weight k on $\mathfrak{H}$ for the full modular group with Fourier expansion

$$f(\tau) = \sum a_n e^{2\pi i n \tau}.$$

Then Hecke [34] associated to f an L-function

$$L(s, f) = \sum a_n n^{-s}$$

and analyzed its analytic properties, namely continuation, order of growth, and functional equation, by writing it as the Mellin transform of f

$$\Lambda(s, f) = (2\pi)^{-s}\Gamma(s)L(s, f) = \int_0^\infty f(iy)y^s d^\times y.$$

An application of the modular transformation law for $f(\tau)$ under the transformation $\tau \mapsto -1/\tau$ gives the functional equation

$$\Lambda(s, f) = (-1)^{k/2}\Lambda(k - s, f).$$

Moreover, if f was an eigenfunction of all Hecke operators then $L(s, f)$ had an Euler product expansion

$$L(s, f) = \prod_p (1 - a_p p^{-s} + p^{k-1-2s})^{-1}.$$

We will present a similar theory for cuspidal representations (π, V_π) of $\mathrm{GL}_n(\mathbb{A})$. For applications to functoriality via the Converse Theorem (see Lecture 6) we will need not only the standard L-functions $L(s, \pi)$ but the twisted L-functions $L(s, \pi \times \pi')$ for $(\pi', V_{\pi'})$ a cuspidal automorphic representation of $\mathrm{GL}_m(\mathbb{A})$ for $m < n$ as well. One point to notice from the outset is that we want to associate a single L-function to an infinite dimensional representation (or pair of representations). The approach we will take will be that of integral representations, but it will broadened in the sense of Tate's thesis [91].

The basic references for the material in this section are Jacquet-Langlands [39], Jacquet, Piatetski-Shapiro, and Shalika [40], and Jacquet and Shalika [45].

2.1. Eulerian integrals for GL_2

Let us first consider the L-functions for cuspidal representations (π, V_π) of $\mathrm{GL}_2(\mathbb{A})$ with twists by an idele class character χ, or what is the same, a (cuspidal) automorphic representation of $\mathrm{GL}_1(\mathbb{A})$, as in Jacquet-Langlands [39].

Following Jacquet and Langlands, who were following Hecke, for each $\varphi \in V_\pi$ we consider the integral

$$I(s;\varphi,\chi) = \int_{k^\times\backslash\mathbb{A}^\times} \varphi\begin{pmatrix} a & \\ & 1\end{pmatrix} \chi(a)|a|^{s-1/2}\ d^\times a.$$

Since a cusp form on $\mathrm{GL}_2(\mathbb{A})$ is rapidly decreasing upon restriction to $\mathbb{A}^\times$ as in the integral, it follows that the integral is absolutely convergent for all s, uniformly for $\mathrm{Re}(s)$ in an interval. Thus $I(s;\varphi,\chi)$ is an entire function of s, bounded in any vertical strip $a \leq \mathrm{Re}(s) \leq b$. Moreover, if we let $\widetilde{\varphi}(g) = \varphi({}^t g^{-1}) = \varphi(w_n\ {}^t g^{-1})$ then $\widetilde{\varphi} \in V_{\widetilde{\pi}}$ and the simple change of variables $a \mapsto a^{-1}$ in the integral shows that each integral satisfies a functional equation of the form

$$I(s;\varphi,\chi) = I(1-s;\widetilde{\varphi},\chi^{-1}).$$

So these integrals individually enjoy rather nice analytic properties.

If we replace φ by its Fourier expansion from Lecture 1 and unfold, we find

$$\begin{aligned} I(s;\varphi,\chi) &= \int_{k^\times\backslash\mathbb{A}^\times} \sum_{\gamma\in k^\times} W_\varphi\begin{pmatrix} \gamma a & \\ & 1\end{pmatrix} \chi(a)|a|^{s-1/2}\ d^\times a \\ &= \int_{\mathbb{A}^\times} W_\varphi\begin{pmatrix} a & \\ & 1\end{pmatrix} \chi(a)|a|^{s-1/2}\ d^\times a \end{aligned}$$

where we have used the fact that the function $\chi(a)|a|^{s-1/2}$ is invariant under $k^\times$. By standard gauge estimates on Whittaker functions [40] this converges for $Re(s) >> 0$ after the unfolding. As we have seen in Lecture 1, if $W_\varphi \in \mathcal{W}(\pi,\psi)$ corresponds to a decomposable vector $\varphi \in V_\pi \simeq \otimes' V_{\pi_v}$ then the Whittaker function factors into a product of local Whittaker functions

$$W_\varphi(g) = \prod_v W_{\varphi_v}(g_v).$$

Since the character χ and the adelic absolute value factor into local components and the domain of integration $\mathbb{A}^\times$ also factors we find that our global integral naturally factors into a product of local integrals

$$\int_{\mathbb{A}^\times} W_\varphi\begin{pmatrix} a & \\ & 1\end{pmatrix} \chi(a)|a|^{s-1/2}\ d^\times a = \prod_v \int_{k_v^\times} W_{\varphi_v}\begin{pmatrix} a_v & \\ & 1\end{pmatrix} \chi_v(a_v)|a_v|^{s-1/2}\ d^\times a_v,$$

with the infinite product still convergent for $\mathrm{Re}(s) >> 0$, or

$$I(s;\varphi,\chi) = \prod_v \Psi_v(s;W_{\varphi_v},\chi_v)$$

with the obvious definition of the local integrals

$$\Psi_v(s;W_{\varphi_v},\chi_v) = \int_{k_v^\times} W_{\varphi_v}\begin{pmatrix} a_v & \\ & 1\end{pmatrix} \chi_v(a_v)|a_v|^{s-1/2}\ d^\times a_v.$$

Thus each of our global integrals is Eulerian.

In this way, to π and χ we have associated a family of global Eulerian integrals with nice analytic properties as well as for each place v a family of local integrals convergent for $\mathrm{Re}(s) >> 0$.

2.2. Eulerian integrals for $\mathrm{GL}_n \times \mathrm{GL}_m$ with $m < n$

Now let (π, V_π) be a cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$ and $(\pi', V_{\pi'})$ a cuspidal representation of $\mathrm{GL}_m(\mathbb{A})$ with $m < n$. Take $\varphi \in V_\pi$ and $\varphi' \in V_{\pi'}$. At first blush, a natural analogue of the integrals we considered for GL_2 with GL_1 twists would be

$$\int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \varphi \begin{pmatrix} h & \\ & I_{n-m} \end{pmatrix} \varphi'(h) |\det(h)|^{s-(n-m)/2}\, dh.$$

This family of integrals would have all the nice analytic properties as before (entire functions of finite order satisfying a functional equation), but they would not be Eulerian except in the case $m = n-1$, which proceeds exactly as in the GL_2 case.

The problem is that the restriction of the form φ to GL_m is too brutal to allow a nice unfolding when the Fourier expansion of φ is inserted. Instead we will introduce projection operators from cusp forms on $\mathrm{GL}_n(\mathbb{A})$ to cuspidal functions on $P_{m+1}(\mathbb{A})$ which are given by part of the unipotent integration through which the Whittaker function is defined.

2.2.1. The projection operator

In GL_n, let $\mathrm{Y}_{n,m}$ be the unipotent radical of the standard parabolic subgroup attached to the partition $(m+1, 1, \dots, 1)$. If ψ is our standard additive character of $k\backslash\mathbb{A}$, then ψ defines a character of $\mathrm{Y}_{n,m}(\mathbb{A})$ trivial on $\mathrm{Y}_{n,m}(k)$ since $\mathrm{Y}_{n,m} \subset \mathrm{N}_n$. The group $\mathrm{Y}_{n,m}$ is normalized by $\mathrm{GL}_{m+1} \subset \mathrm{GL}_n$ and the mirabolic subgroup $\mathrm{P}_{m+1} \subset \mathrm{GL}_{m+1}$ is the stabilizer in GL_{m+1} of the character ψ.

Definition . If $\varphi(g)$ is a cusp form on $\mathrm{GL}_n(\mathbb{A})$ define the projection operator $\mathbb{P}^n_m$ from cusp forms on $\mathrm{GL}_n(\mathbb{A})$ to cuspidal functions on $\mathrm{P}_{m+1}(\mathbb{A})$ by

$$\mathbb{P}^n_m\varphi(p) = |\det(p)|^{-\left(\frac{n-m-1}{2}\right)} \int_{\mathrm{Y}_{n,m}(k)\backslash \mathrm{Y}_{n,m}(\mathbb{A})} \varphi\left(y \begin{pmatrix} p & \\ & I_{n-m-1} \end{pmatrix}\right) \psi^{-1}(y)\, dy$$

for $p \in \mathrm{P}_{m+1}(\mathbb{A})$.

As the integration is over a compact domain, the integral is absolutely convergent. We first analyze the behavior on $\mathrm{P}_{m+1}(\mathbb{A})$.

Lemma 2.1. The function $\mathbb{P}^n_m\varphi(p)$ is a cuspidal function on $\mathrm{P}_{m+1}(\mathbb{A})$.

Proof: Let us let $\varphi'(p)$ denote the non-normalized projection, i.e., for $p \in \mathrm{P}_{m+1}(\mathbb{A})$ set

$$\varphi'(p) = |\det(p)|^{\left(\frac{n-m-1}{2}\right)} \mathbb{P}^n_m\varphi(p).$$

It suffices to show this function is cuspidal. Since $\varphi(g)$ was a smooth function on $\mathrm{GL}_n(\mathbb{A})$, $\varphi'(p)$ will remain smooth on $\mathrm{P}_{m+1}(\mathbb{A})$. To see that $\varphi'(p)$ is automorphic, let $\gamma \in \mathrm{P}_{m+1}(k)$. Then

$$\varphi'(\gamma p) = \int_{\mathrm{Y}_{n,m}(k)\backslash \mathrm{Y}_{n,m}(\mathbb{A})} \varphi\left(y \begin{pmatrix} \gamma & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & I \end{pmatrix}\right) \psi^{-1}(y)\, dy.$$

Since $\gamma \in \mathrm{P}_{m+1}(k)$ and P_{m+1} normalizes $\mathrm{Y}_{n,m}$ and stabilizes ψ we may make the change of variable $y \mapsto \begin{pmatrix} \gamma & 0 \\ 0 & I \end{pmatrix} y \begin{pmatrix} \gamma & 0 \\ 0 & I \end{pmatrix}^{-1}$ in this integral to obtain

$$\varphi'(\gamma p) = \int_{\mathrm{Y}_{n,m}(k)\backslash \mathrm{Y}_{n,m}(\mathbb{A})} \varphi\left(\begin{pmatrix} \gamma & 0 \\ 0 & I \end{pmatrix} y \begin{pmatrix} p & 0 \\ 0 & I \end{pmatrix}\right) \psi^{-1}(y)\ dy.$$

Since $\varphi(g)$ is automorphic on $\mathrm{GL}_n(\mathbb{A})$ it is left invariant under $\mathrm{GL}_n(k)$ and we find that $\varphi'(\gamma p) = \varphi'(p)$ so that φ' is indeed automorphic on $\mathrm{P}_{m+1}(\mathbb{A})$.

We next need to see that φ' is cuspidal on $\mathrm{P}_{m+1}(\mathbb{A})$. To this end, let $\mathrm{U} \subset \mathrm{P}_{m+1}$ be the standard unipotent subgroup associated to the partition $(n_1, \dots, n_r)$ of $m+1$. Then we must compute the integral

$$\int_{\mathrm{U}(k)\backslash \mathrm{U}(\mathbb{A})} \varphi'(up)\ du.$$

Inserting the definition of φ' we find

$$\begin{aligned} &\int_{\mathrm{U}(k)\backslash \mathrm{U}(\mathbb{A})} \varphi'(up)\ du \\ &\qquad = \int_{\mathrm{U}(k)\backslash \mathrm{U}(\mathbb{A})} \int_{\mathrm{Y}_{n,m}(k)\backslash \mathrm{Y}_{n,m}(\mathbb{A})} \varphi\left(y \begin{pmatrix} u & 0 \\ 0 & I \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & I \end{pmatrix}\right) \psi^{-1}(y)\ dy\ du. \end{aligned}$$

The group $\mathrm{U}' = \mathrm{U} \ltimes \mathrm{Y}_{n,m}$ is the standard unipotent subgroup of GL_n associated to the partition $(n_1, \dots, n_r, 1, \dots, 1)$ of n. We may decompose this group in a second manner. If we let U'' be the standard unipotent subgroup of GL_n associated to the partition $(n_1, \dots, n_{r-1}, n_r + n - m - 1)$ of n and let Y'' be the subgroup of GL_n obtained by embedding unipotent subgroup of $\mathrm{GL}_{n_r+n-m-1}$ associated to the partition $(n_r, 1, \dots 1)$ into GL_n by

$$y'' \mapsto \begin{pmatrix} I_{n_1+\cdots+n_{r-1}} & 0 \\ 0 & y'' \end{pmatrix}$$

then $\mathrm{U}' = \mathrm{Y}'' \ltimes \mathrm{U}''$. If we extend the character ψ of $\mathrm{Y}_{m,n}$ to U' by making it trivial on U, then in the decomposition $\mathrm{U}' = \mathrm{Y}'' \ltimes \mathrm{U}''$, ψ is dependent only on the Y'' component and there it is the standard character ψ on Y''. Hence we may rearrange the integration to give

$$\begin{aligned} &\int_{\mathrm{U}(k)\backslash \mathrm{U}(\mathbb{A})} \varphi'(up)\ du \\ &\quad = \int_{\mathrm{Y}''(k)\backslash \mathrm{Y}''(\mathbb{A})} \int_{\mathrm{U}''(k)\backslash \mathrm{U}''(\mathbb{A})} \varphi\left(u'' \begin{pmatrix} I & 0 \\ 0 & y'' \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & I \end{pmatrix}\right)\ du'' \psi^{-1}(y'')\ dy''. \end{aligned}$$

But since φ is cuspidal on GL_n and U'' is a standard unipotent subgroup of GL_n then

$$\int_{\mathrm{U}''(k)\backslash \mathrm{U}''(\mathbb{A})} \varphi\left(u'' \begin{pmatrix} I & 0 \\ 0 & y'' \end{pmatrix} \begin{pmatrix} p & 0 \\ 0 & I \end{pmatrix}\right)\ du'' \equiv 0$$

from which it follows that

$$\int_{\mathrm{U}(k)\backslash \mathrm{U}(\mathbb{A})} \varphi'(up)\ du \equiv 0$$

so that φ' is a cuspidal function on $\mathrm{P}_{m+1}(\mathbb{A})$. $\square$

From Lecture 1, we know that cuspidal functions on $\mathrm{P}_{m+1}(\mathbb{A})$ have a Fourier expansion summed over $\mathrm{N}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})$. Applying this expansion to our projected cusp form on $\mathrm{GL}_n(\mathbb{A})$ we are led to the following result.

Lemma 2.2. Let φ be a cusp form on $\mathrm{GL}_n(\mathbb{A})$. Then for $h \in \mathrm{GL}_m(\mathbb{A})$ we have the Fourier expansion

$$\mathbb{P}_m^n\varphi\begin{pmatrix} h & \\ & 1\end{pmatrix} = |\det(h)|^{-\left(\frac{n-m-1}{2}\right)} \sum_{\gamma\in \mathrm{N}_m(k)\backslash \mathrm{GL}_m(k)} W_\varphi\left(\begin{pmatrix}\gamma & 0\\ 0 & I_{n-m}\end{pmatrix}\begin{pmatrix} h & \\ & I_{n-m}\end{pmatrix}\right)$$

with convergence absolute and uniform on compact subsets.

Proof: Once again let

$$\varphi'(p) = |\det(p)|^{\left(\frac{n-m-1}{2}\right)}\mathbb{P}_m^n\varphi(p)$$

with $p \in \mathrm{P}_{m+1}(\mathbb{A})$. Since we have verified that $\varphi'(p)$ is a cuspidal function on $\mathrm{P}_{m+1}(\mathbb{A})$ we know that it has a Fourier expansion of the form

$$\varphi'(p) = \sum_{\gamma\in \mathrm{N}_m(k)\backslash \mathrm{GL}_m(k)} W_{\varphi'}\left(\begin{pmatrix}\gamma & 0\\ 0 & 1\end{pmatrix} p\right)$$

where

$$W_{\varphi'}(p) = \int_{\mathrm{N}_{m+1}(k)\backslash \mathrm{N}_{m+1}(\mathbb{A})} \varphi'(np)\psi^{-1}(n)\, dn.$$

To obtain our expansion for $\mathbb{P}_m^n\varphi$ we need to express the right hand side in terms of φ rather than φ'.

We have

$$\begin{aligned} W_{\varphi'}(p) &= \int_{\mathrm{N}_{m+1}(k)\backslash \mathrm{N}_{m+1}(\mathbb{A})} \varphi'(n'p)\psi^{-1}(n')\, dn' \\ &= \int_{\mathrm{N}_{m+1}(k)\backslash \mathrm{N}_{m+1}(\mathbb{A})}\int_{\mathrm{Y}_{n,m}(k)\backslash \mathrm{Y}_{n,m}(\mathbb{A})} \varphi\left(y\begin{pmatrix} n'p & 0\\ 0 & I\end{pmatrix}\right)\psi^{-1}(y)\, dy\ \psi^{-1}(n')\, dn'. \end{aligned}$$

It is elementary to see that the maximal unipotent subgroup N_n of GL_n can be factored as $\mathrm{N}_n = \mathrm{N}_{m+1} \ltimes \mathrm{Y}_{n,m}$ and if we write $n = n'y$ with $n' \in \mathrm{N}_{m+1}$ and $y \in \mathrm{Y}_{n,m}$ then $\psi(n) = \psi(n')\psi(y)$. Hence the above integral may be written as

$$W_{\varphi'}(p) = \int_{\mathrm{N}_n(k)\backslash \mathrm{N}_n(\mathbb{A})} \varphi\left(n\begin{pmatrix} p & 0\\ 0 & I_{n-m-1}\end{pmatrix}\right)\psi^{-1}(n)\, dn = W_\varphi\begin{pmatrix} p & 0\\ 0 & I_{n-m-1}\end{pmatrix}.$$

Substituting this expression into the above we find that

$$\mathbb{P}_m^n\varphi\begin{pmatrix} h & \\ & 1\end{pmatrix} = |\det(h)|^{-\left(\frac{n-m-1}{2}\right)} \sum_{\gamma\in \mathrm{N}_m(k)\backslash \mathrm{GL}_m(k)} W_\varphi\left(\begin{pmatrix}\gamma & 0\\ 0 & I_{n-m}\end{pmatrix}\begin{pmatrix} h & \\ & I_{n-m}\end{pmatrix}\right)$$

and the convergence is absolute and uniform for h in compact subsets of $\mathrm{GL}_m(\mathbb{A})$. $\square$

2.2.2. The global integrals

We now have the prerequisites for writing down a family of Eulerian integrals for cusp forms φ on GL_n twisted by automorphic forms on GL_m for $m < n$. Let $\varphi \in V_\pi$ be a cusp form on $\mathrm{GL}_n(\mathbb{A})$ and $\varphi' \in V_{\pi'}$ a cusp form on $\mathrm{GL}_m(\mathbb{A})$. (Actually, we could take φ' to be an arbitrary automorphic form on $\mathrm{GL}_m(\mathbb{A})$.) Consider the integrals

$$I(s;\varphi,\varphi') = \int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \mathbb{P}^n_m\varphi \begin{pmatrix} h & 0 \\ 0 & 1 \end{pmatrix} \varphi'(h)|\det(h)|^{s-1/2}\ dh.$$

The integral $I(s;\varphi,\varphi')$ is absolutely convergent for all values of the complex parameter s, uniformly in compact subsets, since the cusp forms are rapidly decreasing. Hence it is entire and bounded in any vertical strip as before.

Let us now investigate the Eulerian properties of these integrals. We first replace $\mathbb{P}^n_m\varphi$ by its Fourier expansion.

$$\begin{aligned} I(s;\varphi,\varphi') &= \int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \mathbb{P}^n_m\varphi \begin{pmatrix} h & 0 \\ 0 & I_{n-m} \end{pmatrix} \varphi'(h)|\det(h)|^{s-1/2}\ dh \\ &= \int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \sum_{\gamma\in \mathrm{N}_m(k)\backslash \mathrm{GL}_m(k)} W_\varphi\left(\begin{pmatrix} \gamma & 0 \\ 0 & I_{n-m} \end{pmatrix}\begin{pmatrix} h & 0 \\ 0 & I_{n-m} \end{pmatrix}\right) \\ &\qquad\qquad \varphi'(h)|\det(h)|^{s-(n-m)/2}\ dh. \end{aligned}$$

Since $\varphi'(h)$ is automorphic on $\mathrm{GL}_m(\mathbb{A})$ and $|\det(\gamma)| = 1$ for $\gamma \in \mathrm{GL}_m(k)$ we may interchange the order of summation and integration for $\mathrm{Re}(s) >> 0$ and then recombine to obtain

$$I(s;\varphi,\varphi') = \int_{\mathrm{N}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} W_\varphi \begin{pmatrix} h & 0 \\ 0 & I_{n-m} \end{pmatrix} \varphi'(h)|\det(h)|^{s-(n-m)/2}\ dh.$$

This integral is absolutely convergent for $\mathrm{Re}(s) >> 0$ by the gauge estimates of [40, Section 13] and this justifies the interchange.

Let us now integrate first over $\mathrm{N}_m(k)\backslash \mathrm{N}_m(\mathbb{A})$. Recall that for $n \in \mathrm{N}_m(\mathbb{A}) \subset \mathrm{N}_n(\mathbb{A})$ we have $W_\varphi(ng) = \psi(n)W_\varphi(g)$. Hence we have

$$\begin{aligned} I(s;\varphi,\varphi') &= \int_{\mathrm{N}_m(\mathbb{A})\backslash \mathrm{GL}_m(\mathbb{A})} \int_{\mathrm{N}_m(k)\backslash \mathrm{N}_m(\mathbb{A})} W_\varphi\left(\begin{pmatrix} n & 0 \\ 0 & I_{n-m} \end{pmatrix}\begin{pmatrix} h & 0 \\ 0 & I_{n-m} \end{pmatrix}\right) \\ &\qquad\qquad \varphi'(nh)\ dn\ |\det(h)|^{s-(n-m)/2}\ dh \\ &= \int_{\mathrm{N}_m(\mathbb{A})\backslash \mathrm{GL}_m(\mathbb{A})} W_\varphi \begin{pmatrix} h & 0 \\ 0 & I_{n-m} \end{pmatrix} \\ &\qquad\qquad \int_{\mathrm{N}_m(k)\backslash \mathrm{N}_m(\mathbb{A})} \psi(n)\varphi'(nh)\ dn\ |\det(h)|^{s-(n-m)/2}\ dh \\ &= \int_{\mathrm{N}_m(\mathbb{A})\backslash \mathrm{GL}_m(\mathbb{A})} W_\varphi \begin{pmatrix} h & 0 \\ 0 & I_{n-m} \end{pmatrix} W'_{\varphi'}(h)|\det(h)|^{s-(n-m)/2}\ dh \\ &= \Psi(s;W_\varphi,W'_{\varphi'}) \end{aligned}$$

where $W'_{\varphi'}(h)$ is the ψ^{-1}-Whittaker function on $\mathrm{GL}_m(\mathbb{A})$ associated to φ', i.e.,

$$W'_{\varphi'}(h) = \int_{\mathrm{N}_m(k)\backslash \mathrm{N}_m(\mathbb{A})} \varphi'(nh)\psi(n)\ dn,$$

and we retain absolute convergence for $\mathrm{Re}(s) >> 0$.

From this point, the fact that the integrals are Eulerian is a consequence of the uniqueness of the Whittaker model for GL_n. Take φ a smooth cusp form in a cuspidal representation π of $\mathrm{GL}_n(\mathbb{A})$. Assume in addition that φ is factorizable, i.e., in the decomposition $\pi = \otimes' \pi_v$ of π into a restricted tensor product of local representations, $\varphi = \otimes \varphi_v$ is a pure tensor. Then as we have seen there is a choice of local Whittaker models so that $W_\varphi(g) = \prod W_{\varphi_v}(g_v)$. Similarly for decomposable φ' we have the factorization $W'_{\varphi'}(h) = \prod W'_{\varphi'_v}(h_v)$.

If we substitute these factorizations into our integral expression, then since the domain of integration factors $\mathrm{N}_m(\mathbb{A})\backslash \mathrm{GL}_m(\mathbb{A}) = \prod \mathrm{N}_m(k_v)\backslash \mathrm{GL}_m(k_v)$ we see that our integral factors into a product of local integrals

$$\Psi(s;W_\varphi, W'_{\varphi'}) \\ = \prod_v \int_{\mathrm{N}_m(k_v)\backslash \mathrm{GL}_m(k_v)} W_{\varphi_v}\begin{pmatrix} h_v & 0 \\ 0 & I_{n-m}\end{pmatrix} W'_{\varphi'_v}(h_v)|\det(h_v)|_v^{s-(n-m)/2}\, dh_v.$$

If we denote the local integrals by

$$\Psi_v(s;W_{\varphi_v}, W'_{\varphi'_v}) \\ = \int_{\mathrm{N}_m(k_v)\backslash \mathrm{GL}_m(k_v)} W_{\varphi_v}\begin{pmatrix} h_v & 0 \\ 0 & I_{n-m}\end{pmatrix} W'_{\varphi'_v}(h_v)|\det(h_v)|_v^{s-(n-m)/2}\, dh_v,$$

which converges for $\mathrm{Re}(s) >> 0$ by the gauge estimate of [40, Prop. 2.3.6], we see that we now have a family of Eulerian integrals.

Now let us return to the question of a functional equation. As in the case of GL_2, the functional equation is essentially a consequence of the existence of the outer automorphism $g \mapsto \iota(g) = g^\iota = {}^t g^{-1}$ of GL_n. If we define the action of this automorphism on automorphic forms by setting $\widetilde{\varphi}(g) = \varphi(g^\iota) = \varphi(w_n g^\iota)$ and let $\widetilde{\mathbb{P}}^n_m = \iota \circ \mathbb{P}^n_m \circ \iota$ then our integrals naturally satisfy the functional equation

$$I(s;\varphi,\varphi') = \widetilde{I}(1-s;\widetilde{\varphi},\widetilde{\varphi}')$$

where

$$\widetilde{I}(s;\varphi,\varphi') = \int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \widetilde{\mathbb{P}}^n_m \varphi\begin{pmatrix} h & \\ & 1\end{pmatrix} \varphi'(h)|\det(h)|^{s-1/2}\, dh.$$

We have established the following result.

Theorem 2.1. Let $\varphi \in V_\pi$ be a cusp form on $\mathrm{GL}_n(\mathbb{A})$ and $\varphi' \in V_{\pi'}$ a cusp form on $\mathrm{GL}_m(\mathbb{A})$ with $m < n$. Then the family of integrals $I(s;\varphi,\varphi')$ define entire functions of s, bounded in vertical strips, and satisfy the functional equation

$$I(s;\varphi,\varphi') = \widetilde{I}(1-s;\widetilde{\varphi},\widetilde{\varphi}').$$

Moreover the integrals are Eulerian and if φ and φ' are factorizable, we have

$$I(s;\varphi,\varphi') = \prod_v \Psi_v(s;W_{\varphi_v}, W'_{\varphi'_v})$$

with convergence absolute and uniform for $\mathrm{Re}(s) >> 0$.

The integrals occurring in the right hand side of our functional equation are again Eulerian. One can unfold the definitions to find first that

$$\widetilde{I}(1-s;\widetilde{\varphi},\widetilde{\varphi}') = \widetilde{\Psi}(1-s;\rho(w_{n,m})\widetilde{W}_\varphi, \widetilde{W}'_{\varphi'})$$

where the unfolded global integral is

$$\widetilde{\Psi}(s;W,W') = \int\int W\begin{pmatrix} h & & \\ x & I_{n-m-1} & \\ & & 1\end{pmatrix} dx\ W'(h)|\det(h)|^{s-(n-m)/2}\ dh$$

with the h integral over $\mathrm{N}_m(\mathbb{A})\backslash \mathrm{GL}_m(\mathbb{A})$ and the x integral over $M_{n-m-1,m}(\mathbb{A})$, the space of $(n-m-1)\times m$ matrices, ρ denoting right translation, and $w_{n,m}$ the Weyl element $w_{n,m} = \begin{pmatrix} I_m & \\ & w_{n-m}\end{pmatrix}$ with $w_{n-m} = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ the standard long Weyl element in GL_{n-m}. Also, for $W\in\mathcal{W}(\pi,\psi)$ we set $\widetilde{W}(g) = W(w_n g^{\iota}) \in \mathcal{W}(\widetilde{\pi},\psi^{-1})$. The extra unipotent integration is the remnant of $\widetilde{\mathbb{P}}^n_m$. As before, $\widetilde{\Psi}(s;W,W')$ is absolutely convergent for $\mathrm{Re}(s) >> 0$. For φ and φ' factorizable as before, these integrals $\widetilde{\Psi}(s;W_\varphi,W'_{\varphi'})$ will factor as well. Hence we have

$$\widetilde{\Psi}(s;W_\varphi,W'_{\varphi'}) = \prod_v \widetilde{\Psi}_v(s;W_{\varphi_v},W'_{\varphi'_v})$$

where

$$\widetilde{\Psi}_v(s;W_v,W'_v) = \int\int W_v\begin{pmatrix} h_v & & \\ x_v & I_{n-m-1} & \\ & & 1\end{pmatrix} dx_v\ W'_v(h_v)|\det(h_v)|^{s-(n-m)/2}\ dh_v$$

where now with the h_v integral is over $\mathrm{N}_m(k_v)\backslash \mathrm{GL}_m(k_v)$ and the x_v integral is over the matrix space $M_{n-m-1,m}(k_v)$. Thus, coming back to our functional equation, we find that the right hand side is Eulerian and factors as

$$\widetilde{I}(1-s;\widetilde{\varphi},\widetilde{\varphi}') = \widetilde{\Psi}(1-s;\rho(w_{n,m})\widetilde{W}_\varphi,\widetilde{W}'_{\varphi'}) = \prod_v \widetilde{\Psi}_v(1-s;\rho(w_{n,m})\widetilde{W}_{\varphi_v},\widetilde{W}'_{\varphi'_v}).$$

2.3. Eulerian integrals for $\mathrm{GL}_n\times\mathrm{GL}_n$

The paradigm for integral representations of L-functions for $\mathrm{GL}_n\times\mathrm{GL}_n$ is not Hecke but rather the classical papers of Rankin [71] and Selberg [73]. These were first interpreted in the framework of automorphic representations by Jacquet for $\mathrm{GL}_2\times\mathrm{GL}_2$ [37] and then Jacquet and Shalika in general [45].

Let (π,V_π) and $(\pi',V_{\pi'})$ be two cuspidal representations of $\mathrm{GL}_n(\mathbb{A})$. Let $\varphi\in V_\pi$ and $\varphi'\in V_{\pi'}$ be two cusp forms. The analogue of the construction above would be simply

$$\int_{\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A})} \varphi(g)\varphi'(g)|\det(g)|^s\ dg.$$

This integral is essentially the L^2-inner product of φ and φ' and is not suitable for defining an L-function, although it will occur as a residue of our integral at a pole. Instead, following Rankin and Selberg, we use an integral representation that involves a third function: an Eisenstein series on $\mathrm{GL}_n(\mathbb{A})$. This family of Eisenstein series is constructed using the mirabolic subgroup once again.

2.3.1. The mirabolic Eisenstein series

To construct our Eisenstein series we return to the observation that $\mathrm{P}_n\backslash\mathrm{GL}_n \simeq k^n - \{0\}$. If we let $\mathcal{S}(\mathbb{A}^n)$ denote the Schwartz–Bruhat functions on $\mathbb{A}^n$, then each $\Phi\in\mathcal{S}$ defines a smooth function on $\mathrm{GL}_n(\mathbb{A})$, left invariant by $\mathrm{P}_n(\mathbb{A})$, by

$g \mapsto \Phi((0,\dots,0,1)g) = \Phi(e_n g)$. Let η be a unitary idele class character. (For our application η will be determined by the central characters of π and π'.) Consider the function

$$F(g,\Phi;s,\eta) = |\det(g)|^s \int_{\mathbb{A}^\times} \Phi(ae_n g)|a|^{ns}\eta(a)\ d^\times a.$$

If we let $\mathrm{P}'_n = \mathrm{Z}_n\,\mathrm{P}_n$ be the parabolic of GL_n associated to the partition $(n-1,1)$ then one checks that for $p' = \begin{pmatrix} h & y \\ 0 & d\end{pmatrix} \in \mathrm{P}'_n(\mathbb{A})$ with $h \in \mathrm{GL}_{n-1}(\mathbb{A})$ and $d \in \mathbb{A}^\times$ we have,

$$\begin{aligned} F(p'g,\Phi;s,\eta) &= |\det(h)|^s|d|^{-(n-1)s}\eta(d)^{-1}F(g,\Phi;s,\eta) \\ &= \delta^s_{\mathrm{P}'_n}(p')\eta^{-1}(d)F(g,\Phi;s,\eta), \end{aligned}$$

with the integral absolutely convergent for $\mathrm{Re}(s) > 1/n$, so that if we extend η to a character of P'_n by $\eta(p') = \eta(d)$ in the above notation we have that $F(g,\Phi;s,\eta)$ is a smooth section of the normalized induced representation $\mathrm{Ind}_{\mathrm{P}'_n(\mathbb{A})}^{\mathrm{GL}_n(\mathbb{A})}(\delta^{s-1/2}_{\mathrm{P}'_n}\eta)$. Since the inducing character $\delta^{s-1/2}_{\mathrm{P}'_n}\eta$ of $\mathrm{P}'_n(\mathbb{A})$ is invariant under $\mathrm{P}'_n(k)$ we may form Eisenstein series from this family of sections by

$$E(g,\Phi;s,\eta) = \sum_{\gamma \in \mathrm{P}'_n(k)\backslash \mathrm{GL}_n(k)} F(\gamma g,\Phi;s,\eta).$$

If we replace F in this sum by its definition we can rewrite this Eisenstein series as

$$\begin{aligned} E(g,\Phi;s,\eta) &= |\det(g)|^s \int_{k^\times\backslash\mathbb{A}^\times} \sum_{\xi \in k^n - \{0\}} \Phi(a\xi g)|a|^{ns}\eta(a)\ d^\times a \\ &= |\det(g)|^s \int_{k^\times\backslash\mathbb{A}^\times} \Theta'_\Phi(a,g)|a|^{ns}\eta(a)\ d^\times a \end{aligned}$$

and this first expression is convergent absolutely for $\mathrm{Re}(s) > 1$ [45].

The second expression essentially gives the Eisenstein series as the Mellin transform of the Theta series

$$\Theta_\Phi(a,g) = \sum_{\xi\in k^n} \Phi(a\xi g),$$

where in the above we have written

$$\Theta'_\Phi(a,g) = \sum_{\xi \in k^n - \{0\}} \Phi(a\xi g) = \Theta_\Phi(a,g) - \Phi(0).$$

This allows us to obtain the analytic properties of the Eisenstein series from the Poisson summation formula for Θ_Φ, namely

$$\begin{aligned} \Theta_\Phi(a,g) &= \sum_{\xi\in k^n}\Phi(a\xi g) = \sum_{\xi\in k^n}\Phi_{a,g}(\xi) \\ &= \sum_{\xi\in k^n}\widehat{\Phi_{a,g}}(\xi) = \sum_{\xi\in k^n}|a|^{-n}|\det(g)|^{-1}\widehat{\Phi}(a^{-1}\xi\,{}^tg^{-1}) \\ &= |a|^{-n}|\det(g)|^{-1}\Theta_{\hat\Phi}(a^{-1},{}^tg^{-1}) \end{aligned}$$

where the Fourier transform $\hat\Phi$ on $\mathcal{S}(\mathbb{A}^n)$ is defined by

$$\hat\Phi(x) = \int_{\mathbb{A}^\times} \Phi(y)\psi(y\,{}^tx)\ dy.$$

This allows us to write the Eisenstein series as

$$E(g,\Phi,s,\eta)=|\det(g)|^s\int_{|a|\geq 1}\Theta'_\Phi(a,g)|a|^{ns}\eta(a)\ d^\times a$$
$$+|\det(g)|^{s-1}\int_{|a|\geq 1}\Theta'_{\hat\Phi}(a,{}^tg^{-1})|a|^{n(1-s)}\eta^{-1}(a)\ d^\times a+\delta(s)$$

where

$$\delta(s)=\begin{cases}0 & \text{if } \eta \text{ is ramified}\\ -c\Phi(0)\frac{|\det(g)|^s}{s+i\sigma}+c\hat\Phi(0)\frac{|\det(g)|^{s-1}}{s-1+i\sigma} & \text{if } \eta(a)=|a|^{in\sigma} \text{ with } \sigma\in\mathbb{R}\end{cases}$$

with c a non-zero constant. From this we derive easily the basic properties of our Eisenstein series [45, Section 4].

Proposition 2.1. The Eisenstein series $E(g,\Phi;s,\eta)$ has a meromorphic continuation to all of $\mathbb{C}$ with at most simple poles at $s=-i\sigma,1-i\sigma$ when η is unramified of the form $\eta(a)=|a|^{in\sigma}$. As a function of g it is smooth of moderate growth and as a function of s it is bounded in vertical strips (away from the possible poles), uniformly for g in compact sets. Moreover, we have the functional equation

$$E(g,\Phi;s,\eta)=E(g^\iota,\hat\Phi;1-s,\eta^{-1})$$

where $g^\iota={}^tg^{-1}$.

Note that under the center the Eisenstein series transforms by the central character η^{-1}.

2.3.2. The global integrals

Now let us return to our Eulerian integrals. Let π and π' be our irreducible cuspidal representations. Let their central characters be ω and ω'. Set $\eta=\omega\omega'$. Then for each pair of cusp forms $\varphi\in V_\pi$ and $\varphi'\in V_{\pi'}$ and each Schwartz-Bruhat function $\Phi\in\mathcal{S}(\mathbb{A}^n)$ set

$$I(s;\varphi,\varphi',\Phi)=\int_{\mathrm{Z}_n(\mathbb{A})\,\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbb{A})}\varphi(g)\varphi'(g)E(g,\Phi;s,\eta)\ dg.$$

Since the two cusp forms are rapidly decreasing on $\mathrm{Z}_n(\mathbb{A})\,\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbb{A})$ and the Eisenstein is only of moderate growth, we see that the integral converges absolutely for all s away from the poles of the Eisenstein series and is hence meromorphic. It will be bounded in vertical strips away from the poles and satisfies the functional equation

$$I(s;\varphi,\varphi',\Phi)=I(1-s;\widetilde\varphi,\widetilde\varphi',\hat\Phi),$$

coming from the functional equation of the Eisenstein series, where we still have $\widetilde\varphi(g)=\varphi(g^\iota)=\varphi(w_ng^\iota)\in V_{\widetilde\pi}$ and similarly for $\widetilde\varphi'$.

These integrals will be entire unless we have $\eta(a)=\omega(a)\omega'(a)=|a|^{in\sigma}$ is unramified. In that case, the residue at $s=-i\sigma$ will be

$$\operatorname*{Res}_{s=-i\sigma}I(s;\varphi,\varphi',\Phi)=-c\Phi(0)\int_{\mathrm{Z}_n(\mathbb{A})\,\mathrm{GL}_n(\mathbb{A})\backslash\mathrm{GL}_n(\mathbb{A})}\varphi(g)\varphi'(g)|\det(g)|^{-i\sigma}\ dg$$

and at $s=1-i\sigma$ we can write the residue as

$$\operatorname*{Res}_{s=1-i\sigma}I(s;\varphi,\varphi',\Phi)=c\hat\Phi(0)\int_{\mathrm{Z}_n(\mathbb{A})\,\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbb{A})}\widetilde\varphi(g)\widetilde\varphi'(g)|\det(g)|^{i\sigma}\ dg.$$

Therefore these residues define $\mathrm{GL}_n(\mathbb{A})$ invariant pairings between π and $\pi'\otimes|\det|^{-i\sigma}$ or equivalently between $\widetilde{\pi}$ and $\widetilde{\pi}'\otimes|\det|^{i\sigma}$. Hence a residues can be non-zero only if $\pi\simeq\widetilde{\pi}'\otimes|\det|^{i\sigma}$ and in this case we can find φ, φ', and Φ such that indeed the residue does not vanish.

We have yet to check that our integrals are Eulerian. To this end we take the integral, replace the Eisenstein series by its definition, and unfold:

$$\begin{aligned}
I(s;\varphi,\varphi',\Phi) &= \int_{\mathrm{Z}_n(\mathbb{A})\,\mathrm{GL}_n(k)\backslash\mathrm{GL}_n(\mathbb{A})} \varphi(g)\varphi'(g)E(g,\Phi;s,\eta)\ dg\\
&= \int_{\mathrm{Z}_n(\mathbb{A})\,\mathrm{P}'_n(k)\backslash\mathrm{GL}_n(\mathbb{A})} \varphi(g)\varphi'(g)F(g,\Phi;s,\eta)\ dg\\
&= \int_{\mathrm{Z}_n(\mathbb{A})\,\mathrm{P}_n(k)\backslash\mathrm{GL}_n(\mathbb{A})} \varphi(g)\varphi'(g)|\det(g)|^s\int_{\mathbb{A}^\times}\Phi(ae_ng)|a|^{ns}\eta(a)\ da\ dg\\
&= \int_{\mathrm{P}_n(k)\backslash\mathrm{GL}_n(\mathbb{A})} \varphi(g)\varphi'(g)\Phi(e_ng)|\det(g)|^s\ dg.
\end{aligned}$$

We next replace φ by its Fourier expansion in the form

$$\varphi(g)=\sum_{\gamma\in\mathrm{N}_n(k)\backslash\mathrm{P}_n(k)} W_\varphi(\gamma g)$$

and unfold to find

$$\begin{aligned}
I(s;\varphi,\varphi',\Phi) &= \int_{\mathrm{N}_n(k)\backslash\mathrm{GL}_n(\mathbb{A})} W_\varphi(g)\varphi'(g)\Phi(e_ng)|\det(g)|^s\ dg\\
&= \int_{\mathrm{N}_n(\mathbb{A})\backslash\mathrm{GL}_n(\mathbb{A})} W_\varphi(g)\int_{\mathrm{N}_n(k)\backslash\mathrm{N}_n(\mathbb{A})}\varphi'(ng)\psi(n)\ dn\ \Phi(e_ng)|\det(g)|^s\ dg\\
&= \int_{\mathrm{N}_n(\mathbb{A})\backslash\mathrm{GL}_n(\mathbb{A})} W_\varphi(g)W'_{\varphi'}(g)\Phi(e_ng)|\det(g)|^s\ dg\\
&= \Psi(s;W_\varphi,W'_{\varphi'},\Phi).
\end{aligned}$$

This expression converges for $\mathrm{Re}(s)>>0$ by the gauge estimates as before.

To continue, we assume that φ, φ' and Φ are decomposable tensors under the isomorphisms $\pi\simeq\otimes'\pi_v$, $\pi'\simeq\otimes'\pi'_v$, and $\mathcal{S}(\mathbb{A}^n)\simeq\otimes'\mathcal{S}(k_v^n)$ so that we have $W_\varphi(g)=\prod_v W_{\varphi_v}(g_v)$, $W'_{\varphi'}(g)=\prod_v W'_{\varphi'_v}(g_v)$ and $\Phi(g)=\prod_v\Phi_v(g_v)$. Then, since the domain of integration also naturally factors we can decompose this last integral into an Euler product and now write

$$\Psi(s;W_\varphi,W'_{\varphi'},\Phi)=\prod_v\Psi_v(s;W_{\varphi_v},W'_{\varphi'_v},\Phi_v),$$

where

$$\Psi_v(s;W_{\varphi_v},W'_{\varphi'_v},\Phi_v)=\int_{\mathrm{N}_n(k_v)\backslash\mathrm{GL}_n(k_v)} W_{\varphi_v}(g_v)W'_{\varphi'_v}(g_v)\Phi_v(e_ng_v)|\det(g_v)|^s\ dg_v,$$

still with convergence for $\mathrm{Re}(s)>>0$ by the local gauge estimates. Once again we see that the Euler factorization is a direct consequence of the uniqueness of the Whittaker models.

Theorem 2.2. Let $\varphi\in V_\pi$ and $\varphi'\in V_{\pi'}$ cusp forms on $\mathrm{GL}_n(\mathbb{A})$ and let $\Phi\in\mathcal{S}(\mathbb{A}^n)$. Then the family of integrals $I(s;\varphi,\varphi',\Phi)$ define meromorphic functions of s, bounded in vertical strips away from the poles. The only possible poles are simple and occur iff $\pi \simeq \widetilde{\pi}' \otimes |\det|^{i\sigma}$ with σ real and are then at $s = -i\sigma$ and $s = 1 - i\sigma$ with residues as above. They satisfy the functional equation

$$I(s;\varphi,\varphi',\Phi) = I(1-s;\widetilde{W}_\varphi,\widetilde{W}'_{\varphi'},\hat{\Phi}).$$

Moreover, for φ, φ', and Φ factorizable we have that the integrals are Eulerian and we have

$$I(s;\varphi,\varphi',\Phi) = \prod_v \Psi_v(s;W_{\varphi_v},W'_{\varphi'_v},\Phi_v)$$

with convergence absolute and uniform for $\mathrm{Re}(s) >> 0$.

We remark in passing that the right hand side of the functional equation also unfolds as

$$\begin{aligned} I(1-s;\widetilde{\varphi},\widetilde{\varphi}',\hat{\Phi}) &= \int_{\mathrm{N}_n(\mathbb{A})\backslash\,\mathrm{GL}_n(\mathbb{A})} \widetilde{W}_\varphi(g)\widetilde{W}'_{\varphi'}(g)\hat{\Phi}(e_n g)|\det(g)|^{1-s}\,dg \\ &= \prod_v \Psi_v(1-s;\widetilde{W}_\varphi,\widetilde{W}'_{\varphi'},\hat{\Phi}) \end{aligned}$$

with convergence for $\mathrm{Re}(s) << 0$.

We note again that if these integrals are not entire, then the residues give us invariant pairings between the cuspidal representations and hence tell us structural facts about the relation between these representations.

LECTURE 3
Local L-functions

If (π, V_π) is a cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$ and $(\pi', V_{\pi'})$ is a cuspidal representation of $\mathrm{GL}_m(\mathbb{A})$ we have associated to the pair (π, π') a family of Eulerian integrals $\{I(s;\varphi,\varphi')\}$ (or $\{I(s;\varphi,\varphi',\Phi)\}$ if $m = n$) and through the Euler factorization we have for each place v of k a family of local integrals $\{\Psi_v(s; W_v, W_v')\}$ (or $\{\Psi_v(s; W_v, W_v', \Phi_v)\}$) attached to the pair of local components (π_v, π_v'). In this lecture we would like to attach a local L-function (or local Euler factor) $L(s, \pi_v \times \pi_v')$ to such a pair of local representations through the family of local integrals and analyze its basic properties, including the local functional equation. The paradigm for such an analysis of local L-functions is Tate's thesis [91]. The mechanics of the archimedean and non-archimedean theories are slightly different so we will treat them separately, beginning with the non-archimedean theory.

3.1. The non-archimedean local factors

For this section we will let k denote a non-archimedean local field. We will let $\mathfrak{o}$ denote the ring of integers of k and $\mathfrak{p}$ the unique prime ideal of $\mathfrak{o}$. Fix a generator ϖ of $\mathfrak{p}$. We let q be the residue degree of k, so $q = |\mathfrak{o}/\mathfrak{p}| = |\varpi|^{-1}$. We fix a nontrivial continuous additive character ψ of k. (π, V_π) and $(\pi', V_{\pi'})$ will now be the smooth vectors in irreducible admissible unitary generic representations of $\mathrm{GL}_n(k)$ and $\mathrm{GL}_m(k)$ respectively, as is true for local components of cuspidal representations. We will let ω and ω' denote their central characters.

The basic reference for this section is the paper of Jacquet, Piatetski-Shapiro, and Shalika [42].

3.1.1. The local L-function

For each pair of Whittaker functions $W \in \mathcal{W}(\pi, \psi)$ and $W' \in \mathcal{W}(\pi', \psi^{-1})$ and in the case $n = m$ each Schwartz-Bruhat function $\Phi \in \mathcal{S}(k^n)$ we have defined local integrals

$$\Psi(s; W, W') = \int W \begin{pmatrix} h & \\ & I_{n-m} \end{pmatrix} W'(h) |\det(h)|^{s-(n-m)/2}\, dh$$

$$\widetilde{\Psi}(s; W, W') = \int\int W \begin{pmatrix} h & & \\ x & I_{n-m-1} & \\ & & 1 \end{pmatrix} dx\ W'(h) |\det(h)|^{s-(n-m)/2}\, dh$$

in the case $m < n$, where the h integration is over $\mathrm{N}_m(k)\backslash \mathrm{GL}_m(k)$ in both integrals and the x integration is over the matrix space $M_{n-m-1,m}(k)$, and in the case $n = m$

$$\Psi(s; W, W', \Phi) = \int_{\mathrm{N}_n(k)\backslash \mathrm{GL}_n(k)} W(g)W'(g)\Phi(e_n g)|\det(g)|^s \, dg$$

all integrals being convergent for $\mathrm{Re}(s) >> 0$. To make the notation more convenient for what follows, in the case $m < n$ for any $0 \leq j \leq n-m-1$ let us set

$$\Psi_j(s : W, W') = \int\int W \begin{pmatrix} h & & \\ x & I_j & \\ & & I_{n-m-j} \end{pmatrix} dx \; W'(h)|\det(h)|^{s-(n-m)/2} \, dh,$$

where the h integral is still over $\mathrm{N}_m(k)\backslash \mathrm{GL}_m(k)$ and now the x integral is over the matrix space $M_{j,m}(k)$, so that $\Psi(s; W, W') = \Psi_0(s; W, W')$ and $\widetilde{\Psi}(s; W, W') = \Psi_{n-m-1}(s; W, W')$, which is still absolutely convergent for $\mathrm{Re}(s) >> 0$.

We need to understand what type of functions of s these local integrals are. To this end, we need to understand the local Whittaker functions. So let $W \in \mathcal{W}(\pi, \psi)$. Since W is smooth, there is a compact open subgroup K', of finite index in the maximal compact subgroup $\mathrm{K}_n = \mathrm{GL}_n(\mathfrak{o})$, so that $W(gk) = W(g)$ for all $k \in \mathrm{K}'$. If we let $\{k_i\}$ be a set of coset representatives of $\mathrm{GL}_n(\mathfrak{o})/\mathrm{K}'$, using that W transforms on the left under $\mathrm{N}_n(k)$ via ψ and the Iwasawa decomposition on $\mathrm{GL}_n(k)$ we see that $W(g)$ is completely determined by the values of $W(ak_i) = W_i(a)$ for $a \in \mathrm{A}_n(k)$, the maximal split (diagonal) torus of $\mathrm{GL}_n(k)$. So it suffices to understand a general Whittaker function on the torus. Let α_i, $i = 1, \ldots, n-1$, denote the standard simple roots of GL_n, so that if $a = \begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \in \mathrm{A}_n(k)$ then $\alpha_i(a) = a_i/a_{i+1}$. By a finite function on $\mathrm{A}_n(k)$ we mean a continuous function whose translates span a finite dimensional vector space [39, 40, Section 2.2]. (For the field $k^\times$ itself the finite functions are spanned by products of characters and powers of the valuation map.) The fundamental result on the asymptotics of Whittaker functions is then the following [40, Prop. 2.2].

Proposition 3.1. Let π be a generic representation of $\mathrm{GL}_n(k)$. Then there is a finite set of finite functions $X(\pi) = \{\chi_i\}$ on $\mathrm{A}_n(k)$, depending only on π, so that for every $W \in \mathcal{W}(\pi, \psi)$ there are Schwartz –Bruhat functions $\phi_i \in \mathcal{S}(k^{n-1})$ such that for all $a \in \mathrm{A}_n(k)$ with $a_n = 1$ we have

$$W(a) = \sum_{X(\pi)} \chi_i(a)\phi_i(\alpha_1(a), \ldots, \alpha_{n-1}(a)).$$

The finite set of finite functions $X(\pi)$ which occur in the asymptotics near 0 of the Whittaker functions come from analyzing the Jacquet module of π in the form $\mathcal{W}(\pi, \psi)/\langle \pi(n)W - W | n \in \mathrm{N}_n\rangle$ which is naturally an $\mathrm{A}_n(k)$–module. Note that due to the Schwartz-Bruhat functions, the Whittaker functions vanish whenever any simple root $\alpha_i(a)$ becomes large. The gauge estimates alluded to in Lecture 2 are a consequence of this expansion and the one in Proposition 3.6.

Several nice consequences follow from inserting these formulas for W and W' into the local integrals $\Psi_j(s; W, W')$ or $\Psi(s; W, W', \Phi)$ [40, 42].

Proposition 3.2. The local integrals $\Psi_j(s;W,W')$ or $\Psi(s;W,W',\Phi)$ satisfy the following properties.

1. Each integral converges for $\mathrm{Re}(s) >> 0$. For π and π' unitary, as we have assumed, they converge absolutely for $\mathrm{Re}(s) \geq 1$. For π and π' tempered, we have absolute convergence for $\mathrm{Re}(s) > 0$.
2. Each integral defines a rational function in q^{-s} and hence meromorphically extends to all of $\mathbb{C}$.
3. Each such rational function can be written with a common denominator which depends only on the finite functions $X(\pi)$ and $X(\pi')$ and hence only on π and π'.

In deriving these when $m < n-1$ note that one has that

$$W\begin{pmatrix} h & & \\ x & I_j & \\ & & I_{n-m-j-1}\end{pmatrix} \neq 0$$

implies that x lies in a compact set independent of $h \in \mathrm{GL}_m(k)$ [42].

Let $\mathcal{I}_j(\pi,\pi')$ denote the complex linear span of the local integrals $\Psi_j(s;W,W')$ if $m < n$ and $\mathcal{I}(\pi,\pi')$ the complex linear span of the $\Psi(s;W,W',\Phi)$ if $m = n$. These are then all subspaces of $\mathbb{C}(q^{-s})$ which have "bounded denominators" in the sense of (3). In fact, these subspaces have more structure – they are modules for $\mathbb{C}[q^s,q^{-s}] \subset \mathbb{C}(q^{-s})$. To see this, note that for any $h \in \mathrm{GL}_m(k)$ we have

$$\Psi_j\left(s;\pi\begin{pmatrix} h & \\ & I_{n-m}\end{pmatrix}W,\pi'(h)W'\right) = |\det(h)|^{-s-j+(n-m)/2}\Psi_j(s;W,W')$$

and

$$\Psi(s;\pi(h)W,\pi'(h)W',\rho(h)\Phi) = |\det(h)|^{-s}\Psi(s;W,W',\Phi).$$

So by varying h and multiplying by scalars, we see that each $\mathcal{I}_j(\pi,\pi')$ and $\mathcal{I}(\pi,\pi')$ is closed under multiplication by $\mathbb{C}[q^s,q^{-s}]$. Since we have bounded denominators, we can conclude:

Proposition 3.3. Each $\mathcal{I}_j(\pi,\pi')$ and $\mathcal{I}(\pi,\pi')$ is a fractional $\mathbb{C}[q^s,q^{-s}]$–ideal of $\mathbb{C}(q^{-s})$.

Note that $\mathbb{C}[q^s,q^{-s}]$ is a principal ideal domain, so that each fractional ideal $\mathcal{I}_j(\pi,\pi')$ has a single generator, which we call $Q_{j,\pi,\pi'}(q^{-s})$, as does $\mathcal{I}(\pi,\pi')$, which we call $Q_{\pi,\pi'}(q^{-s})$. However, we can say more. In the case $m < n$ recall that from what we have said about the Kirillov model that when we restrict Whittaker functions in $\mathcal{W}(\pi,\psi)$ to the embedded $\mathrm{GL}_m(k) \subset \mathrm{P}_n(k)$ we get all functions of compact support on $\mathrm{GL}_m(k)$ transforming by ψ. Using this freedom for our choice of $W \in \mathcal{W}(\pi,\psi)$ one can show that in fact the constant function 1 lies in $\mathcal{I}_j(\pi,\pi')$. In the case $m = n$ one can reduce to a sum of integrals over $\mathrm{P}_n(k)$ and then use the freedom one has in the Kirillov model, plus the complete freedom in the choice of Φ to show that once again $1 \in \mathcal{I}(\pi,\pi')$. The consequence of this is that our generator can be taken to be of the form $Q_{j,\pi,\pi'}(q^{-s}) = P_{j,\pi,\pi'}(q^s,q^{-s})^{-1}$ for $m < n$ or $Q_{\pi,\pi'}(q^{-s}) = P_{\pi,\pi'}(q^s,q^{-s})^{-1}$ for appropriate polynomials in $\mathbb{C}[q^s,q^{-s}]$. Moreover, since q^s and q^{-s} are units in $\mathbb{C}[q^s,q^{-s}]$ we can always normalize our generator to be of the form $P_{j,\pi,\pi'}(q^{-s})^{-1}$ or $P_{\pi,\pi'}(q^{-s})^{-1}$ where the polynomial $P(X)$ satisfies $P(0) = 1$.

Finally, in the case $m < n$ one can show by a rather elementary although somewhat involved manipulation of the integrals that all of the ideals $\mathcal{I}_j(\pi,\pi')$ are the same [42, Section 2.7]. We will write this ideal as $\mathcal{I}(\pi,\pi')$ and its generator as $P_{\pi,\pi'}(q^{-s})^{-1}$.

This gives us the definition of our local L-function.

Definition . Let π and π' be as above. Then $L(s,\pi\times\pi') = P_{\pi,\pi'}(q^{-s})^{-1}$ is the normalized generator of the fractional ideal $\mathcal{I}(\pi,\pi')$ formed by the family of local integrals. If $\pi' = \mathbf{1}$ is the trivial representation of $\mathrm{GL}_1(k)$ then we write $L(s,\pi) = L(s,\pi\times\mathbf{1})$.

One can show easily that the ideal $\mathcal{I}(\pi,\pi')$ is independent of the character ψ used in defining the Whittaker models, so that $L(s,\pi\times\pi')$ is independent of the choice of ψ. So it is not included in the notation. Also, note that for $\pi' = \chi$ an automorphic representation (character) of $\mathrm{GL}_1(\mathbb{A})$ we have the identity $L(s,\pi\times\chi) = L(s,\pi\otimes\chi)$ where $\pi\otimes\chi$ is the representation of $\mathrm{GL}_n(\mathbb{A})$ on V_π given by $\pi\otimes\chi(g)\xi = \chi(\det(g))\pi(g)\xi$.

We summarize the above in the following theorem.

Theorem 3.1. Let π and π' be as above. The family of local integrals form a $\mathbb{C}[q^s,q^{-s}]$–fractional ideal $\mathcal{I}(\pi,\pi')$ in $\mathbb{C}(q^{-s})$ with generator the local L-function $L(s,\pi\times\pi')$.

Another useful way of thinking of the local L-function is the following. The function $L(s,\pi\times\pi')$ is the minimal (in terms of degree) function of the form $P(q^{-s})^{-1}$, with $P(X)$ a polynomial satisfying $P(0)=1$, such that the ratios

$$\frac{\Psi(s;W,W')}{L(s,\pi\times\pi')} \quad \text{or} \quad \frac{\Psi(s;W,W',\Phi)}{L(s,\pi\times\pi')}$$

are entire for all $W\in\mathcal{W}(\pi,\psi)$ and $W'\in\mathcal{W}(\pi',\psi^{-1})$, and if necessary $\Phi\in\mathcal{S}(k^n)$. That is, $L(s,\pi\times\pi')$ is the standard Euler factor determined by the poles of the functions in $\mathcal{I}(\pi,\pi')$.

One should note that since the L-factor is a generator of the ideal $\mathcal{I}(\pi,\pi')$, then in particular it lies in $\mathcal{I}(\pi,\pi')$. Since this ideal is spanned by our local integrals, we have the following useful Corollary.

Corollary . There are a finite collection of $W_i\in\mathcal{W}(\pi,\psi)$, $W_i'\in\mathcal{W}(\pi',\psi^{-1})$, and if necessary $\Phi_i\in\mathcal{S}(k^n)$ such that

$$L(s,\pi\times\pi') = \sum_i \Psi(s;W_i,W_i') \qquad \text{or} \qquad L(s,\pi\times\pi') = \sum_i \Psi(s;W_i,W_i',\Phi_i).$$

For future reference, let us set

$$e(s;W,W') = \frac{\Psi(s;W,W')}{L(s,\pi\times\pi')}, \quad e_j(s;W,W') = \frac{\Psi_j(s;W,W')}{L(s,\pi\times\pi')},$$

$$\tilde{e}(s;W,W') = \frac{\widetilde{\Psi}(s;W,W')}{L(s,\pi\times\pi')}, \quad \text{and} \quad e(s;W,W',\Phi) = \frac{\Psi(s;W,W',\Phi)}{L(s,\pi\times\pi')}.$$

Then all of these functions are Laurent polynomials in $q^{\pm s}$, that is, elements of $\mathbb{C}[q^s,q^{-s}]$. As such they are entire and bounded in vertical strips. As above, there are choices of W_i, W_i', and if necessary Φ_i such that $\sum e(s;W_i,W_i')\equiv 1$ or $\sum e(s;W_i,W_i',\Phi_i)\equiv 1$. In particular we have the following result.

Corollary . The functions $e(s;W,W')$ and $e(s;W,W',\Phi)$ are entire functions, bounded in vertical strips, and for each $s_0 \in \mathbb{C}$ there is a choice of W, W', and if necessary Φ such that $e(s_0;W,W') \neq 0$ or $e(s_0;W,W',\Phi) \neq 0$.

3.1.2. The local functional equation

Either by analogy with Tate's thesis or from the corresponding global statement, we would expect our local integrals to satisfy a local functional equation. From the functional equations for our global integrals, we would expect these to relate the integrals $\Psi(s;W,W')$ and $\widetilde{\Psi}(1-s;\rho(w_{n,m})\widetilde{W},\widetilde{W}')$ when $m<n$ and $\Psi(s;W,W',\Phi)$ and $\Psi(1-s;\widetilde{W},\widetilde{W}',\hat{\Phi})$ when $m=n$. This will indeed be the case. These functional equations will come from interpreting the local integrals as families (in s) of quasi-invariant bilinear forms on $\mathcal{W}(\pi,\psi)\times\mathcal{W}(\pi',\psi^{-1})$ or trilinear forms on $\mathcal{W}(\pi,\psi)\times\mathcal{W}(\pi',\psi^{-1})\times\mathcal{S}(k^n)$ depending on the case.

First, consider the case when $m<n$. In this case we have seen that

$$\Psi\left(s;\pi\begin{pmatrix} h & \\ & I_{n-m}\end{pmatrix}W,\pi'(h)W'\right)=|\det(h)|^{-s+(n-m)/2}\Psi(s;W,W')$$

and one checks that $\widetilde{\Psi}(1-s;\rho(w_{n,m})\widetilde{W},\widetilde{W}')$ has the same quasi-invariance as a bilinear form on $\mathcal{W}(\pi,\psi)\times\mathcal{W}(\pi',\psi^{-1})$. In addition, if we let $\mathrm{Y}_{n,m}$ denote the unipotent radical of the standard parabolic subgroup associated to the partition $(m+1,1,\dots,1)$ as before then we have the quasi-invariance

$$\Psi(s;\pi(y)W,W')=\psi(y)\Psi(s;W,W')$$

for all $y\in\mathrm{Y}_{n,m}$. One again checks that $\widetilde{\Psi}(1-s;\rho(w_{n,m})\widetilde{W},\widetilde{W}')$ satisfies the same quasi-invariance as a bilinear form on $\mathcal{W}(\pi,\psi)\times\mathcal{W}(\pi',\psi^{-1})$.

For $n=m$ we have seen that

$$\Psi(s;\pi(h)W,\pi'(h)W',\rho(h)\Phi)=|\det(h)|^{-s}\Psi(s;W,W',\Phi)$$

and it is elementary to check that $\Psi(1-s;\widetilde{W},\widetilde{W}',\hat{\Phi})$ satisfies the same quasi-invariance as a trilinear form on $\mathcal{W}(\pi,\psi)\times\mathcal{W}(\pi',\psi^{-1})\times\mathcal{S}(k^n)$. Our local functional equations will now follow from the following result [42, Propositions 2.10 and 2.11].

Proposition 3.4. (i) If $m<n$, then except for a finite number of exceptional values of q^{-s} there is a unique bilinear form B_s on $\mathcal{W}(\pi,\psi)\times\mathcal{W}(\pi',\psi^{-1})$ satisfying

$$B_s\left(\pi\begin{pmatrix} h & \\ & I_{n-m}\end{pmatrix}W,\pi'(h)W'\right)=|\det(h)|^{-s+(n-m)/2}B_s(W,W')$$
$$\text{and}\quad B_s(\pi(y)W,W')=\psi(y)B_s(W,W')$$

for all $h\in\mathrm{GL}_m(k)$ and $y\in Y_{n,m}(k)$.

(ii) If $n=m$, then except for a finite number of exceptional values of q^{-s} there is a unique trilinear form T_s on $\mathcal{W}(\pi,\psi)\times\mathcal{W}(\pi',\psi^{-1})\times\mathcal{S}(k^n)$ satisfying

$$T_s(\pi(h)W,\pi'(h)W',\rho(h)\Phi)=|\det(h)|^{-s}T_s(W,W',\Phi)$$

for all $h\in\mathrm{GL}_n(k)$.

Let us say a few words about the proof of this proposition, because it is another application of the analysis of the restriction of representations of GL_n to the mirabolic subgroup P_n [42, Sections 2.10 and 2.11]. In the case where $m < n$ the local integrals involve the restriction of the Whittaker functions in $\mathcal{W}(\pi,\psi)$ to $\mathrm{GL}_m(k) \subset \mathrm{P}_n$, that is, the Kirillov model $\mathcal{K}(\pi,\psi)$ of π. In the case $m = n$ one notes that $\mathcal{S}_0(k^n) = \{\Phi \in \mathcal{S}(k^n) \mid \Phi(0) = 0\}$, which has co-dimension one in $\mathcal{S}(k^n)$, is isomorphic to the compactly induced representation $\mathrm{ind}_{\mathrm{P}_n(k)}^{\mathrm{GL}_n(k)}(\delta_{\mathrm{P}_n}^{-1/2})$ so that by Frobenius reciprocity a $\mathrm{GL}_n(k)$ quasi-invariant trilinear form on $\mathcal{W}(\pi,\psi) \times \mathcal{W}(\pi',\psi^{-1}) \times \mathcal{S}_0(k^n)$ reduces to a $\mathrm{P}_n(k)$-quasi-invariant bilinear form on $\mathcal{K}(\pi,\psi) \times \mathcal{K}(\pi',\psi^{-1})$. So in both cases we are naturally working in the restriction to $\mathrm{P}_n(k)$. The restrictions of irreducible representations of $\mathrm{GL}_n(k)$ to $\mathrm{P}_n(k)$ are no longer irreducible, but do have composition series of finite length. One of the tools for analyzing the restrictions of representations of GL_n to P_n, or analyzing the irreducible representations of P_n, are the *derivatives* of Bernstein and Zelevinsky [2,15]. These derivatives $\pi^{(n-r)}$ are naturally representations of $\mathrm{GL}_r(k)$ for $r \leq n$. $\pi^{(0)} = \pi$ and since π is generic the highest derivative $\pi^{(n)}$ corresponds to the irreducible common submodule (τ, V_τ) of all Kirillov models, and is hence the non-zero irreducible representation of the trivial group $\mathrm{GL}_0(k)$. The poles of our local integrals can be interpreted as giving quasi-invariant pairings between derivatives of π and π' [15]. The s for which such pairings exist for all but the highest derivatives are the exceptional s of the proposition. There is always a unique pairing between the highest derivatives $\pi^{(n)}$ and $\pi'^{(m)}$, which are necessarily non-zero since they since these correspond to the common irreducible subspace (τ, V_τ) of any Kirillov model, and this is the unique B_s or T_s of the proposition.

As a consequence of this Proposition, we can define the local γ-factor which gives the local functional equation for our integrals.

Theorem 3.2. There is a rational function $\gamma(s, \pi \times \pi', \psi) \in \mathbb{C}(q^{-s})$ such that we have

$$\widetilde{\Psi}(1-s; \rho(w_{n,m})\widetilde{W}, \widetilde{W}') = \omega'(-1)^{n-1}\gamma(s, \pi \times \pi', \psi)\Psi(s; W, W') \quad \text{if } m < n$$

or

$$\Psi(1-s; \widetilde{W}, \widetilde{W}', \hat{\Phi}) = \omega'(-1)^{n-1}\gamma(s, \pi \times \pi', \psi)\Psi(s; W, W', \Phi) \quad \text{if } m = n$$

for all $W \in \mathcal{W}(\pi,\psi)$, $W' \in \mathcal{W}(\pi',\psi^{-1})$, and if necessary all $\Phi \in \mathcal{S}(k^n)$.

Again, if $\pi' = \mathbf{1}$ is the trivial representation of $\mathrm{GL}_1(k)$ we write $\gamma(s,\pi,\psi) = \gamma(s, \pi \times \mathbf{1}, \psi)$. The fact that $\gamma(s, \pi \times \pi', \psi)$ is rational follows from the fact that it is a ratio of local integrals.

An equally important local factor, which occurs in the current formulations of the local Langlands correspondence [32,35], is the local ε-factor.

Definition . The local factor $\varepsilon(s, \pi \times \pi', \psi)$ is defined as the ratio

$$\varepsilon(s, \pi \times \pi', \psi) = \frac{\gamma(s, \pi \times \pi', \psi)L(s, \pi \times \pi')}{L(1-s, \widetilde{\pi} \times \widetilde{\pi}')}.$$

With the local ε-factor the local functional equation can be written in the form

$$\frac{\widetilde{\Psi}(1-s; \rho(w_{n,m})\widetilde{W}, \widetilde{W}')}{L(1-s, \widetilde{\pi} \times \widetilde{\pi}')} = \omega'(-1)^{n-1}\varepsilon(s, \pi \times \pi', \psi)\frac{\Psi(s; W, W')}{L(s, \pi \times \pi')} \quad \text{if } m < n$$

or

$$\frac{\Psi(1-s;\widetilde{W},\widetilde{W}',\hat{\Phi})}{L(1-s,\widetilde{\pi}\times\widetilde{\pi}')} = \omega'(-1)^{n-1}\varepsilon(s,\pi\times\pi',\psi)\frac{\Psi(s;W,W',\Phi)}{L(s,\pi\times\pi')} \quad \text{if } m=n\ .$$

This can also be expressed in terms of the $e(s;W,W')$, etc.. In fact, since we know we can choose a finite set of W_i, W_i', and if necessary Φ_i so that

$$\sum_i \frac{\Psi(s,;W_i,W_i')}{L(s,\pi\times\pi')} = \sum_i e(s;W_i,W_i') = 1$$

or

$$\sum_i \frac{\Psi(s;W_i,W_i',\Phi_i)}{L(s,\pi\times\pi')} = \sum_i e(s;W_i,W_i',\Phi_i) = 1$$

we see that we can write either

$$\varepsilon(s,\pi\times\pi',\psi) = \omega'(-1)^{n-1}\sum_i \tilde{e}(1-s;\rho(w_{n,m})\widetilde{W}_i,\widetilde{W}_i')$$

or

$$\varepsilon(s,\pi\times\pi',\psi) = \omega'(-1)^{n-1}\sum_i e(1-s;\widetilde{W}_i,\widetilde{W}_i',\hat{\Phi}_i)$$

and hence $\varepsilon(s,\pi\times\pi',\psi)\in\mathbb{C}[q^s,q^{-s}]$. On the other hand, applying the functional equation twice we get

$$\varepsilon(s,\pi\times\pi',\psi)\varepsilon(1-s,\widetilde{\pi}\times\widetilde{\pi}',\psi^{-1}) = 1$$

so that $\varepsilon(s,\pi\times\pi',\psi)$ is a unit in $\mathbb{C}[q^s,q^{-s}]$. This can be restated as:

Proposition 3.5. $\varepsilon(s,\pi\times\pi',\psi)$ is a monomial function of the form cq^{-fs}.

Let us make a few remarks on the meaning of the number f occurring in the ε–factor in the case of a single representation. Assume that ψ is unramified. In this case write $\varepsilon(s,\pi,\psi)=\varepsilon(0,\pi,\psi)q^{-f(\pi)s}$. In [43] it is shown that $f(\pi)$ is a non-negative integer, $f(\pi)=0$ iff π is unramified, that in general the space of vectors in V_π which is fixed by the compact open subgroup

$$\mathrm{K}_1(\mathfrak{p}^{f(\pi)}) = \left\{ g\in \mathrm{GL}_n(\mathfrak{o}) \middle| g\equiv \begin{pmatrix} & & & * \\ & * & & \vdots \\ & & & * \\ 0 & \cdots & 0 & 1\end{pmatrix} \pmod{\mathfrak{p}^{f(\pi)}} \right\}$$

has dimension exactly 1, and that if $t<f(\pi)$ then the dimension of the space of fixed vectors for $\mathrm{K}_1(\mathfrak{p}^t)$ is 0. Depending on the context, either the integer $f(\pi)$ or the ideal $\mathfrak{f}(\pi)=\mathfrak{p}^{f(\pi)}$ is called the *conductor of* π. Note that the analytically defined ε-factor carries structural information about π.

3.1.3. The unramified calculation

Let us now turn to the calculation of the local L-functions. The first case to consider is that where both π and π' are unramified. Since they are assumed generic, they are both full induced representations from unramified characters of the Borel subgroup [97]. So let us write $\pi\simeq \mathrm{Ind}_{\mathrm{B}_n}^{\mathrm{GL}_n}(\mu_1\otimes\cdots\otimes\mu_n)$ and $\pi'\simeq\mathrm{Ind}_{\mathrm{B}_m}^{\mathrm{GL}_m}(\mu_1'\otimes\cdots\otimes\mu_m')$ with the μ_i and μ_j' unramified characters of $k^\times$. The Satake parameterization of unramified representations associates to each of these representation the semi-simple conjugacy classes $[A_\pi] \in \mathrm{GL}_n(\mathbb{C})$ and $[A_{\pi'}] \in \mathrm{GL}_m(\mathbb{C})$ given by

$$A_\pi = \begin{pmatrix} \mu_1(\varpi) & & \\ & \ddots & \\ & & \mu_n(\varpi) \end{pmatrix} \qquad A_{\pi'} = \begin{pmatrix} \mu_1'(\varpi) & & \\ & \ddots & \\ & & \mu_m'(\varpi) \end{pmatrix}.$$

(Recall that ϖ is a uniformizing parameter for k, that is, a generator of $\mathfrak{p}$.)

In the Whittaker models there will be unique normalized $\mathrm{K} = \mathrm{GL}(\mathfrak{o})$– fixed Whittaker functions, $W_\circ \in \mathcal{W}(\pi, \psi)$ and $W_\circ' \in \mathcal{W}(\pi', \psi^{-1})$, normalized by $W_\circ(e) = W_\circ'(e) = 1$. Let us concentrate on $W_\circ$ for the moment. Since this function is right K_n–invariant and transforms on the left by ψ under N_n we have that its values are completely determined by its values on diagonal matrices of the form

$$\varpi^J = \begin{pmatrix} \varpi^{j_1} & & \\ & \ddots & \\ & & \varpi^{j_n} \end{pmatrix}$$

for $J = (j_1, \dots, j_n) \in \mathbb{Z}^n$. There is an explicit formula for $W_\circ(\varpi^J)$ in terms of the Satake parameter A_π due to Shintani [87] for GL_n and generalized to arbitrary reductive groups by Casselman and Shalika [4].

Let $T^+(n)$ be the set of n–tuples $J = (j_1, \dots, j_n) \in \mathbb{Z}^n$ with $j_1 \geq \cdots \geq j_n$. Let ρ_J be the rational representation of $\mathrm{GL}_n(\mathbb{C})$ with dominant weight Λ_J defined by

$$\Lambda_J \begin{pmatrix} t_1 & & \\ & \ddots & \\ & & t_n \end{pmatrix} = t_1^{j_1} \cdots t_n^{j_n}.$$

Then the formula of Shintani says that

$$W_\circ(\varpi^J) = \begin{cases} 0 & \text{if } J \notin T^+(n) \\ \delta_{\mathrm{B}_n}^{1/2}(\varpi^J)\,\mathrm{tr}(\rho_J(A_\pi)) & \text{if } J \in T^+(n) \end{cases}$$

under the assumption that ψ is unramified. This is proved by analyzing the recursion relations coming from the action of the unramified Hecke algebra on $W_\circ$.

We have a similar formula for $W_\circ'(\varpi^J)$ for $J \in \mathbb{Z}^m$.

If we use these formulas in our local integrals, we find [45, I, Prop. 2.3]

$$\begin{aligned} \Psi(s; W_\circ, W_\circ') &= \sum_{J \in T^+(m),\ j_m \geq 0} W_\circ \begin{pmatrix} \varpi^J & \\ & I_{n-m} \end{pmatrix} W_\circ'(\varpi^J) |\det(\varpi^J)|^{s - \frac{(n-m)}{2}} \delta_{\mathrm{B}_m}^{-1}(\varpi^J) \\ &= \sum_{J \in T^+(m),\ j_m \geq 0} \mathrm{tr}(\rho_{(J,0)}(A_\pi))\,\mathrm{tr}(\rho_J(A_{\pi'})) q^{-|J|s} \\ &= \sum_{J \in T^+(m),\ j_m \geq 0} \mathrm{tr}(\rho_{(J,0)}(A_\pi) \otimes \rho_J(A_{\pi'})) q^{-|J|s} \end{aligned}$$

where we let $|J| = j_1 + \cdots + j_m$ and we embed $\mathbb{Z}^m \hookrightarrow \mathbb{Z}^n$ by $J = (j_1, \cdots, j_m) \mapsto (J, 0) = (j_1, \cdots, j_m, 0, \cdots, 0)$. We now use the invariant theory facts that

$$\sum_{J \in T^+(m),\ j_m \geq 0,\ |J| = r} \mathrm{tr}(\rho_{(J,0)}(A_\pi) \otimes \rho_J(A_{\pi'})) = \mathrm{tr}(S^r(A_\pi \otimes A_{\pi'})),$$

where $S^r(A)$ is the r^{th}-symmetric power of the matrix A, and

$$\sum_{r=0}^{\infty} \operatorname{tr}(S^r(A))z^r = \det(I - Az)^{-1}$$

for any matrix A. Then we quickly arrive at

$$\Psi(s; W_\circ, W'_\circ) = \det(I - q^{-s}A_\pi \otimes A_{\pi'})^{-1} = \prod_{i,j}(1 - \mu_i(\varpi)\mu'_j(\varpi)q^{-s})^{-1}$$

a standard Euler factor of degree mn. Since the L-function cancels all poles of the local integrals, we know at least that $\det(I - q^{-s}A_\pi \otimes A_{\pi'})$ divides $L(s, \pi \times \pi')^{-1}$. Either of the methods discussed below for the general calculation of local factors then shows that in fact these are equal.

There is a similar calculation when $n = m$ and $\Phi = \Phi_\circ$ is the characteristic function of the lattice $\mathfrak{o}^n \subset k^n$. Also, since π unramified implies that its contragredient $\widetilde{\pi}$ is also unramified, with $\widetilde{W}_\circ$ as its normalized unramified Whittaker function, then from the functional equation we can conclude that in this situation we have $\varepsilon(s, \pi \times \pi', \psi) \equiv 1$.

Theorem 3.3. If π, π', and ψ are all unramified, then

$$L(s, \pi \times \pi') = \det(I - q^{-s}A_\pi \otimes A_{\pi'})^{-1} = \begin{cases} \Psi(s; W_\circ, W'_\circ) & m < n \\ \Psi(s; W_\circ, W'_\circ, \Phi_\circ) & m = n \end{cases}$$

and $\varepsilon(s, \pi \times \pi', \psi) \equiv 1$.

For future use, let us recall a consequence of this calculation due to Jacquet and Shalika [45].

Corollary . Suppose π is irreducible unitary generic admissible (our usual assumptions on π) and unramified. The the eigenvalues $\mu_i(\varpi)$ of A_π all satisfy $q^{-1/2} < |\mu_i(\varpi)| < q^{1/2}$.

To see this, we apply the above calculation to the case where $\pi' = \bar{\pi}$ the complex conjugate representation. Then $A_{\pi'} = \overline{A_\pi}$, the complex conjugate matrix, and we have from the above

$$\det(I - q^{-s}A_\pi \otimes \overline{A_\pi})\Psi(s; W_\circ, \overline{W_\circ}, \Phi_\circ) = 1.$$

The local integral in this case is absolutely convergent for $\mathrm{Re}(s) \geq 1$ and so the factor $\det(I - q^{-s}A_\pi \otimes \overline{A_\pi})$ cannot vanish for $\mathrm{Re}(s) \geq 1$. If $\mu_i(\varpi)$ is an eigenvalue of A_π then we have $1 - q^{-\sigma}|\mu_i(\varpi)|^2 \neq 0$ for $\sigma \geq 1$. Hence $|\mu_i(\varpi)| < q^{1/2}$. Note that if we apply this to the contragredient representation $\widetilde{\pi}$ as well we conclude that $q^{-1/2} < |\mu_i(\varpi)| < q^{1/2}$.

3.1.4. The supercuspidal calculation

The other basic case is when both π and π' are supercuspidal. In this case the restriction of W to P_n or W' to P_m lies in the Kirillov model and is hence compactly supported mod N. In the case of $m < n$ we find that in our integral we have W evaluated along $\mathrm{GL}_m(k) \subset \mathrm{P}_n(k)$. Since W is smooth, and hence stabilized by some compact open subgroup, we find that the local integral always reduces to a finite sum and and hence lies in $\mathbb{C}[q^s, q^{-s}]$. In particular it is always entire. Thus in this case $L(s, \pi \times \pi') \equiv 1$. In the case $n = m$ the calculation is a bit more involved and can be found in [15,23]. In essence, in the family of integrals $\Psi(s;W,W',\Phi)$, if $\Phi(0)=0$ then the integral will again reduce to a finite sum and hence be entire. If $\Phi(0)\neq 0$ and if s_0 is a pole of $\Psi(s;W,W',\Phi)$ then the residue of the pole at $s=s_0$ will be of the form

$$c\Phi(0)\int_{\mathrm{Z}_n(k)\,\mathrm{N}_n(k)\backslash\,\mathrm{GL}_n(k)} W(g)W'(g)|\det(g)|^{s_0}\ dg$$

which is the Whittaker form of an invariant pairing between π and $\pi'\otimes|\det|^{s_0}$. Thus we must have s_0 is pure imaginary and $\widetilde{\pi}\simeq\pi'\otimes|\det|^{s_0}$ for the residue to be nonzero. This condition is also sufficient.

Theorem 3.4. If π and π' are both (unitary) supercuspidal, then $L(s,\pi\times\pi')\equiv 1$ if $m<n$ and if $m=n$ we have

$$L(s,\pi\times\pi')=\prod(1-\alpha q^{-s})^{-1}$$

with the product over all $\alpha=q^{s_0}$ with $\widetilde{\pi}\simeq\pi'\otimes|\det|^{s_0}$.

3.1.5. Remarks on the general calculation

In the other cases, we must rely on the Bernstein–Zelevinsky classification of generic representations of $\mathrm{GL}_n(k)$ [97]. All generic representations can be realized as prescribed constituents of representations parabolically induced from supercuspidals. One can proceed by analyzing the Whittaker functions of induced representations in terms of Whittaker functions of the inducing data as in [42] or by analyzing the poles of the local integrals in terms of quasi invariant pairings of derivatives of π and π' as in [15] to compute $L(s,\pi\times\pi')$ in terms of L-functions of pairs of supercuspidal representations. We refer you to those papers or [58] for the explicit formulas.

3.1.6. Multiplicativity and stability of γ–factors

To conclude this section, let us mention two results on the γ-factors. One is used in the computations of L-factors in the general case. This is the *multiplicativity of γ-factors* [42]. The second is the *stability of γ-factors* [46]. Both of these results are necessary in applications of the Converse Theorem to liftings, which we discuss in Lecture 6.

Proposition (Multiplicativity of γ-factors)**.** If $\pi=\mathrm{Ind}(\pi_1\otimes\pi_2)$, with π_i and irreducible admissible representation of $\mathrm{GL}_{r_i}(k)$, then

$$\gamma(s,\pi\times\pi',\psi)=\gamma(s,\pi_1\times\pi',\psi)\gamma(s,\pi_2\times\pi',\psi)$$

and similarly for π'. Moreover $L(s,\pi\times\pi')^{-1}$ divides $[L(s,\pi_1\times\pi')L(s,\pi_2\times\pi')]^{-1}$.

Proposition (Stability of γ-factors)**.** If π_1 and π_2 are two irreducible admissible generic representations of $\mathrm{GL}_n(k)$, having the same central character, then for every sufficiently highly ramified character η of $\mathrm{GL}_1(k)$ we have

$$\gamma(s,\pi_1\times\eta,\psi)=\gamma(s,\pi_2\times\eta,\psi)$$

and

$$L(s,\pi_1\times\eta)=L(s,\pi_2\times\eta)\equiv 1.$$

More generally, if in addition π' is an irreducible generic representation of $\mathrm{GL}_m(k)$ then for all sufficiently highly ramified characters η of $\mathrm{GL}_1(k)$ we have

$$\gamma(s,(\pi_1\otimes\eta)\times\pi',\psi)=\gamma(s,(\pi_2\otimes\eta)\times\pi',\psi)$$

and

$$L(s,(\pi_1\otimes\eta)\times\pi')=L(s,(\pi_2\otimes\eta)\times\pi')\equiv 1.$$

3.2. The archimedean local factors

We now take k to be an archimedean local field, i.e., $k=\mathbb{R}$ or $\mathbb{C}$. We take (π,V_π) to be the space of smooth vectors in an irreducible admissible unitary generic representation of $\mathrm{GL}_n(k)$ and similarly for the representation $(\pi',V_{\pi'})$ of $\mathrm{GL}_m(k)$. We take ψ a non-trivial continuous additive character of k.

The treatment of the archimedean local factors parallels that of the non-archimedean in many ways, but there are some significant differences. The major work on these factors is that of Jacquet and Shalika in [47], which we follow for the most part without further reference, and in the archimedean parts of [45].

One significant difference in the development of the archimedean theory is that the local Langlands correspondence was already in place when the theory was developed [62]. The correspondence is very explicit in terms of the usual Langlands classification. Thus to π is associated an n dimensional semi-simple representation $\tau=\tau(\pi)$ of the Weil group W_k of k and to π' an m-dimensional semi-simple representation $\tau'=\tau(\pi')$ of W_k. Then $\tau(\pi)\otimes\tau(\pi')$ is an nm dimensional representation of W_k and to this representation of the Weil group is attached Artin-Weil $L-$ and ε–factors [92], denoted $L(s,\tau\otimes\tau')$ and $\varepsilon(s,\tau\otimes\tau',\psi)$. In essence, Jacquet and Shalika *define*

$$L(s,\pi\times\pi')=L(s,\tau(\pi)\otimes\tau(\pi'))\quad\text{and}\quad\varepsilon(s,\pi\times\pi',\psi)=\varepsilon(s,\tau(\pi)\otimes\tau(\pi'),\psi)$$

and then set

$$\gamma(s,\pi\times\pi',\psi)=\frac{\varepsilon(s,\pi\times\pi',\psi)L(1-s,\widetilde{\pi}\times\widetilde{\pi}')}{L(s,\pi\times\pi')}.$$

For example, if π is unramified, and hence of the form $\pi\simeq\mathrm{Ind}(\mu_1\otimes\cdots\otimes\mu_n)$ with unramified characters of the form $\mu_i(x)=|x|^{r_i}$ then

$$L(s,\pi)=L(s,\tau(\pi))=\prod_{i=1}^n\Gamma_v(s+r_i)$$

is a standard archimedean Euler factor of degree n, where

$$\Gamma_v(s)=\begin{cases}\pi^{-s/2}\Gamma(\frac{s}{2}) & \text{if } k_v=\mathbb{R}\\ 2(2\pi)^{-s}\Gamma(s) & \text{if } k_v=\mathbb{C}\end{cases}.$$

They then proceed to show that these functions have the expected relation to the local integrals. Their methods of analyzing the local integrals $\Psi_j(s;W,W')$ and $\Psi(s;W,W',\Phi)$, defined as in the non-archimedean case for $W\in\mathcal{W}(\pi,\psi)$, $W'\in\mathcal{W}(\pi',\psi^{-1})$, and $\Phi\in\mathcal{S}(k^n)$, are direct analogues of those used in [42] for the non-archimedean case. Once again, a most important fact is [47, Proposition 2.2]

Proposition 3.6. Let π be a generic representation of $\mathrm{GL}_n(k)$. Then there is a finite set of finite functions $X(\pi) = \{\chi_i\}$ on $\mathrm{A}_n(k)$, depending only on π, so that for every $W \in \mathcal{W}(\pi, \psi)$ there are Schwartz functions $\phi_i \in \mathcal{S}(k^{n-1} \times \mathrm{K}_n)$ such that for all $a \in \mathrm{A}_n(k)$ with $a_n = 1$ we have

$$W(nak) = \psi(n) \sum_{X(\pi)} \chi_i(a)\phi_i(\alpha_1(a), \ldots, \alpha_{n-1}(a), k).$$

Now the finite functions are related to the exponents of the representation π and through the Langlands classification to the representation $\tau(\pi)$ of W_k. We retain the same convergence statements as in the non-archimedean case [45, I, Proposition 3.17; II, Proposition 2.6], [47, Proposition 5.3].

Proposition 3.7. The integrals $\Psi_j(s; W, W')$ and $\Psi(s; W, W', \Phi)$ converge absolutely in the half plane $\mathrm{Re}(s) \geq 1$ under the unitarity assumption and for $\mathrm{Re}(s) > 0$ if π and π' are tempered.

The meromorphic continuation and "bounded denominator" statement in the case of a non-archimedean local field is now replaced by the following. Define $\mathcal{M}(\pi \times \pi')$ to be the space of all meromorphic functions $\phi(s)$ with the property that if $P(s)$ is a polynomial function such that $P(s)L(s, \pi \times \pi')$ is holomorphic in a vertical strip $S[a, b] = \{s \mid a \leq \mathrm{Re}(s) \leq b\}$ then $P(s)\phi(s)$ is bounded in $S[a, b]$. Note in particular that if $\phi \in \mathcal{M}(\pi \times \pi')$ then the quotient $\phi(s)L(s, \pi \times \pi')^{-1}$ is entire.

Theorem 3.5. The integrals $\Psi_j(s; W, W')$ or $\Psi(s; W, W', \Phi)$ extend to meromorphic functions of s which lie in $\mathcal{M}(\pi \times \pi')$. In particular, the ratios

$$e_j(s; W, W') = \frac{\Psi_j(s; W, W')}{L(s, \pi \times \pi')} \quad \text{or} \quad e(s; W, W', \Phi) = \frac{\Psi(s; W, W', \Phi)}{L(s, \pi \times \pi')}$$

are entire.

This statement has more content than just the continuation and "bounded denominator" statements in the non-archimedean case. Since it prescribes the "denominator" to be the L factor $L(s, \pi \times \pi')^{-1}$ it is bound up with the actual computation of the poles of the local integrals. In fact, a significant part of the paper of Jacquet and Shalika [47] is taken up with the simultaneous proof of this and the local functional equations:

Theorem 3.6. We have the local functional equations

$$\Psi_{n-m-j-1}(1 - s; \rho(w_{n,m})\widetilde{W}, \widetilde{W}') = \omega'(-1)^{n-1}\gamma(s, \pi \times \pi', \psi)\Psi_j(s; W, W')$$

or

$$\Psi(1 - s; \widetilde{W}, \widetilde{W}', \hat{\Phi}) = \omega'(-1)^{n-1}\gamma(s, \pi \times \pi', \psi)\Psi(s; W, W', \Phi).$$

The one fact that we are missing is the statement of "minimality" of the L-factor. That is, we know that $L(s, \pi \times \pi')$ is a standard archimedean Euler factor (i.e., a product of Γ-functions of the standard type) and has the property that the poles of all the local integrals are contained in the poles of the L-factor, even with multiplicity. But we have not established that the L-factor cannot have extraneous poles. In particular, we do not know that we can achieve the local L-function as a finite linear combination of local integrals.

Towards this end, Jacquet and Shalika enlarge the allowable space of local integrals. Let Λ and Λ' be the Whittaker functionals on V_π and $V_{\pi'}$ associated with the Whittaker models $\mathcal{W}(\pi,\psi)$ and $\mathcal{W}(\pi',\psi^{-1})$. Then $\hat{\Lambda}=\Lambda\otimes\Lambda'$ defines a continuous linear functional on the algebraic tensor product $V_\pi\otimes V_{\pi'}$ which extends continuously to the topological tensor product $V_{\pi\hat{\otimes}\pi'}=V_\pi\hat{\otimes}V_{\pi'}$, viewed as representations of $\mathrm{GL}_n(k)\times\mathrm{GL}_m(k)$.

Before proceeding, let us make a few remarks on smooth representations. If (π,V_π) is the space of smooth vectors in an irreducible admissible unitary representation, then the underlying Harish-Chandra module is the space of K_n-finite vectors $V_{\pi,\mathrm{K}}$. V_π then corresponds to the (Casselman-Wallach) canonical completion of $V_{\pi,\mathrm{K}}$ [94]. The category of Harish-Chandra modules is appropriate for the algebraic theory of representations, but it is useful to work in the category of smooth admissible representations for automorphic forms. If in our context we take the underlying Harish-Chandra modules $V_{\pi,\mathrm{K}}$ and $V_{\pi',\mathrm{K}}$ then their algebraic tensor product is an admissible Harish-Chandra module for $\mathrm{GL}_n(k)\times\mathrm{GL}_m(k)$. The associated smooth admissible representation is the canonical completion of this tensor product, which is in fact $V_{\pi\hat{\otimes}\pi'}$, the topological tensor product of the smooth representations π and π'. It is also the space of smooth vectors in the unitary tensor product of the unitary representations associated to π and π'. So this completion is a natural place to work in the category of smooth admissible representations.

Now let

$$\mathcal{W}(\pi\hat{\otimes}\pi',\psi)=\{W(g,h)=\hat{\Lambda}(\pi(g)\otimes\pi'(h)\xi)|\xi\in V_{\pi\hat{\otimes}\pi'}\}.$$

Then $\mathcal{W}(\pi\hat{\otimes}\pi',\psi)$ contains the algebraic tensor product $\mathcal{W}(\pi,\psi)\otimes\mathcal{W}(\pi',\psi^{-1})$ and is again equal to the topological tensor product. Now we can extend all out local integrals to the space $\mathcal{W}(\pi\hat{\otimes}\pi',\psi)$ by setting

$$\Psi_j(s;W)=\int\int W\left(\begin{pmatrix} h & & \\ x & I_j & \\ & & I_{n-m-j}\end{pmatrix},h\right)\,dx\,|\det(h)|^{s-(n-m)/2}\,dh$$

and

$$\Psi(s;W,\Phi)=\int W(g,g)\Phi(e_ng)|\det(g)|^s\,dh$$

for $W\in\mathcal{W}(\pi\hat{\otimes}\pi',\psi)$. Since the local integrals are continuous with respect to the topology on the topological tensor product, all of the above facts remain true, in particular the convergence statements, the local functional equations, and the fact that these integrals extend to functions in $\mathcal{M}(\pi\times\pi')$.

At this point, let $\mathcal{I}_j(\pi,\pi')=\{\Psi_j(s;W)|W\in\mathcal{W}(\pi\hat{\otimes}\pi')\}$ and let $\mathcal{I}(\pi,\pi')$ be the span of the local integrals $\{\Psi(s;W,\Phi)|W\in\mathcal{W}(\pi\hat{\otimes}\pi',\psi),\ \Phi\in\mathcal{S}(k^n)\}$. Once again, in the case $m<n$ we have that the space $\mathcal{I}_j(\pi,\pi')$ is independent of j and we denote it also by $\mathcal{I}(\pi,\pi')$. These are still independent of ψ. So we know from above that $\mathcal{I}(\pi,\pi')\subset\mathcal{M}(\pi\times\pi')$. The remainder of [47] is then devoted to showing the following.

Theorem 3.7. $\mathcal{I}(\pi,\pi')=\mathcal{M}(\pi\times\pi')$.

As a consequence of this, we draw the following useful Corollary.

Corollary . There is a choice of Whittaker function W in $\mathcal{W}(\pi\hat{\otimes}\pi',\psi)$ such that $L(s,\pi\times\pi')=\Psi(s;W)$ if $m<n$ or finite collection of functions $W_i\in\mathcal{W}(\pi\hat{\otimes}\pi',\psi)$ and $\Phi_i\in\mathcal{S}(k^n)$ such that $L(s,\pi\times\pi')=\sum_i\Psi(s;W_i,\Phi_i)$ if $m=n$.

As a final result, let us note that in [16] it is established that the linear functionals $e(s;W) = \Psi(s;W)L(s,\pi\times\pi')^{-1}$ and $e(s;W,\Phi) = \Psi(s;W,\Phi)L(s,\pi\times\pi')^{-1}$ are continuous on $\mathcal{W}(\pi\hat{\otimes}\pi',\psi)$, uniformly for s in compact sets. Since there is a choice of $W \in \mathcal{W}(\pi\hat{\otimes}\pi',\psi)$ such that $e(s;W)\equiv 1$ or $W_i \in \mathcal{W}(\pi\hat{\otimes}\pi',\psi)$ and $\Phi_i \in \mathcal{S}(k^n)$ such that $\sum e(s;W_i,\Phi_i)\equiv 1$, as a result of this continuity and the fact that the algebraic tensor product $\mathcal{W}(\pi,\psi)\otimes\mathcal{W}(\pi',\psi^{-1})$ is dense in $\mathcal{W}(\pi\hat{\otimes}\pi',\psi)$ we have the following result [16].

Proposition 3.8. For any $s_0 \in \mathbb{C}$ there are choices of $W \in \mathcal{W}(\pi,\psi)$, $W' \in \mathcal{W}(\pi',\psi^{-1})$ and if necessary Φ such that $e(s_0;W,W')\neq 0$ or $e(s_0;W,W',\Phi)\neq 0$.

The continuity of the local integrals plays a role in proving the following result of Stade [89, 90] and Jacquet and Shalika (unpublished).

Theorem 3.8. In the cases $m = n$ and $m = n-1$ there exist a finite collection of K–finite functions $W_i \in \mathcal{W}(\pi,\psi)$, $W_i' \in \mathcal{W}(\pi',\psi^{-1})$, and $\Phi_i \in \mathcal{S}(k^n)$ if necessary such that

$$L(s,\pi\times\pi') = \sum \Psi(s;W_i,W_i') \quad \text{or} \quad L(s,\pi\times\pi') = \sum \Psi(s;W_i,W_i',\Phi_i).$$

In the case where both π and π' are unramified, Stade shows that one obtains the L-function exactly with the K–invariant Whittaker functions (and Schwartz function if necessary). In the general case, Jacquet has provided us with a sketch of his argument with Shalika. First one proves that the integrals involving K–finite functions are equal to the product of a polynomial and the L-factor. It suffices to prove this for principal series, since the other representations embed into principal series. For principal series one proceeds by an induction argument on n, however one must prove the $m = n$ and $m = n-1$ cases simultaneously. The (essentially formal) arguments needed are to be found in the published papers of Jacquet and Shalika. The polynomials in question then form an ideal and the point now is to show this ideal is the full polynomial ring. This is then implied by Proposition 3.8 above.

LECTURE 4
Global L-functions

Once again, we let k be a global field, $\mathbb{A}$ its ring of adeles, and fix a non-trivial continuous additive character $\psi = \otimes\psi_v$ of $\mathbb{A}$ trivial on k.

Let (π, V_π) be an cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$ (see Lecture 1 for all the implied assumptions in this terminology) and $(\pi', V_{\pi'})$ a cuspidal representation of $\mathrm{GL}_m(\mathbb{A})$. Since they are irreducible we have restricted tensor product decompositions $\pi \simeq \otimes'\pi_v$ and $\pi' \simeq \otimes'\pi'_v$ with (π_v, V_{π_v}) and $(\pi'_v, V_{\pi'_v})$ irreducible admissible smooth generic unitary representations of $\mathrm{GL}_n(k_v)$ and $\mathrm{GL}_m(k_v)$ [19, 26]. Let $\omega = \otimes'\omega_v$ and $\omega' = \otimes'\omega'_v$ be their central characters. These are both continuous characters of $k^\times\backslash\mathbb{A}^\times$.

Let S be a finite set of places of k, containing the archimedean places S_∞, such that for all $v \notin S$ we have that π_v, π'_v, and ψ_v are unramified.

For each place v of k we have defined the local factors $L(s, \pi_v \times \pi'_v)$ and $\varepsilon(s, \pi_v \times \pi'_v, \psi_v)$. Then we can at least formally define

$$L(s, \pi \times \pi') = \prod_v L(s, \pi_v \times \pi'_v) \quad \text{and} \quad \varepsilon(s, \pi \times \pi') = \prod_v \varepsilon(s, \pi_v \times \pi'_v, \psi_v).$$

We need to discuss convergence of these products. Let us first consider the convergence of $L(s, \pi \times \pi')$. For those $v \notin S$, so π_v, π'_v, and ψ_v are unramified, we know that $L(s, \pi_v \times \pi'_v) = \det(I - q_v^{-s} A_{\pi_v} \otimes A_{\pi'_v})^{-1}$ and that the eigenvalues of A_{π_v} and $A_{\pi'_v}$ are all of absolute value less than $q_v^{1/2}$. Thus the partial (or incomplete) L-function

$$L^S(s, \pi \times \pi') = \prod_{v \notin S} L(s, \pi_v \times \pi'_v) = \prod_{v \notin S} \det(I - q^{-s} A_{\pi_v} \otimes A_{\pi'_v})^{-1}$$

is absolutely convergent for $\mathrm{Re}(s) >> 0$. Thus the same is true for $L(s, \pi \times \pi')$.

For the ε–factor, we have seen that $\varepsilon(s, \pi_v \times \pi'_v, \psi_v) \equiv 1$ for $v \notin S$ so that the product is in fact a finite product and there is no problem with convergence. The fact that $\varepsilon(s, \pi \times \pi')$ is independent of ψ can either be checked by analyzing how the local ε–factors vary as you vary ψ, as is done in [9, Lemma 2.1], or it will follow from the global functional equation presented below.

4.1. The basic analytic properties

Our first goal is to show that these L-functions have nice analytic properties.

Theorem 4.1. The global L–functions $L(s,\pi\times\pi')$ are nice in the sense that

1. $L(s,\pi\times\pi')$ has a meromorphic continuation to all of $\mathbb{C}$,
2. the extended function is bounded in vertical strips (away from its poles),
3. they satisfy the functional equation

$$L(s,\pi\times\pi')=\varepsilon(s,\pi\times\pi')L(1-s,\widetilde{\pi}\times\widetilde{\pi}').$$

To do so, we relate the L-functions to the global integrals.

Let us begin with continuation. In the case $m<n$ for every $\varphi\in V_\pi$ and $\varphi'\in V_{\pi'}$ we know the integral $I(s;\varphi,\varphi')$ converges absolutely for all s. From the unfolding in Lecture 2 and the local calculation of Lecture 3 we know that for $\mathrm{Re}(s)>>0$ and for appropriate choices of φ and φ' we have

$$\begin{aligned}
I(s;\varphi,\varphi') &= \prod_v \Psi_v(s;W_{\varphi_v},W_{\varphi'_v})\\
&= \left(\prod_{v\in S}\Psi_v(s;W_{\varphi_v},W_{\varphi'_v})\right)L^S(s,\pi\times\pi')\\
&= \left(\prod_{v\in S}\frac{\Psi_v(s;W_{\varphi_v},W_{\varphi'_v})}{L(s,\pi_v\times\pi'_v)}\right)L(s,\pi\times\pi')\\
&= \left(\prod_{v\in S}e_v(s;W_{\varphi_v},W_{\varphi'_v})\right)L(s,\pi\times\pi')
\end{aligned}$$

We know that each $e_v(s;W_v,W'_v)$ is entire. Hence $L(s,\pi\times\pi')$ has a meromorphic continuation. If $m=n$ then for appropriate $\varphi\in V_\pi$, $\varphi'\in V_{\pi'}$, and $\Phi\in\mathcal{S}(\mathbb{A}^n)$ we again have

$$I(s;\varphi,\varphi',\Phi)=\left(\prod_{v\in S}e_v(s;W_{\varphi_v},W'_{\varphi'_v},\Phi_v)\right)L(s,\pi\times\pi').$$

Once again, since each $e_v(s;W_v,W'_v,\Phi_v)$ is entire, $L(s,\pi\times\pi')$ has a meromorphic continuation.

Let us next turn to the functional equation. This will follow from the functional equation for the global integrals and the local functional equations. We will consider only the case where $m<n$ since the other case is entirely analogous. The functional equation for the global integrals is simply

$$I(s;\varphi,\varphi')=\tilde{I}(1-s;\widetilde{\varphi},\widetilde{\varphi}').$$

Once again we have for appropriate φ and φ'

$$I(s;\varphi,\varphi')=\left(\prod_{v\in S}e_v(s;W_{\varphi_v},W'_{\varphi'_v})\right)L(s,\pi\times\pi')$$

while on the other side

$$\tilde{I}(1-s;\widetilde{\varphi},\widetilde{\varphi}')=\left(\prod_{v\in S}\tilde{e}_v(1-s;\rho(w_{n,m})\widetilde{W}_{\varphi_v},\widetilde{W}'_{\varphi'_v})\right)L(1-s,\widetilde{\pi}\times\widetilde{\pi}').$$

However, by the local functional equations, for each $v \in S$ we have

$$\begin{aligned}\tilde{e}_v(1-s;\rho(w_{n,m})\widetilde{W}_v,\widetilde{W}'_v) &= \frac{\widetilde{\Psi}(1-s;\rho(w_{n,m})\widetilde{W}_v,\widetilde{W}'_v)}{L(1-s,\widetilde{\pi}\times\widetilde{\pi}')} \\ &= \omega'_v(-1)^{n-1}\varepsilon(s,\pi_v\times\pi'_v,\psi_v)\frac{\Psi(s;W_v,W'_v)}{L(s,\pi\times\pi')} \\ &= \omega'_v(-1)^{n-1}\varepsilon(s,\pi_v\times\pi'_v,\psi_v)e_v(s,W_v,W'_v)\end{aligned}$$

Combining these, we have

$$L(s,\pi\times\pi') = \left(\prod_{v\in S}\omega'_v(-1)^{n-1}\varepsilon(s,\pi_v\times\pi'_v,\psi_v)\right)L(1-s,\widetilde{\pi}\times\widetilde{\pi}').$$

Now, for $v \notin S$ we know that π'_v is unramified, so $\omega'_v(-1) = 1$, and also that $\varepsilon(s,\pi_v\times\pi'_v,\psi_v)\equiv 1$. Therefore

$$\begin{aligned}\prod_{v\in S}\omega'_v(-1)^{n-1}\varepsilon(s,\pi_v\times\pi'_v,\psi_v) &= \prod_{v}\omega'_v(-1)^{n-1}\varepsilon(s,\pi_v\times\pi'_v,\psi_v) \\ &= \omega'(-1)^{n-1}\varepsilon(s,\pi\times\pi') \\ &= \varepsilon(s,\pi\times\pi')\end{aligned}$$

and we indeed have

$$L(s,\pi\times\pi') = \varepsilon(s,\pi\times\pi')L(1-s,\widetilde{\pi}\times\widetilde{\pi}').$$

Note that this implies that $\varepsilon(s,\pi\times\pi')$ is independent of ψ as well.

Let us now turn to the boundedness in vertical strips. For the global integrals $I(s;\varphi,\varphi')$ or $I(s;\varphi,\varphi,\Phi)$ this simply follows from the absolute convergence. For the L-function itself, the paradigm is the following. For every finite place $v \in S$ we know that there is a choice of $W_{v,i}$, $W'_{v,i}$, and $\Phi_{v,i}$ if necessary such that

$$L(s,\pi_v\times\pi'_v) = \sum\Psi(s;W_{v,i},W'_{v'}{}_i) \quad \text{or} \quad L(s,\pi_v\times\pi'_v) = \sum\Psi(s;W_{v,i},W'_{v'}{}_i,\Phi_{v,i}).$$

If $m = n-1$ or $m = n$ then by Theorem 3.8 we know that we have similar statements for $v \in S_\infty$. Hence if $m = n-1$ or $m = n$ there are global choices φ_i, φ'_i, and if necessary Φ_i such that

$$L(s,\pi\times\pi') = \sum I(s;\varphi_i,\varphi'_i) \quad \text{or} \quad L(s,\pi\times\pi') = \sum I(s;\varphi_i,\varphi'_i,\Phi_i).$$

Then the boundedness in vertical strips for the L-functions follows from that of the global integrals.

However, if $m < n-1$ then all we know at those $v \in S_\infty$ is that there is a function $W_v \in \mathcal{W}(\pi_v\hat{\otimes}\pi'_v,\psi_v) = \mathcal{W}(\pi_v,\psi_v)\hat{\otimes}\mathcal{W}(\pi'_v,\psi_v^{-1})$ or a finite collection of such functions $W_{v,i}$ and of $\Phi_{v,i}$ such that

$$L(s,\pi_v\times\pi'_v) = I(s;W_v) \quad \text{or} \quad L(s,\pi_v\times\pi'_v) = \sum I(s;W_{v,i},\Phi_{v,i}).$$

To make the above paradigm work for $m < n-1$ we would need to rework the theory of global Eulerian integrals for cusp forms in $V_\pi\hat{\otimes}V_{\pi'}$. This is naturally the space of smooth vectors in an irreducible unitary cuspidal representation of $\mathrm{GL}_n(\mathbb{A})\times\mathrm{GL}_m(\mathbb{A})$. So we would need extend the global theory of integrals parallel to Jacquet and Shalika's extension of the local integrals in the archimedean theory. There seems to be no obstruction to carrying this out, and then we obtain boundedness in vertical strips for $L(s,\pi\times\pi')$ in general.

We should point out that if one approaches these L-function by the method of constant terms and Fourier coefficients of Eisenstein series, then Gelbart and Shahidi have shown a wide class of automorphic L-functions, including ours, to be bounded in vertical strips [25].

4.2. Poles of L-functions

Let us determine where the global L-functions can have poles. The poles of the L-functions will be related to the poles of the global integrals. Recall from Lecture 2 that in the case of $m < n$ we have that the global integrals $I(s;\varphi,\varphi')$ are entire and that when $m = n$ then $I(s;\varphi,\varphi',\Phi)$ can have at most simple poles and they occur at $s = -i\sigma$ and $s = 1 - i\sigma$ for σ real when $\pi \simeq \widetilde{\pi}' \otimes |\det|^{i\sigma}$. As we have noted above, the global integrals and global L-functions are related, for appropriate φ, φ', and Φ, by

$$I(s;\varphi,\varphi') = \left(\prod_{v\in S} e_v(s;W_{\varphi_v},W'_{\varphi'_v})\right) L(s,\pi\times\pi')$$

or

$$I(s;\varphi,\varphi',\Phi) = \left(\prod_{v\in S} e_v(s;W_{\varphi_v},W'_{\varphi'_v},\Phi_v)\right) L(s,\pi\times\pi').$$

On the other hand, we have seen that for any $s_0 \in \mathbb{C}$ and any v there is a choice of local W_v, W'_v, and Φ_v such that the local factors $e_v(s_0;W_v,W'_v) \neq 0$ or $e_v(s_0;W_v,W'_v,\Phi_v) \neq 0$. So as we vary φ, φ' and Φ at the places $v \in S$ we see that division by these factors can introduce no extraneous poles in $L(s,\pi\times\pi')$, that is, in keeping with the local characterization of the L-factor in terms of poles of local integrals, globally the poles of $L(s,\pi\times\pi')$ are precisely the poles of the family of global integrals $\{I(s;\varphi,\varphi')\}$ or $\{I(s;\varphi,\varphi',\Phi)\}$. Hence from Theorems 2.1 and 2.2 we have.

Theorem 4.2. If $m < n$ then $L(s,\pi\times\pi')$ is entire. If $m = n$, then $L(s,\pi\times\pi')$ has at most simple poles and they occur iff $\pi \simeq \widetilde{\pi}' \otimes |\det|^{i\sigma}$ with σ real and are then at $s = -i\sigma$ and $s = 1 - i\sigma$.

If we apply this with $\pi' = \widetilde{\pi}$ we obtain the following useful corollary.

Corollary . $L(s,\pi\times\widetilde{\pi})$ has simple poles at $s = 0$ and $s = 1$.

4.3. Strong Multiplicity One

Let us return to the Strong Multiplicity One Theorem for cuspidal representations. First, recall the statement:

Theorem (Strong Multiplicity One)**.** Let (π, V_π) and $(\pi', V_{\pi'})$ be two cuspidal representations of $GL_n(\mathbb{A})$. Suppose there is a finite set of places S such that for all $v \notin S$ we have $\pi_v \simeq \pi'_v$. Then $\pi = \pi'$.

We will now present Jacquet and Shalika's proof of this statement via L-functions [45]. First note the following observation, which follows from our analysis of the location of the poles of the L-functions.

Observation . For π and π' cuspidal representations of $\mathrm{GL}_n(\mathbb{A})$, $L(s, \pi \times \widetilde{\pi}')$ has a pole at $s = 1$ iff $\pi \simeq \pi'$.

Thus the L-function gives us an analytic method of testing when two cuspidal representations are isomorphic, and so by the Multiplicity One Theorem, the same.

Proof: If we take π and π' as in the statement of Strong Multiplicity One, we have that $\pi_v \simeq \pi'_v$ for $v \notin S$ and hence

$$L^S(s, \pi \times \widetilde{\pi}) = \prod_{v \notin S} L(s, \pi_v \times \widetilde{\pi}_v) = \prod_{v \notin S} L(s, \pi_v \times \widetilde{\pi}'_v) = L^S(s, \pi \times \widetilde{\pi}')$$

Since the local L-factors never vanish and for unitary representations they have no poles in $\mathrm{Re}(s) \geq 1$ (since the local integrals have no poles in this region) we see that for $s = 1$ that $L(s, \pi \times \widetilde{\pi}')$ has a pole at $s = 1$ iff $L^S(s, \pi \times \widetilde{\pi}')$ does. Hence we have that since $L(s, \pi \times \widetilde{\pi})$ has a pole at $s = 1$, so does $L^S(s, \pi \times \widetilde{\pi})$. But $L^S(s, \pi \times \widetilde{\pi}) = L^S(s, \pi \times \widetilde{\pi}')$, so that both $L^S(s, \pi \times \widetilde{\pi}')$ and then $L(s, \pi \times \widetilde{\pi}')$ have poles at $s = 1$. But then the L-function criterion above gives that $\pi \simeq \pi'$. Now apply Multiplicity One. $\square$

In fact, Jacquet and Shalika push this method much further. If π is an irreducible automorphic representation of $\mathrm{GL}_n(\mathbb{A})$, but not necessarily cuspidal, then it is a theorem of Langlands [61] that there are cuspidal representations, say $\tau_1, \ldots, \tau_r$ of $\mathrm{GL}_{n_1}, \ldots, \mathrm{GL}_{n_r}$ with $n = n_1 + \cdots + n_r$, such that π is a constituent of $\mathrm{Ind}(\tau_1 \otimes \cdots \otimes \tau_r)$. Similarly, π' is a constituent of $\mathrm{Ind}(\tau'_1 \otimes \cdots \otimes \tau'_{r'})$. Then the generalized version of the Strong Multiplicity One theorem that Jacquet and Shalika establish in [45] is the following.

Theorem (Generalized Strong Multiplicity One)**.** Given π and π' irreducible automorphic representations of $\mathrm{GL}_n(\mathbb{A})$ as above, suppose that there is a finite set of places S such that $\pi_v \simeq \pi'_v$ for all $v \notin S$. Then $r = r'$ and there is a permutation σ of the set $\{1, \ldots, r\}$ such that $n_i = n'_{\sigma(i)}$ and $\tau_i = \tau'_{\sigma(i)}$.

Note, the cuspidal representations τ_i and τ'_i need not be unitary in this statement.

4.4. Non-vanishing results

Of interest for questions from analytic number theory, for example questions of equidistribution, are results on the non-vanishing of L-functions. The fundamental non-vanishing result for GL_n is the following theorem of Jacquet and Shalika [44] and Shahidi [75, 76].

Theorem 4.3. Let π and π' be cuspidal representations of $\mathrm{GL}_n(\mathbb{A})$ and $\mathrm{GL}_m(\mathbb{A})$. Then the L-function $L(s, \pi \times \pi')$ is non-vanishing for $\mathrm{Re}(s) \geq 1$.

The proof of non-vanishing for $\mathrm{Re}(s) > 1$ is in keeping with the spirit of these notes [45, I, Theorem 5.3]. Since the local L-functions are never zero, to establish the non-vanishing of the Euler product for $\mathrm{Re}(s) > 1$ it suffices to show that the Euler product is absolutely convergent for $\mathrm{Re}(s) > 1$, and for this it is sufficient to work with the incomplete L-function $L^S(s,\pi\times\pi')$ where S is as at the beginning of this Lecture. Then we can write

$$L^S(s,\pi\times\pi')=\prod_{v\notin S}L(s,\pi_v\times\pi'_v)=\prod_{v\notin S}\det(I-q_v^{-s}A_{\pi_v}\otimes A_{\pi'_v})^{-1}$$

with absolute convergence for $\mathrm{Re}(s)>>0$.

Recall that an infinite product $\prod(1+a_n)$ is absolutely convergent iff the associated series $\sum\log(1+a_n)$ is absolutely convergent.

Let us write

$$A_{\pi_v}=\begin{pmatrix}\mu_{v,1}&&\\&\ddots&\\&&\mu_{v,n}\end{pmatrix}\quad\text{and}\quad A_{\pi'_v}=\begin{pmatrix}\mu'_{v,1}&&\\&\ddots&\\&&\mu'_{v,m}\end{pmatrix}.$$

We have seen that $|\mu_{v,i}|<q_v^{1/2}$ and $|\mu'_{v,j}|<q_v^{1/2}$. Then

$$\begin{aligned}\log L(s,\pi_v\times\pi'_v)&=-\sum_{i,j}\log(1-\mu_{v,i}\mu'_{v,j}q_v^{-s})\\&=\sum_{i,j}\sum_{d=1}^{\infty}\frac{(\mu_{v,i}\mu'_{v,j})^d}{dq_v^{ds}}=\sum_{d=1}^{\infty}\frac{\mathrm{tr}(A^d_{\pi_v})\,\mathrm{tr}(A^d_{\pi'_v})}{dq_v^{ds}}\end{aligned}$$

with the sum absolutely convergent for $\mathrm{Re}(s)>>0$. Then, still for $\mathrm{Re}(s)>>0$,

$$\log(L^S(s,\pi\times\pi'))=\sum_{v\notin S}\sum_{d=1}^{\infty}\frac{\mathrm{tr}(A^d_{\pi_v})\,\mathrm{tr}(A^d_{\pi'_v})}{dq_v^{ds}}.$$

If we apply this to $\pi'=\overline{\pi}=\widetilde{\pi}$ we find

$$\log(L^S(s,\pi\times\overline{\pi}))=\sum_{v\notin S}\sum_{d=1}^{\infty}\frac{|\,\mathrm{tr}(A^d_{\pi_v})|^2}{dq_v^{ds}}$$

which is a Dirichlet series with non-negative coefficients. By Landau's Lemma this will be absolutely convergent up to the its first pole, which we know is at $s=1$. Hence this series, and the associated Euler product $L(s,\pi\times\widetilde{\pi})$, is absolutely convergent for $\mathrm{Re}(s)>1$.

An application of the Cauchy–Schwatrz inequality then implies that the series

$$\log(L^S(s,\pi\times\pi'))=\sum_{v\notin S}\sum_{d=1}^{\infty}\frac{\mathrm{tr}(A^d_{\pi_v})\,\mathrm{tr}(A^d_{\pi'_v})}{dq_v^{ds}}$$

is also absolutely convergent for $\mathrm{Re}(s)>1$. Thus $L(s,\pi\times\pi')$ is absolutely convergent and hence non-vanishing for $\mathrm{Re}(s)>1$.

To obtain the non-vanishing on the line $\mathrm{Re}(s)=1$ requires the technique of analyzing L-functions via their occurrence in the constant terms and Fourier coefficients of Eisenstein series, which we have not discussed. They can be found in the references [44] and [75, 76] mentioned above.

4.5. The Generalized Ramanujan Conjecture (GRC)

The current version of the GRC is a conjecture about the structure of cuspidal representations.

Conjecture (GRC)**.** Let π be a (unitary) cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$ with decomposition $\pi \simeq \otimes' \pi_v$. Then the local components π_v are tempered representations.

However, it has an interesting interpretation in terms of L-functions which is more in keeping with the origins of the conjecture. If π is cuspidal, then at every finite place v where π_v is unramified we have associated a semisimple conjugacy class, say $A_{\pi_v} = \begin{pmatrix} \mu_{v,1} & & \\ & \ddots & \\ & & \mu_{v,n} \end{pmatrix}$ so that

$$L(s, \pi_v) = \det(I - q_v^{-s} A_{\pi_v})^{-1} = \prod_{i=1}^{n} (1 - \mu_{v,i} q_v^{-s})^{-1}.$$

If v is an archimedean place where π_v is unramified, then we can similarly write

$$L(s, \pi) = \prod_{i=1}^{n} \Gamma_v(s + \mu_{v,i})$$

where

$$\Gamma_v(s) = \begin{cases} \pi^{-s/2}\Gamma(\frac{s}{2}) & \text{if } k_v \simeq \mathbb{R} \\ 2(2\pi)^{-s}\Gamma(s) & \text{if } k_v \simeq \mathbb{C} \end{cases}.$$

Then the statement of the GRC in these terms is

Conjecture (GRC for L-functions)**.** If π is a cuspidal representation of $\mathrm{GL}_n(\mathbb{A})$ and if v is a place where π_v is unramified, then $|\mu_{v,i}| = 1$ for v non-archimedean and $\mathrm{Re}(\mu_{v,i}) = 0$ for v archimedean.

Note that by including the archimedean places, this conjecture encompasses not only the classical Ramanujan conjectures but also the various versions of the Selberg eigenvalue conjecture [36].

Recall that by the Corollary to Theorem 3.3 we have the bounds $q_v^{-1/2} < |\mu_{v,i}| < q_v^{1/2}$ for v non-archimedean, and a similar local analysis for v archimedean would give $|\,\mathrm{Re}(\mu_{v,i})| < \frac{1}{2}$. The best bound for general GL_n over a number field is due to Luo, Rudnick, and Sarnak [63]. They are the uniform bounds

$$q_v^{-(\frac{1}{2} - \frac{1}{n^2+1})} \leq |\mu_{v,i}| \leq q_v^{\frac{1}{2} - \frac{1}{n^2+1}} \quad \text{if } v \text{ is non-archimedean}$$

and

$$|\,\mathrm{Re}(\mu_{v,i})| \leq \frac{1}{2} - \frac{1}{n^2+1} \quad \text{for } v \text{ archimedean.}$$

Their techniques are global and rely on the theory of Rankin–Selberg L-functions as presented here, a technique of persistence of zeros in families of L-functions, and a positivity argument.

For function fields over a finite field, the Ramanujan Conjecture for GL_n follows from Lafforgue's proof of the Global Langlands Correspondence [59].

For GL_2 over a number field there has been much recent progress. The best general estimates at present are due to Kim and Shahidi [56], who use the holomorphy of the symmetric ninth power L-function for $\mathrm{Re}(s) > 1$ to obtain

$$q_v^{-\frac{1}{9}} < |\mu_{v,i}| < q_v^{\frac{1}{9}} \quad \text{for } i = 1, 2, \text{ and } v \text{ non-archimedean.}$$

A simple generalization of the method used in [56] gives the analogous estimate at the archimedean places over any number field [54]. On the other hand, [53] and its appendix gives that one can replace $\frac{1}{9}$ with $\frac{7}{64}$ at all places, but only for $k = \mathbb{Q}$.

For some applications, the notion of *weakly Ramanujan* [10] can replace knowing the full GRC.

Definition . A cuspidal representation π of $\mathrm{GL}_n(\mathbb{A})$ is called weakly Ramanujan if for every $\epsilon > 0$ there is a constant $c_\epsilon > 0$ and an infinite sequence of places $\{v_m\}$ with the property that each π_{v_m} is unramified and the Satake parameters $\mu_{v_m,i}$ satisfy

$$c_\epsilon^{-1} q_{v_m}^{-\epsilon} < |\mu_{v_m,i}| < c_\epsilon q_{v_m}^{\epsilon}.$$

For example, if we knew that all cuspidal representations on $\mathrm{GL}_n(\mathbb{A})$ were weakly Ramanujan, then we would know that under Langlands liftings between general linear groups, the property of occurrence in the spectral decomposition is preserved [10].

For $n = 2, 3$ our techniques let us show the following.

Proposition 4.1. For $n = 2$ or $n = 3$ all cuspidal representations are weakly Ramanujan.

Proof: First, let π be a cuspidal representation or $\mathrm{GL}_n(\mathbb{A})$. Recall that from the absolute convergence of the Euler product for $L(s, \pi \times \overline{\pi})$ we know that the series $\displaystyle\sum_{v \notin S} \sum_d \frac{|\operatorname{tr}(A_{\pi_v}^d)|^2}{d q_v^{ds}}$ is absolutely convergent for $\mathrm{Re}(s) > 1$, so that in particular we have that $\displaystyle\sum_{v \notin S} \frac{|\operatorname{tr}(A_{\pi_v})|^2}{q_v^s}$ is absolutely convergent for $\mathrm{Re}(s) > 1$. Thus, for a set of places of positive density, we have the estimate $|\operatorname{tr}(A_{\pi_v})|^2 < q_v^\epsilon$ for each ϵ. Since $\overline{A_{\pi_v}} = A_{\pi_v}^{-1}$ for components of cuspidal representations, we have the same estimate for $|\operatorname{tr}(A_{\pi_v}^{-1})|$.

In the case of $n = 2$ and $n = 3$, these estimates and the fact that $|\det A_{\pi_v}| = |\omega_v(\varpi_v)| = 1$ give us estimates on the coefficients of the characteristic polynomial for A_{π_v}. For example, if $n = 3$ and the characteristic polynomial of A_{π_v} is $X^3 + aX^2 + bX + c$ then we know $|a| = |\operatorname{tr}(A_{\pi_v})| < q_v^{\epsilon/2}$, $|b| = |\operatorname{tr}(A_{\pi_v}^{-1})\det(A_{\pi_v})| < q_v^{\epsilon/2}$, and $|c| = |\det(A_{\pi_v})| = 1$. Then an application of Rouche's theorem gives that the roots of this polynomial all lie in the circle of radius q_v^ϵ as long as $q_v > 3$. Applying this to both A_{π_v} and $A_{\pi_v}^{-1}$ we find that for our set primes of positive density above we have the estimate $q_v^{-\epsilon} < |\mu_{v_m,i}| < q_v^\epsilon$. Thus we find that for $n = 2, 3$ cuspidal representations of GL_n are weakly Ramanujan. $\square$

4.6. The Generalized Riemann Hypothesis (GRH)

This is one of the most important conjectures in the analytic theory of L-functions. Simply stated, it is

Conjecture (GRH). For any cuspidal representation π, all the zeros of the L-function $L(s, \pi)$ lie on the line $\mathrm{Re}(s) = \frac{1}{2}$.

Even in the simplest case of $n = 1$ and $\pi = \mathbf{1}$ the trivial representation this reduces to the Riemann hypothesis for the Riemann zeta function!

For an interesting survey on these and other conjectures on L-functions and their relation to number theoretic problems, we refer the reader to the survey of Iwaniec and Sarnak [36].

LECTURE 5
Converse Theorems

Let us return first to Hecke. Recall that to a modular form

$$f(\tau) = \sum_{n-1}^{\infty} a_n e^{2\pi i n\tau}$$

for say $\mathrm{SL}_2(\mathbb{Z})$ Hecke attached an L function $L(s, f)$ and they were related via the Mellin transform

$$\Lambda(s, f) = (2\pi)^{-s}\Gamma(s)L(s, f) = \int_0^\infty f(iy)y^s \, d^\times y$$

and derived the functional equation for $L(s, f)$ from the modular transformation law for $f(\tau)$ under the transformation $\tau \mapsto -1/\tau$. In his fundamental paper [33] he inverted this process by taking a Dirichlet series

$$D(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

and assuming that it converged in a half plane, had an entire continuation to a function of finite order, and satisfied the same functional equation as the L-function of a modular form of weight k, then he could actually reconstruct a modular form from $D(s)$ by Mellin inversion

$$f(iy) = \sum_i a_n e^{-2\pi n y} = \frac{1}{2\pi i}\int_{\sigma_0 - i\infty}^{\sigma_0 + i\infty} (2\pi)^{-s}\Gamma(s)D(s)y^{-s} \, ds$$

and obtain the modular transformation law for $f(\tau)$ under $\tau \mapsto -1/\tau$ from the functional equation for $D(s)$ under $s \mapsto k - s$. This is Hecke's Converse Theorem.

In this Lecture we will present some analogues of Hecke's theorem in the context of L-functions for GL_n. Surprisingly, the technique is exactly the same as Hecke's, i.e., inverting the integral representation. This was first done in the context of automorphic representation for GL_2 by Jacquet and Langlands [39] and then extended and significantly strengthened for GL_3 by Jacquet, Piatetski-Shapiro, and Shalika [40]. For a more extensive bibliography and history, see [13].

This section is taken mainly from our survey [13]. Further details can be found in [9, 12].

5.1. The results

Once again, let k be a global field, $\mathbb{A}$ its adele ring, and ψ a fixed non-trivial continuous additive character of $\mathbb{A}$ which is trivial on k. We will take $n \geq 3$ to be an integer.

To state these Converse Theorems, we begin with an irreducible admissible representation Π of $\mathrm{GL}_n(\mathbb{A})$. In keeping with the conventions of these notes, we will assume that Π is unitary and generic, but this is not necessary. It has a decomposition $\Pi = \otimes' \Pi_v$, where Π_v is an irreducible admissible generic representation of $\mathrm{GL}_n(k_v)$. By the local theory of Lecture 3, to each Π_v is associated a local L-function $L(s, \Pi_v)$ and a local ε-factor $\varepsilon(s, \Pi_v, \psi_v)$. Hence formally we can form

$$L(s,\Pi) = \prod_v L(s,\Pi_v) \qquad \text{and} \qquad \varepsilon(s,\Pi,\psi) = \prod_v \varepsilon(s,\Pi_v,\psi_v).$$

We will always assume the following two things about Π:

1. $L(s,\Pi)$ converges in some half plane $Re(s) >> 0$,
2. the central character ω_Π of Π is automorphic, that is, invariant under $k^\times$.

Under these assumptions, $\varepsilon(s,\Pi,\psi) = \varepsilon(s,\Pi)$ is independent of our choice of ψ [9].

Our Converse Theorems will involve twists by cuspidal automorphic representations of $\mathrm{GL}_m(\mathbb{A})$ for certain m. Let $\pi' = \otimes' \pi'_v$ be a cuspidal representation of $\mathrm{GL}_m(\mathbb{A})$ with $m < n$. Then again we can formally define

$$L(s,\Pi \times \pi') = \prod_v L(s,\Pi_v \times \pi'_v) \qquad \text{and} \qquad \varepsilon(s,\Pi \times \pi') = \prod_v \varepsilon(s,\Pi_v \times \pi'_v,\psi_v)$$

since again the local factors make sense whether Π is automorphic or not. A consequence of (1) and (2) above and the cuspidality of π' is that both $L(s,\Pi \times \pi')$ and $L(s,\widetilde{\Pi} \times \widetilde{\pi}')$ converge absolutely for $Re(s) >> 0$, where $\widetilde{\Pi}$ and $\widetilde{\pi}'$ are the contragredient representations, and that $\varepsilon(s,\Pi \times \pi')$ is independent of the choice of ψ.

We say that $L(s,\Pi \times \pi')$ is *nice* if it satisfies the same analytic properties it would if Π were cuspidal, i.e.,

1. $L(s,\Pi \times \pi')$ and $L(s,\widetilde{\Pi} \times \widetilde{\pi}')$ have analytic continuations to entire functions of s,
2. these entire continuations are bounded in vertical strips of finite width,
3. they satisfy the standard functional equation

$$L(s,\Pi \times \pi') = \varepsilon(s,\Pi \times \pi') L(1-s,\widetilde{\Pi} \times \widetilde{\pi}').$$

For convenience, let us set $\mathcal{A}(m)$ to be the set of automorphic representations of $\mathrm{GL}_m(\mathbb{A})$, $\mathcal{A}_0(m)$ the set of cuspidal representations of $\mathrm{GL}_m(\mathbb{A})$, and $\mathcal{T}(m) = \coprod_{d=1}^{m} \mathcal{A}_0(d)$. If we fix a finite set of S of finite places, then let $\mathcal{T}^S(m)$ denote the subset of $\mathcal{T}(m)$ consisting of representations that are *unramified* at all places $v \in S$.

The basic Converse Theorem for GL_n is the following.

Theorem 5.1. Let Π be an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A})$ as above. Let S be a finite set of finite places of k. Suppose that $L(s,\Pi \times \pi')$ is nice for all $\pi' \in \mathcal{T}^S(n-1)$.

1. If $S = \emptyset$ then Π is a cuspidal automorphic representation.

2. If $S \neq \emptyset$ then Π is quasi-automorphic in the sense that there is an automorphic representation Π' such that $\Pi_v \simeq \Pi'_v$ for all $v \notin S$.

In this theorem we twist by the maximal amount and obtain the strongest possible conclusion about Π. The proof of part 1 of this theorem essentially follows that of Hecke [33] and Weil [95] and Jacquet–Langlands [39]. It is of course valid for $n = 2$ as well. Note that as soon as we restrict the ramification of our twisting representations we lose information about Π at those places. In applications we usually choose S to contain the set of finite places v where Π_v is ramified.

For applications, it is desirable to twist by as little as possible. There are essentially two ways to restrict the twisting. One is to restrict the rank of the groups that the twisting representations live on. The other is to further restrict ramification.

When we restrict the rank of our twists, we can obtain the following result.

Theorem 5.2. Let Π be an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A})$ as above. Let S be a finite set of finite places of k. Suppose that $L(s, \Pi \times \pi')$ is nice for all $\pi' \in \mathcal{T}^S(n-2)$.

1. If $S = \emptyset$ then Π is a cuspidal automorphic representation.
2. If $S \neq \emptyset$ then Π is quasi-automorphic in the sense that there is an automorphic representation Π' such that $\Pi_v \simeq \Pi'_v$ for all $v \notin S$.

This result is stronger than Theorem 5.1, but its proof is a bit more delicate.

The second way to restrict our twists is to restrict the ramification at all but a finite number of places. Now fix a non-empty finite set of places S which in the case of a number field contains the set S_∞ of all archimedean places. Let $\mathcal{T}_S(m)$ denote the subset consisting of all representations π' in $\mathcal{T}(m)$ which are *unramified for all* $v \notin S$. Note that we are placing a grave restriction on the ramification of these representations.

Theorem 5.3. Let Π be an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A})$ as above. Let S be a non-empty finite set of places, containing S_∞, such that the class number of the ring $\mathfrak{o}_S$ of S-integers is one. Suppose that $L(s, \Pi \times \pi')$ is nice for all $\pi' \in \mathcal{T}_S(n-1)$. Then Π is quasi-automorphic in the sense that there is an automorphic representation Π' such that $\Pi_v \simeq \Pi'_v$ for all $v \in S$ and all $v \notin S$ such that both Π_v and Π'_v are unramified.

There are several things to note here. First, there is a class number restriction. However, if $k = \mathbb{Q}$ then we may take $S = S_\infty$ and we have a Converse Theorem with "level 1" twists. As a practical consideration, if we let S_Π be the set of finite places v where Π_v is ramified, then for applications we usually take S and S_Π to be disjoint. Once again, we are losing all information at those places $v \notin S$ where we have restricted the ramification unless Π_v was already unramified there.

The proof of part 1 of Theorem 5.1 essentially follows the lead of Hecke, Weil, and Jacquet–Langlands. It is based on the integral representations of L-functions, Fourier expansions, Mellin inversion, and finally a use of the weak form of Langlands' spectral theory. For part 2 of Theorem 5.1 and Theorems 5.2, and 5.3, where we have restricted our twists, we must impose certain local conditions to compensate for our limited twists. For Theorem 5.1 and 5.2 there are a finite number of local conditions and for Theorem 5.3 an infinite number of local conditions. We must then work around these by using results on generation of congruence subgroups and either weak or strong approximation.

5.2. Inverting the integral representation

Let Π be as above and let $\xi \in V_\Pi$ be a decomposable vector in the space V_Π of Π. Since Π is generic, then fixing local Whittaker models $\mathcal{W}(\Pi_v, \psi_v)$ at all places, compatibly normalized at the unramified places, we can associate to ξ a non-zero function $W_\xi(g) = \prod W_{\xi_v}(g_v)$ on $\mathrm{GL}_n(\mathbb{A})$ which transforms by the global character ψ under left translation by $\mathrm{N}_n(\mathbb{A})$, i.e., $W_\xi(ng) = \psi(n) W_\xi(g)$. Since ψ is trivial on rational points, we see that $W_\xi(g)$ is left invariant under $\mathrm{N}_n(k)$. We would like to use W_ξ to construct an embedding of V_Π into the space of (smooth) automorphic forms on $\mathrm{GL}_n(\mathbb{A})$. The simplest idea is to average W_ξ over $\mathrm{N}_n(k)\backslash \mathrm{GL}_n(k)$, but this will not be convergent. However, if we average over the rational points of the mirabolic $\mathrm{P} = \mathrm{P}_n$ then the sum

$$U_\xi(g) = \sum_{\mathrm{N}_n(k)\backslash \mathrm{P}(k)} W_\xi(pg)$$

is absolutely convergent. For the relevant growth properties of U_ξ see [9]. Since Π is assumed to have automorphic central character, we see that $U_\xi(g)$ is left invariant under both $\mathrm{P}(k)$ and the center $\mathrm{Z}_n(k)$.

Suppose now that we know that $L(s, \Pi \times \pi')$ is nice for all $\pi' \in \mathcal{T}(m)$. Then we will hope to obtain the remaining invariance of U_ξ from the $\mathrm{GL}_n \times \mathrm{GL}_m$ functional equation by inverting the integral representation for $L(s, \Pi \times \pi')$. With this in mind, let $\mathrm{Q} = \mathrm{Q}_m$ be the mirabolic subgroup of GL_n which stabilizes the standard unit vector ${}^t e_{m+1}$, that is the column vector all of whose entries are 0 except the $(m+1)^{th}$, which is 1. Note that if $m = n-1$ then Q is nothing more than the opposite mirabolic $\overline{\mathrm{P}} = {}^t\mathrm{P}^{-1}$ to P. If we let α_m be the permutation matrix in $\mathrm{GL}_n(k)$ given by

$$\alpha_m = \begin{pmatrix} & 1 & \\ I_m & & \\ & & I_{n-m-1} \end{pmatrix}$$

then $\mathrm{Q}_m = \alpha_m^{-1} \alpha_{n-1} \overline{\mathrm{P}} \alpha_{n-1}^{-1} \alpha_m$ is a conjugate of $\overline{\mathrm{P}}$ and for any m we have that $\mathrm{P}(k)$ and $\mathrm{Q}(k)$ generate all of $\mathrm{GL}_n(k)$. So now set

$$V_\xi(g) = \sum_{\mathrm{N}'(k)\backslash \mathrm{Q}(k)} W_\xi(\alpha_m q g)$$

where $\mathrm{N}' = \alpha_m^{-1}\, \mathrm{N}_n\, \alpha_m \subset \mathrm{Q}$. This sum is again absolutely convergent and is invariant on the left by $\mathrm{Q}(k)$ and $\mathrm{Z}(k)$. Thus, to embed Π into the space of automorphic forms it suffices to show $U_\xi = V_\xi$, for then we get invariance of U_ξ under all of $\mathrm{GL}_n(k)$. It is this that we will attempt to do using the integral representations.

Now let $(\pi', V_{\pi'})$ be an irreducible subrepresentation of the space of automorphic forms on $\mathrm{GL}_m(\mathbb{A})$ and assume $\varphi' \in V_{\pi'}$ is also factorizable. Let

$$I(s; U_\xi, \varphi') = \int_{\mathrm{GL}_m(k)\backslash \mathrm{GL}_m(\mathbb{A})} \mathbb{P}_m^n U_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \varphi'(h) |\det(h)|^{s-1/2}\, dh.$$

This integral is always absolutely convergent for $Re(s) >> 0$, and for all s if π' is cuspidal. As with the usual integral representation we have that this unfolds into the Euler product

$$\begin{aligned} I(s;U_\xi,\varphi') &= \int_{\mathrm{N}_m(\mathbb{A})\backslash\,\mathrm{GL}_m(\mathbb{A})} W_\xi \begin{pmatrix} h & 0 \\ 0 & I_{n-m} \end{pmatrix} W'_{\varphi'}(h)|\det(h)|^{s-(n-m)/2}\ dh \\ &= \prod_v \int_{\mathrm{N}_m(k_v)\backslash\,\mathrm{GL}_m(k_v)} W_{\xi_v} \begin{pmatrix} h_v & 0 \\ 0 & I_{n-m} \end{pmatrix} W'_{{\varphi'}_v}(h_v)|\det(h_v)|_v^{s-(n-m)/2}\ dh_v \\ &= \prod_v \Psi_v(s;W_{\xi_v},W'_{{\varphi'}_v}). \end{aligned}$$

Note that unless π' is generic, this integral vanishes.

Assume first that π' is cuspidal. Then from the local theory of L-functions from Lecture 3, for almost all finite places we have $\Psi_v(s;W_{\xi_v},W'_{{\varphi'}_v}) = L(s,\Pi_v\times{\pi'}_v)$ and for the other places $\Psi_v(s;W_{\xi_v},W'_{{\varphi'}_v}) = e_v(s;W_{\xi_v},W'_{{\varphi'}_v})L(s,\Pi_v\times{\pi'}_v)$ with the $e_v(s;W_{\xi_v},W'_{{\varphi'}_v})$ entire. So in this case $I(s;U_\xi,\varphi') = e(s)L(s,\Pi\times\pi')$ with $e(s)$ entire. Since $L(s;\Pi\times\pi')$ is assumed nice we may conclude that $I(s;U_\xi,\varphi')$ has an analytic continuation to an entire function. When π' is not cuspidal, it is a subrepresentation of a representation that is induced from (possibly non-unitary) cuspidal representations σ_i of $\mathrm{GL}_{r_i}(\mathbb{A})$ for $r_i < m$ with $\sum r_i = m$ and is in fact, if our integral doesn't vanish, the unique generic constituent of this induced representation. Then we can make a similar argument using this induced representation and the fact that the $L(s,\Pi\times\sigma_i)$ are nice to again conclude that for all π', $I(s;U_\xi,\varphi') = e(s)L(s,\Pi\times\pi') = e'(s)\prod L(s,\Pi\times\sigma_i)$ is entire. (See [9] for more details on this point.)

Similarly, consider $I(s;V_\xi,\varphi')$ for $\varphi'\in V_{\pi'}$ with π' an irreducible subrepresentation of the space of automorphic forms on $\mathrm{GL}_m(\mathbb{A})$, still with

$$I(s;V_\xi,\varphi') = \int_{\mathrm{GL}_m(k)\backslash\,\mathrm{GL}_m(\mathbb{A})} \mathbb{P}^n_m V_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \varphi'(h)|\det(h)|^{s-1/2}\ dh.$$

Now this integral converges for $Re(s) << 0$. However, when we unfold, we find

$$I(s;V_\xi,\varphi') = \prod \widetilde{\Psi}_v(1-s;\rho(w_{n,m})\widetilde{W}_{\xi_v},\widetilde{W}'_{{\varphi'}_v}) = \tilde{e}(1-s)L(1-s,\widetilde{\Pi}\times\widetilde{\pi}')$$

as above. Thus $I(s;V_\xi,\varphi')$ also has an analytic continuation to an entire function of s.

Now, utilizing the assumed global functional equation for $L(s,\Pi\times\pi')$ in the case where π' is cuspidal, or for the $L(s,\Pi\times\sigma_i)$ in the case π' is not cuspidal, as well as the local functional equations at $v\in S_\infty\cup S_\Pi\cup S_{\pi'}\cup S_\psi$ as in Lecture 3 one finds

$$I(s;U_\xi,\varphi') = e(s)L(s,\Pi\times\pi') = \tilde{e}(1-s)L(1-s,\widetilde{\Pi}\times\widetilde{\pi}') = I(s;V_\xi,\varphi')$$

for all φ' in all irreducible subrepresentations π' of $\mathrm{GL}_m(\mathbb{A})$, in the sense of analytic continuation. Then an application of the Phragmen–Lindelöf principle implies that these functions are bounded in vertical strips of finite width. This concludes our use of the L-function.

We now rewrite our integrals $I(s;U_\xi,\varphi')$ and $I(s;V_\xi,\varphi')$ as follows. We first stratify $\mathrm{GL}_m(\mathbb{A})$. For each $a\in\mathbb{A}^\times$ let $\mathrm{GL}^a_m(\mathbb{A}) = \{g\in\mathrm{GL}_m(\mathbb{A})\mid\det(g)=a\}$. We can, and will, always take $\mathrm{GL}^a_m(\mathbb{A}) = \mathrm{SL}_m(\mathbb{A})\cdot\begin{pmatrix} a & \\ & I_{m-1}\end{pmatrix}$. Let

$$\langle\mathbb{P}^n_m U_\xi,\varphi'\rangle_a = \int_{\mathrm{SL}_m(k)\backslash\,\mathrm{GL}^a_m(\mathbb{A})} \mathbb{P}^n_m U_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix} \varphi'(h)\ dh$$

and similarly for $\langle \mathbb{P}^n_m V_\xi, \varphi' \rangle_a$. These are both absolutely convergent for all a and define continuous functions of a on $k^\times \backslash \mathbb{A}^\times$. We now have that $I(s; U_\xi, \varphi')$ is the Mellin transform of $\langle \mathbb{P}^n_m U_\xi, \varphi' \rangle_a$,

$$I(s; U_\xi, \varphi') = \int_{k^\times \backslash \mathbb{A}^\times} \langle \mathbb{P}^n_m U_\xi, \varphi' \rangle_a \ |a|^{s-1/2} \ d^\times a,$$

similarly for $I(s; V_\xi, \varphi')$, and that these two Mellin transforms are equal in the sense of analytic continuation. By Mellin inversion as in Lemma 11.3.1 of Jacquet-Langlands [39], we have that $\langle \mathbb{P}^n_m U_\xi, \varphi' \rangle_a = \langle \mathbb{P}^n_m V_\xi, \varphi' \rangle_a$ for all a, and in particular for $a = 1$. Since this is true for all φ' in all irreducible subrepresentations of automorphic forms on $\mathrm{GL}_m(\mathbb{A})$, then by the weak form of Langlands' spectral theory for SL_m we may conclude that $\mathbb{P}^n_m U_\xi = \mathbb{P}^n_m V_\xi$ as functions on $\mathrm{P}_{m+1}(\mathbb{A})$. More specifically, we have the following result.

Proposition 5.1. Let Π be an irreducible admissible representation of $GL_n(\mathbb{A})$ as above. Suppose that $L(s, \Pi \times \pi')$ is nice for all $\pi' \in \mathcal{T}(m)$. Then for each $\xi \in V_\Pi$ we have $\mathbb{P}^n_m U_\xi(I_{m+1}) = \mathbb{P}^n_m V_\xi(I_{m+1})$.

All of our Converse Theorems take Proposition 5.1 as their starting point. The first part of Theorem 5.1 follows almost immediately. In all others we must add local conditions to compensate for the fact that we do not have the full family of twists from Theorem 5.1 and then work around them.

5.3. Remarks on the proof of Theorem 5.1

Let us first look at the proof of Theorem 5.1. Details can be found in [9] and [7].

5.3.1. The case of S empty

We now assume that Π is as above and that $L(s, \Pi \times \pi')$ is nice for all $\pi' \in \mathcal{T}(n-1)$. Then we have that for all $\xi \in V_\Pi$, $\mathbb{P}^n_{n-1} U_\xi(I_n) = \mathbb{P}^n_{n-1} V_\xi(I_n)$. But for $m = n-1$ the projection operator $\mathbb{P}^n_{n-1}$ is nothing more than restriction to P_n. Hence we have $U_\xi(I_n) = V_\xi(I_n)$ for all $\xi \in V_\Pi$. Then for each $g \in \mathrm{GL}_n(\mathbb{A})$, we have $U_\xi(g) = U_{\Pi(g)\xi}(I_n) = V_{\Pi(g)\xi}(I_n) = V_\xi(g)$. So the map $\xi \mapsto U_\xi(g)$ gives our embedding of Π into the space of automorphic forms on $\mathrm{GL}_n(\mathbb{A})$, since now U_ξ is left invariant under $\mathrm{P}(k)$, $\mathrm{Q}(k)$, and hence all of $\mathrm{GL}_n(k)$. Since we still have

$$U_\xi(g) = \sum_{\mathrm{N}_n(k) \backslash \mathrm{P}(k)} W_\xi(pg)$$

we can compute that U_ξ is cuspidal along any parabolic subgroup of GL_n. Hence Π embeds in the space of cusp forms on $\mathrm{GL}_n(\mathbb{A})$ as desired.

5.3.2. The case of non-empty S

Now let S be a non-empty set of finite places of k. Since we are only twisting by representations which are unramified at places in S, we will only be able to prove the equality $U_\xi(g) = V_\xi(g)$ for a restricted set of ξ and only on a subset of $\mathrm{GL}_n(\mathbb{A})$. Since we have not twisted by all of $\mathcal{T}(n-1)$ we are not in a position to apply Proposition 5.1. To be able to apply that, we will now have to place local conditions at all $v \in S$.

Let $v \in S$. Let $\mathrm{K}_v = \mathrm{GL}_n(\mathfrak{o}_v)$ be the standard maximal compact subgroup of $\mathrm{GL}_n(k_v)$. Let $\mathfrak{p}_v \subset \mathfrak{o}_v$ be the unique prime ideal of $\mathfrak{o}_v$ and for each integer $m_v \geq 0$ set

$$\mathrm{K}_{0,v}(\mathfrak{p}_v^{m_v}) = \left\{ g \in \mathrm{GL}_n(\mathfrak{o}_v) \middle| g \equiv \begin{pmatrix} & & & * \\ & * & & \vdots \\ & & & * \\ 0 & \cdots & 0 & * \end{pmatrix} \pmod{\mathfrak{p}^{m_v}} \right\}$$

and $\mathrm{K}_{1,v}(\mathfrak{p}_v^{m_v}) = \{ g \in \mathrm{K}_{0,v}(\mathfrak{p}_v^{m_v}) \mid g_{n,n} \equiv 1 \pmod{\mathfrak{p}_v^{m_v}})\}$. Note that for $m_v = 0$ we have $\mathrm{K}_{1,v}(\mathfrak{p}_v^0) = \mathrm{K}_{0,v}(\mathfrak{p}_v^0) = \mathrm{K}_v$. Then, as noted at the end of Section 3.1.2, for each local component Π_v with $v \in S$ there is a unique integer $m_v \geq 0$ such that the dimension of the space of $\mathrm{K}_{1,v}(\mathfrak{p}_v^{m_v})$–fixed vectors in Π_v is exactly one. For every place $v \in S$ we choose a vector ξ_v° such that ξ_v° is invariant under the compact open subgroup $\mathrm{K}_1(\mathfrak{p}_v^{m_v})$ for this value of m_v. This vector will necessarily transform by the character ω_{Π_v} under the action of $\mathrm{K}_0(\mathfrak{p}_v^{m_v})$.

As is standard, we will let $\mathrm{G}_S = \prod_{v \in S} \mathrm{GL}_n(k_v)$, $\mathrm{G}^S = \prod_{v \notin S} \mathrm{GL}_n(k_v)$, $\Pi_S = \otimes_{v \in S} \Pi_v$, $\Pi^S = \otimes'_{v \notin S} \Pi_v$, etc.. Let $\mathrm{K}_{0,S}(\mathfrak{n}) = \prod_{v \in S} K_{0,v}(\mathfrak{p}_v^{m_v}) \subset \mathrm{G}_S$ where $\mathfrak{n} = \prod_{v \in S} \mathfrak{p}_v^{m_v}$. Let $\xi_S^\circ = \otimes_{v \in S} \xi_v^\circ \in V_{\Pi_S}$. Let ξ^S be any vector in V_{Π^S}. Then for ξ of the form $\xi = \xi_S^\circ \otimes \xi^S$ the functions $U_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix}$ and $V_\xi \begin{pmatrix} h & \\ & 1 \end{pmatrix}$ are unramified at the places $v \in S$, so that the integrals $I(s; U_\xi, \varphi')$ and $I(s; V_\xi, \varphi')$ vanish unless $\varphi'(h)$ is also unramified at those places in S. In particular, if $\pi' \in \mathcal{T}(n-1)$ but $\pi' \notin \mathcal{T}^S(n-1)$ these integrals will vanish for all $\varphi' \in V_{\pi'}$. So now, for this fixed class of ξ we actually have $I(s; U_\xi, \varphi') = I(s; V_\xi, \varphi')$ for all $\varphi' \in V_{\pi'}$ for all $\pi' \in \mathcal{T}(n-1)$. Hence, as in Proposition 5.1, $U_\xi(I_n) = V_\xi(I_n)$ for all such ξ. Then the previous argument now lets us conclude that $U_\xi(g) = V_\xi(g)$ for all $g \in \mathrm{K}_{0,S}(\mathfrak{n})\,\mathrm{G}^S$.

Let $\mathrm{P}_0(\mathfrak{n}) = \mathrm{P}(k) \cap \mathrm{K}_{0,S}(\mathfrak{n})\,\mathrm{G}^S$, which in fact is simply $\mathrm{P}(k)$, and $\mathrm{Q}_0(\mathfrak{n}) = \mathrm{Q}(k) \cap \mathrm{K}_{0,S}(\mathfrak{n})\,\mathrm{G}^S$. Then a simple matrix computation shows that $\mathrm{P}_0(\mathfrak{n})$ and $\mathrm{Q}_0(\mathfrak{n})$ generate the congruence type subgroup $\Gamma_0(\mathfrak{n}) = \mathrm{GL}_n(k) \cap \mathrm{K}_{0,S}(\mathfrak{n})\,\mathrm{G}^S$ of $\mathrm{G}' = \mathrm{K}_{0,S}(\mathfrak{n})\,\mathrm{G}^S$. Hence the mapping $\xi^S \mapsto U_{\xi_S^\circ \otimes \xi^S}(g)$ embeds V_{Π^S} into the space of automorphic forms $\mathcal{A}(\Gamma_0(\mathfrak{n}) \backslash \mathrm{G}'; \omega_\Pi)$ as a representation of G'. Since by approximation $\mathrm{GL}_n(\mathbb{A}) = \mathrm{GL}_n(k)\,\mathrm{G}'$ and $\Gamma_0(\mathfrak{n}) = \mathrm{GL}_n(k) \cap \mathrm{G}'$ we see that $\mathcal{A}(\mathrm{GL}_n(k) \backslash \mathrm{GL}_n(\mathbb{A}); \omega_\Pi) = \mathcal{A}(\Gamma_0(\mathfrak{n}) \backslash \mathrm{G}'; \omega_\Pi)$ so that Π^S determines an automorphic representation Π_1 of $\mathrm{GL}_n(\mathbb{A})$. Then by construction, $\Pi_{1,v} \simeq \Pi_v$ for all $v \notin S$. For our Π' of the theorem we now take any irreducible constituent of Π_1.

5.4. Remarks on the proof of Theorem 5.2

Details for this section can be found in [12].

5.4.1. The case of S empty

Now suppose that $n \geq 3$, and that $L(s, \Pi \times \pi')$ is nice for all $\pi' \in \mathcal{T}(n-2)$. Then from Proposition 5.1 we may conclude that $\mathbb{P}_{n-2}^n U_\xi(I_{n-1}) = \mathbb{P}_{n-2}^n V_\xi(I_{n-1})$ for all $\xi \in V_\Pi$. Since the projection operator $\mathbb{P}_{n-2}^n$ now involves a non-trivial integration over $k^{n-1} \backslash \mathbb{A}^{n-1}$ we can no longer argue as in the proof of Theorem 5.1. To get to that point we will have to impose a local condition on the vector ξ at one place.

Before we place our local condition, let us write $F_\xi = U_\xi - V_\xi$. Then F_ξ is rapidly decreasing as a function on P_{n-1}. We have $\mathbb{P}_{n-2}^n F_\xi(I_{n-1}) = 0$ and we would like to have simply that $F_\xi(I_n) = 0$. Let $u = (u_1, \dots, u_{n-1}) \in \mathbb{A}^{n-1}$ and consider the function

$$f_\xi(u) = F_\xi \begin{pmatrix} I_{n-1} & {}^t u \\ & 1 \end{pmatrix}.$$

Now $f_\xi(u)$ is a function on $k^{n-1}\backslash\mathbb{A}^{n-1}$ and as such has a Fourier expansion

$$f_\xi(u) = \sum_{\alpha \in k^{n-1}} \hat{f}_\xi(\alpha)\psi_\alpha(u)$$

where $\psi_\alpha(u) = \psi(\alpha \cdot {}^t u)$ and

$$\hat{f}_\xi(\alpha) = \int_{k^{n-1}\backslash\mathbb{A}^{n-1}} f_\xi(u)\psi_{-\alpha}(u)\, du.$$

In this language, the statement $\mathbb{P}^n_{n-2}F_\xi(I_{n-1}) = 0$ becomes $\hat{f}_\xi(e_{n-1}) = 0$, where as always, e_k is the standard unit vector with 0's in all places except the k^{th} where there is a 1.

Note that $F_\xi(g) = U_\xi(g) - V_\xi(g)$ is left invariant under $(\mathrm{Z}(k)\,\mathrm{P}(k)) \cap (\mathrm{Z}(k)\,\mathrm{Q}(k))$ where $\mathrm{Q} = \mathrm{Q}_{n-2}$. This contains the subgroup

$$\mathrm{R}(k) = \left\{ r = \begin{pmatrix} I_{n-2} & & \\ \alpha' & \alpha_{n-1} & \alpha_n \\ & & 1 \end{pmatrix} \big| \alpha' \in k^{n-2}, \alpha_{n-1} \neq 0 \right\}.$$

Using this invariance of F_ξ, it is now elementary to compute that, with this notation, $\hat{f}_{\Pi(r)\xi}(e_{n-1}) = \hat{f}_\xi(\alpha)$ where $\alpha = (\alpha', \alpha_{n-1}) \in k^{n-1}$. Since $\hat{f}_\xi(e_{n-1}) = 0$ for all ξ, and in particular for $\Pi(r)\xi$, we see that for every ξ we have $\hat{f}_\xi(\alpha) = 0$ whenever $\alpha_{n-1} \neq 0$. Thus

$$f_\xi(u) = \sum_{\alpha \in k^{n-1}} \hat{f}_\xi(\alpha)\psi_\alpha(u) = \sum_{\alpha' \in k^{n-2}} \hat{f}_\xi(\alpha', 0)\psi_{(\alpha',0)}(u).$$

Hence $f_\xi(0, \dots, 0, u_{n-1}) = \sum_{\alpha' \in k^{n-2}} \hat{f}_\xi(\alpha', 0)$ is constant as a function of u_{n-1}. Moreover, this constant is $f_\xi(0) = F_\xi(I_n)$, which we want to be 0. This is what our local condition will guarantee.

If v is a finite place of k, let $\mathfrak{o}_v$ denote the ring of integers of k_v, and let $\mathfrak{p}_v$ denote the prime ideal of $\mathfrak{o}_v$. We may assume that we have chosen v so that the local additive character ψ_v is normalized, i.e., that ψ_v is trivial on $\mathfrak{o}_v$ and non-trivial on $\mathfrak{p}_v^{-1}$. Given an integer $n_v \geq 1$ we consider the open compact group

$$\begin{aligned} \mathrm{K}_{00,v}(\mathfrak{p}_v^{n_v}) = \{g = (g_{i,j}) \in \mathrm{GL}_n(\mathfrak{o}_v)\ |\ &(i)\ g_{i,n-1} \in \mathfrak{p}_v^{n_v} \text{ for } 1 \leq i \leq n-2; \\ &(ii)\ g_{n,j} \in \mathfrak{p}_v^{n_v} \text{ for } 1 \leq j \leq n-2; \\ &(iii)\ g_{n,n-1} \in \mathfrak{p}_v^{2n_v}\}. \end{aligned}$$

(As usual, $g_{i,j}$ represents the entry of g in the i-th row and j-th column.)

Lemma 5.1. Let v be a finite place of k as above and let (Π_v, V_{Π_v}) be an irreducible admissible generic representation of $\mathrm{GL}_n(k_v)$. Then there is a vector $\xi_v' \in V_{\Pi_v}$ and a non-negative integer n_v such that

1. for any $g \in \mathrm{K}_{00,v}(\mathfrak{p}_v^{n_v})$ we have $\Pi_v(g)\xi_v' = \omega_{\Pi_v}(g_{n,n})\xi_v'$
2. $\displaystyle\int_{\mathfrak{p}_v^{-1}} \Pi_v \begin{pmatrix} I_{n-2} & & \\ & 1 & u \\ & & 1 \end{pmatrix} \xi_v'\, du = 0.$

The proof of this Lemma is simply an exercise in the Kirillov model of Π_v and can be found in [12].

If we now fix such a place v_0 and assume that our vector ξ is chosen so that $\xi_{v_0} = \xi'_{v_0}$, then we have

$$\begin{aligned} F_\xi(I_n) = f_\xi(0) &= \mathrm{Vol}(\mathfrak{p}_{v_0}^{-1})^{-1} \int_{\mathfrak{p}_{v_0}^{-1}} f_\xi(0,\dots,0,u_{v_0})\, du_{v_0} \\ &= \mathrm{Vol}(\mathfrak{p}_{v_0}^{-1})^{-1} \int_{\mathfrak{p}_{v_0}^{-1}} F_\xi \begin{pmatrix} I_{n-2} & & \\ & 1 & u_{v_0} \\ & & 1 \end{pmatrix} du_{v_0} = 0 \end{aligned}$$

for such ξ.

Hence we now have $U_\xi(I_n) = V_\xi(I_n)$ for all $\xi \in V_\Pi$ such that $\xi_{v_0} = \xi'_{v_0}$ at our fixed place. If we let $\mathrm{G}' = \mathrm{K}_{00,v_0}(\mathfrak{p}_{v_0}^{n_{v_0}})\,\mathrm{G}^{v_0}$, where we set $\mathrm{G}^{v_0} = \prod'_{v\neq v_0} \mathrm{GL}_n(k_v)$, then we have this group preserves the local component ξ'_{v_0} up to a constant factor so that for $g \in \mathrm{G}'$ we have $U_\xi(g) = U_{\Pi(g)\xi}(I_n) = V_{\Pi(g)\xi}(I_n) = V_\xi(g)$.

We now use a fact about generation of congruence type subgroups. Let $\Gamma_1 = (\mathrm{P}(k)\,\mathrm{Z}(k)) \cap \mathrm{G}'$, $\Gamma_2 = (\mathrm{Q}(k)\,\mathrm{Z}(k)) \cap \mathrm{G}'$, and $\Gamma = \mathrm{GL}_n(k) \cap \mathrm{G}'$. Then $U_\xi(g)$ is left invariant under Γ_1 and $V_\xi(g)$ is left invariant under Γ_2. It is essentially a matrix calculation that together Γ_1 and Γ_2 generate Γ. So, as a function on G', $U_\xi(g) = V_\xi(g)$ is left invariant under Γ. So if we let $\Pi^{v_0} = \otimes'_{v\neq v_0}\Pi_v$ then the map $\xi^{v_0} \mapsto U_{\xi'_{v_o}\otimes\xi^{v_0}}(g)$ embeds $V_{\Pi^{v_0}}$ into $\mathcal{A}(\Gamma\backslash \mathrm{G}')$, the space of automorphic forms on G' relative to Γ. Now, by weak approximation, $\mathrm{GL}_n(\mathbb{A}) = \mathrm{GL}_n(k)\cdot \mathrm{G}'$ and $\Gamma = \mathrm{GL}_n(k) \cap \mathrm{G}'$, so we can extend Π^{v_0} to an automorphic representation of $\mathrm{GL}_n(\mathbb{A})$. Let Π_0 be an irreducible component of the extended representation. Then Π_0 is automorphic and coincides with Π at all places except possible v_0.

One now repeats the entire argument using a second place $v_1 \neq v_0$. Then we have two automorphic representations Π_1 and Π_0 of $\mathrm{GL}_n(\mathbb{A})$ which agree at all places except possibly v_0 and v_1. By the generalized Strong Multiplicity One for GL_n we know that Π_0 and Π_1 are both constituents of the same induced representation $\Xi = \mathrm{Ind}(\sigma_1 \otimes \cdots \otimes \sigma_r)$ where each σ_i is a cuspidal representation of some $\mathrm{GL}_{m_i}(\mathbb{A})$, each $m_i \geq 1$ and $\sum m_i = n$. We can write each $\sigma_i = \sigma_i^\circ \otimes |\det|^{t_i}$ with σ_i° *unitary* cuspidal and $t_i \in \mathbb{R}$ and assume $t_1 \geq \cdots \geq t_r$. If $r > 1$, then either $m_1 \leq n-2$ or $m_r \leq n-2$ (or both). For simplicity assume $m_r \leq n-2$. Let S be a finite set of places containing all archimedean places, v_0, v_1, S_Π, and S_{σ_i} for each i. Taking $\pi' = \widetilde{\sigma}_r \in \mathcal{T}(n-2)$, we have the equality of partial L-functions

$$\begin{aligned} L^S(s,\Pi\times\pi') &= L^S(s,\Pi_0\times\pi') = L^S(s,\Pi_1\times\pi') \\ &= \prod_i L^S(s,\sigma_i\times\pi') = \prod_i L^S(s+t_i-t_r,\sigma_i^\circ\times\widetilde{\sigma}_r^\circ). \end{aligned}$$

Now $L^S(s,\sigma_r^\circ\times\widetilde{\sigma}_r^\circ)$ has a pole at $s=1$ and all other terms are non-vanishing at $s=1$. Hence $L(s,\Pi\times\pi')$ has a pole at $s=1$ contradicting the fact that $L(s,\Pi\times\pi')$ is nice. If $m_1 \leq n-2$, then we can make a similar argument using $L(s,\widetilde{\Pi}\times\sigma_1)$. So in fact we must have $r=1$ and $\Pi_0 = \Pi_1 = \Xi$ is cuspidal. Since Π_0 agrees with Π at v_1 and Π_1 agrees with Π at v_0 we see that in fact $\Pi = \Pi_0 = \Pi_1$ and Π is indeed cuspidal automorphic.

5.4.2. The case of non-empty S

Let S be our non-empty set of finite places of k. Since we have restricted our ramification at the places in S, we no longer know that $L(s,\Pi\times\pi')$ is nice for all $\pi'\in\mathcal{T}(n-2)$ and so Proposition 5.1 above is not immediately applicable. In this case, for each place $v\in S$ we fix a vector $\xi'_v\in V_{\Pi_v}$ as in the above Lemma. (So we must assume we have chosen ψ so it is unramified at the places in S.) Let $\xi'_S=\prod_{v\in S}\xi'_v\in\Pi_S$. Consider now only vectors ξ of the form $\xi^S\otimes\xi'_S$ with ξ^S arbitrary in V_{Π^S} and ξ'_S fixed. For these vectors, the functions $\mathbb{P}^n_{n-2}U_\xi\begin{pmatrix}h&\\&1\end{pmatrix}$ and $\mathbb{P}^n_{n-2}V_\xi\begin{pmatrix}h&\\&1\end{pmatrix}$ are unramified at the places $v\in S$, so that the integrals $I(s;U_\xi,\varphi')$ and $I(s;V_\xi,\varphi')$ vanish unless $\varphi'(h)$ is also unramified at those places in S. In particular, if $\pi'\in\mathcal{T}(n-2)$ but $\pi'\notin\mathcal{T}^S(n-2)$ these integrals will vanish for all $\varphi'\in V_{\pi'}$. So now, for this fixed class of ξ we actually have $I(s;U_\xi,\varphi')=I(s;V_\xi,\varphi')$ for all $\varphi'\in V_{\pi'}$ for all $\pi'\in\mathcal{T}(n-2)$. Hence, as before, $\mathbb{P}^n_{n-2}U_\xi(I_{n-1})=\mathbb{P}^n_{n-2}V_\xi(I_{n-1})$ for all such ξ.

Now we proceed as before. Our Fourier expansion argument is a bit more subtle since we have to work around our local conditions, which now have been imposed before this step, but we do obtain that $U_\xi(g)=V_\xi(g)$ for all $g\in\mathrm{G}'=(\prod_{v\in S}\mathrm{K}_{00,v}(\mathfrak{p}_v^{n_v}))\,\mathrm{G}^S$. The generation of congruence subgroups goes as before. We then use weak approximation as above, but then take for Π' any constituent of the extension of Π^S to an automorphic representation of $\mathrm{GL}_n(\mathbb{A})$.There no use of strong multiplicity one nor any further use of the L-function in this case. More details can be found in [12].

5.5. Remarks on the proof of Theorem 5.3

Let us now sketch the proof of Theorem 5.3. Details can be found in [9].

We fix a non-empty finite set of places S, containing all archimedean places, such that the ring $\mathfrak{o}_S$ of S-integer has class number one. Recall that we are now twisting by all cuspidal representations $\pi'\in\mathcal{T}_S(n-1)$, that is, π' which are unramified at all places $v\notin S$. Since we have not twisted by all of $\mathcal{T}(n-1)$ we are not in a position to apply Proposition 5.1. To be able to apply that, we will now have to place local conditions *at all* $v\notin S$.

We begin by recalling the definition of the conductor of a representation Π_v of $\mathrm{GL}_n(k_v)$ and the conductor (or level) of Π itself. Let $\mathrm{K}_v=\mathrm{GL}_n(\mathfrak{o}_v)$ be the standard maximal compact subgroup of $\mathrm{GL}_n(k_v)$. Let $\mathfrak{p}_v\subset\mathfrak{o}_v$ be the unique prime ideal of $\mathfrak{o}_v$ and for each integer $m_v\geq 0$ recall that

$$\mathrm{K}_{0,v}(\mathfrak{p}_v^{m_v})=\left\{g\in\mathrm{GL}_n(\mathfrak{o}_v)\,\middle|\,g\equiv\begin{pmatrix}&&*\\&*&\vdots\\&&*\\0&\cdots\;0&*\end{pmatrix}\pmod{\mathfrak{p}^{m_v}}\right\}$$

and $\mathrm{K}_{1,v}(\mathfrak{p}_v^{m_v})=\{g\in\mathrm{K}_{0,v}(\mathfrak{p}_v^{m_v})\mid g_{n,n}\equiv 1 \pmod{\mathfrak{p}_v^{m_v}})\}$. Note that for $m_v=0$ we have $\mathrm{K}_{1,v}(\mathfrak{p}_v^0)=\mathrm{K}_{0,v}(\mathfrak{p}_v^0)=\mathrm{K}_v$. Then for each local component Π_v of Π there is a unique integer $m_v\geq 0$ such that the space of $\mathrm{K}_{1,v}(\mathfrak{p}_v^{m_v})$–fixed vectors in Π_v is exactly one. For almost all v, $m_v=0$. We take the ideal $\mathfrak{p}_v^{m_v}=\mathfrak{f}(\Pi_v)$ as the conductor of Π_v. Then the ideal $\mathfrak{n}=\mathfrak{f}(\Pi)=\prod_v\mathfrak{p}_v^{m_v}\subset\mathfrak{o}$ is called the conductor of

Π. For each place v we fix a non-zero vector $\xi_v^\circ \in \Pi_v$ which is fixed by $\mathrm{K}_{1,v}(\mathfrak{p}_v^{m_v})$, which at the unramified places is taken to be the vector with respect to which the restricted tensor product $\Pi = \otimes' \Pi_v$ is taken. Note that for $g \in \mathrm{K}_{0,v}(\mathfrak{p}_v^{m_v})$ we have $\Pi_v(g)\xi_v^\circ = \omega_{\Pi_v}(g_{n,n})\xi_v^\circ$.

Now fix a non-empty finite set of places S, containing the archimedean places if there are any. Then the compact subring $\mathfrak{n}^S = \prod_{v \notin S} \mathfrak{p}_v^{m_v} \subset k^S$, or the ideal it determines $\mathfrak{n}_S = k \cap k_S \mathfrak{n}^S \subset \mathfrak{o}_S$, is called the S–conductor of Π. Let $\mathrm{K}_1^S(\mathfrak{n}) = \prod_{v \notin S} \mathrm{K}_{1,v}(\mathfrak{p}_v^{m_v})$ and similarly for $\mathrm{K}_0^S(\mathfrak{n})$. Let $\xi^\circ = \otimes_{v \notin S} \xi_v^\circ \in \Pi^S$. Then this vector is fixed by $\mathrm{K}_1^S(\mathfrak{n})$ and transforms by a character under $\mathrm{K}_0^S(\mathfrak{n})$. In particular, since $\prod_{v \notin S} \mathrm{GL}_{n-1}(\mathfrak{o}_v)$ embeds in $\mathrm{K}_1^S(\mathfrak{n})$ via $h \mapsto \begin{pmatrix} h & \\ & 1 \end{pmatrix}$ we see that when we restrict Π^S to GL_{n-1} the vector ξ° is *unramified.*

Now let us return to the proof of Theorem 5.3 and in particular the version of Proposition 5.1 we can salvage. For every vector $\xi_S \in \Pi_S$ consider the functions $U_{\xi_S \otimes \xi^\circ}$ and $V_{\xi_S \otimes \xi^\circ}$. When we restrict these functions to GL_{n-1} they become unramified for all places $v \notin S$. Hence we see that the integrals $I(s; U_{\xi_S \otimes \xi^\circ}, \varphi')$ and $I(s; V_{\xi_S \otimes \xi^\circ}, \varphi')$ vanish identically if the function $\varphi' \in V_{\pi'}$ is not unramified for $v \notin S$, and in particular if $\varphi' \in V_{\pi'}$ for $\pi' \in \mathcal{T}(n-1)$ but $\pi' \notin \mathcal{T}_S(n-1)$. Hence, for vectors of the form $\xi = \xi_S \otimes \xi^\circ$ we do indeed have that $I(s; U_{\xi_S \otimes \xi^\circ}, \varphi') = I(s; V_{\xi_S \otimes \xi^\circ}, \varphi')$ for all $\varphi' \in V_{\pi'}$ and all $\pi' \in \mathcal{T}(n-1)$. Hence, as in Proposition 5.1 we may conclude that $U_{\xi_S \otimes \xi^\circ}(I_n) = V_{\xi_S \otimes \xi^\circ}(I_n)$ for all $\xi_S \in V_{\Pi_S}$. Moreover, since ξ_S was arbitrary in V_{Π_S} and the fixed vector ξ° transforms by a character of $\mathrm{K}_0^S(\mathfrak{n})$ we may conclude that $U_{\xi_S \otimes \xi^\circ}(g) = V_{\xi_S \otimes \xi^\circ}(g)$ for all $\xi_S \in V_{\Pi_S}$ and all $g \in \mathrm{G}_S \, \mathrm{K}_0^S(\mathfrak{n})$.

What invariance properties of the function $U_{\xi_S \otimes \xi^\circ}$ have we gained from our equality with $V_{\xi_S \otimes \xi^\circ}$. Let us let $\Gamma_i(\mathfrak{n}_S) = \mathrm{GL}_n(k) \cap \mathrm{G}_S \, \mathrm{K}_i^S(\mathfrak{n})$ which we may view naturally as congruence subgroups of $\mathrm{GL}_n(\mathfrak{o}_S)$. Now, as a function on $\mathrm{G}_S \, \mathrm{K}_0^S(\mathfrak{n})$, $U_{\xi_S \otimes \xi^\circ}(g)$ is naturally left invariant under $\Gamma_{0,\mathrm{P}}(\mathfrak{n}_S) = \mathrm{Z}(k) \, \mathrm{P}(k) \cap \mathrm{G}_S \, \mathrm{K}_0^S(\mathfrak{n})$ while $V_{\xi_S \otimes \xi^\circ}(g)$ is naturally left invariant under $\Gamma_{0,\mathrm{Q}}(\mathfrak{n}_S) = \mathrm{Z}(k) \, \mathrm{Q}(k) \cap \mathrm{G}_S \, \mathrm{K}_0^S(\mathfrak{n})$ where $\mathrm{Q} = \mathrm{Q}_{n-1}$. Similarly we set $\Gamma_{1,\mathrm{P}}(\mathfrak{n}_S) = \mathrm{Z}(k) \, \mathrm{P}(k) \cap \mathrm{G}_S \, \mathrm{K}_1^S(\mathfrak{n})$ and $\Gamma_{1,\mathrm{Q}}(\mathfrak{n}_S) = \mathrm{Z}(k) \, \mathrm{Q}(k) \cap \mathrm{G}_S \, \mathrm{K}_1^S(\mathfrak{n})$. The crucial observation for this Theorem is the following result.

Proposition 5.2. The congruence subgroup $\Gamma_i(\mathfrak{n}_S)$ is generated by the subgroups $\Gamma_{i,\mathrm{P}}(\mathfrak{n}_S)$ and $\Gamma_{i,\mathrm{Q}}(\mathfrak{n}_S)$ for $i = 0, 1$.

This proposition is a consequence of results in the stable algebra of GL_n due to Bass which were crucial to the solution of the congruence subgroup problem for SL_n by Bass, Milnor, and Serre. This is reason for the restriction to $n \geq 3$ in the statement of Theorem 5.3.

From this we get not an embedding of Π into a space of automorphic forms on $\mathrm{GL}_n(\mathbb{A})$, but rather an embedding of Π_S into a space of classical automorphic forms on G_S. To this end, for each $\xi_S \in V_{\Pi_S}$ let us set

$$\Phi_{\xi_S}(g_S) = U_{\xi_S \otimes \xi^\circ}((g_S, 1^S)) = V_{\xi_S \otimes \xi^\circ}((g_S, 1^S))$$

for $g_S \in \mathrm{G}_S$. Then Φ_{ξ_S} will be left invariant under $\Gamma_1(\mathfrak{n}_S)$ and transform by a Nebentypus character χ_S under $\Gamma_0(\mathfrak{n}_S)$ determined by the central character ω_{Π^S} of Π^S. Furthermore, it will transform by a character $\omega_S = \omega_{\Pi_S}$ under the center $\mathrm{Z}(k_S)$ of G_S. The requisite growth properties are satisfied and hence the map

$\xi_S \mapsto \Phi_{\xi_S}$ defines an embedding of Π_S into the space $\mathcal{A}(\Gamma_0(\mathfrak{n}_S)\backslash \mathrm{G}_S;\omega_S,\chi_S)$ of classical automorphic forms on G_S relative to the congruence subgroup $\Gamma_0(\mathfrak{n}_S)$ with Nebentypus χ_S and central character ω_S.

We now need to lift our classical automorphic representation back to an adelic one and hopefully recover the rest of Π. By strong approximation for GL_n and our class number assumption we have the isomorphism between the space of classical automorphic forms $\mathcal{A}(\Gamma_0(\mathfrak{n}_S)\backslash \mathrm{G}_S;\omega_S,\chi_S)$ and the $\mathrm{K}_1^S(\mathfrak{n})$ invariants in the space $\mathcal{A}(\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A});\omega)$ where ω is the central character of Π. Hence Π_S will generate an automorphic subrepresentation of the space of automorphic forms $\mathcal{A}(\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A});\omega)$. To compare this to our original Π, we must check that, in the space of classical forms, the $\Phi_{\xi_S\otimes\xi^\circ}$ are Hecke eigenforms for a classical Hecke algebra and that their Hecke eigenvalues agree with those from Π. We do this only for those $v \notin S$ which are unramified, where it is a rather standard calculation. As we have not talked about Hecke algebras, we refer the reader to [9] for the details.

Now if we let Π' be any irreducible subrepresentation of the representation generated by the image of Π_S in $\mathcal{A}(\mathrm{GL}_n(k)\backslash \mathrm{GL}_n(\mathbb{A});\omega)$, then Π' is automorphic and we have $\Pi'_v \simeq \Pi_v$ for all $v \in S$ by construction and $\Pi'_v \simeq \Pi_v$ for all $v \notin S'$ by the Hecke algebra calculation. Thus we have proven Theorem 5.3.

5.6. A useful variant

For the applications of any of these Converse Theorems to the problem of lifting of automorphic representations to GL_n, which we will take up in the next Lecture, the following simple variant of these theorems is extremely useful [13]. If $\mathcal{T}$ is one of the twisting sets from above and η is a fixed idele class character, we set $\mathcal{T}\otimes\eta = \{\pi' \mid \pi' = \pi'_0\otimes\eta \quad \text{with} \quad \pi'_0 \in \mathcal{T}\}$ where we view η as a character of any GL_m by composition with the determinant.

Observation 5.1. Let Π be as in Theorem 5.1, 5.2, or 5.3. Suppose that η is a fixed character of $k^\times\backslash\mathbb{A}^\times$. Suppose that $L(s,\Pi\times\pi')$ is nice for all $\pi'\in\mathcal{T}\otimes\eta$, where $\mathcal{T}$ is any of the twisting sets of those theorems. Then Π is cuspidal automorphic or quasi-automorphic as in those theorems.

The only thing to observe, say by looking at the local or global integrals, is that if $\pi'_0\in\mathcal{T}$ then $L(s,\Pi\times(\pi'_0\otimes\eta)) = L(s,(\Pi\otimes\eta)\times\pi'_0)$ so that applying the Converse Theorem for Π with twisting set $\mathcal{T}\otimes\eta$ is equivalent to applying the Converse Theorem for $\Pi\otimes\eta$ with the twisting set $\mathcal{T}$. So, by either Theorem 5.1, 5.2, or 5.3, whichever is appropriate, $\Pi\otimes\eta$ is cuspidal automorphic or quasi-automorphic and hence Π is as well.

5.7. Global fields of characteristic $p \neq 0$

When the global field k is of characteristic 0, that is, is a number field, the statements of the Converse Theorems we have given in terms of the analytic properties of the L-functions are the most appropriate and applicable. However when the global field k is of characteristic $p \neq 0$, that is, the field of functions of a curve over a finite field, while the statements we have presented are still true, it is most appropriate and useful to have the Converse Theorems stated in terms of the global L-functions as rational functions, as was done in Piatetski-Shapiro's original paper [65]. While it would take us too far afield to present this in this setting, Lafforgue needed this formulation in his proof of the Global Langlands Correspondence for GL_n in this context [59]. In Appendix B of that paper Lafforgue reformulated our Theorem 5.1 in terms of L-functions as rational functions and showed how to modify our proof above to apply in this context. We recommend Lafforgue's Appendix B to the interested reader.

5.8. Conjectures

What are the optimal statements that one could hope for in a Converse Theorem? At this point in time there seem to be two more or less accepted conjectures [12].

The first is credited to Jacquet. It assumes the least amount of twisting one could hope for and still be able to control the cuspidality of Π. The heuristics for this conjecture can be found in the last section of [12]. Notations are as in Section 5.1.

Conjecture 5.1. Let Π be an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A})$ as above. Let S be a finite set of finite places of k. Suppose that $L(s, \Pi \times \pi')$ is nice for all $\pi' \in \mathcal{T}^S\left(\left[\frac{n}{2}\right]\right)$.

1. If $S = \emptyset$ then Π is a cuspidal automorphic representation.
2. If $S \neq \emptyset$ then Π is quasi-automorphic in the sense that there is an automorphic representation Π' such that $\Pi_v \simeq \Pi'_v$ for all $v \notin S$.

The most ambitious conjecture we know of is due to Piatetski–Shapiro and is explained in [12].

Conjecture 5.2. Let Π be an irreducible admissible representation of $\mathrm{GL}_n(\mathbb{A})$ as above. Suppose that $L(s, \Pi \otimes \omega)$ is nice for all $\omega \in \mathcal{T}(1)$, that is, for all idele class characters ω. Then Π is quasi-automorphic in the sense that there is an automorphic representation Π' such that $\Pi_v \simeq \Pi'_v$ for all finite places of k where both Π and Π' are unramified and such that $L(s, \Pi \otimes \omega) = L(s, \Pi' \otimes \omega)$ and $\varepsilon(s, \Pi \otimes \omega) = \varepsilon(s, \Pi' \otimes \omega)$ for all ω.

This conjecture would have many applications to number theoretic questions. We refer the reader to Taylor's recent ICM talk for a discussion of some of these [93].

LECTURE 6
Converse Theorems and Functoriality

In this section we would like to make some general remarks on how to apply these Converse Theorems to the problem of functorial liftings [3]. Other surveys of this topic can be found in [14, 83].

In order to apply these these theorems, you must be able to control the global properties of the L-function. However, for the most part, the way we have of controlling global L-functions is to associate them to automorphic forms or representations. A minute's thought will then lead one to the conclusion that the primary application of these results will be to the lifting of automorphic representations from some group H to GL_n. This has traditionally been the case, for example in Shimura's original proof of the Shimura correspondence [86], the Doi-Naganuma analysis of quadratic base change for GL_2 [18], and the symmetric square lifting from GL_2 to GL_3 by Gelbart and Jacquet [23]. More explicitly number-theoretic applications then come as consequences of these liftings.

In the recent cases in which the Converse Theorem has been used to establish Functorial liftings, the group H has been split and the field k has been of characteristic zero. To simplify our exposition we will work in this context throughout this lecture. So let k be a number field and H a split connected reductive algebraic group over k.

6.1. Functoriality

Langlands' Principle of Functoriality is a natural philosophy governing the lifting or transfer of automorphic representations, having its origins in viewing the Langlands Conjectures as giving an arithmetic parameterization of local admissible or global automorphic representations.

6.1.1. Langlands Conjectures

Let ${}^L\mathrm{H}$ be the Langlands L-group of H. Since we are assuming that H is split, the Galois structure will play no role and we can simply use the connected component ${}^L\mathrm{H}^0$ as the full L-group without loss of information. This connected component is essentially the complex analytic group determined by the root data which is dual to that of H [3, 6]. The Langlands Conjectures can be viewed as giving an arithmetic parameterization of either the admissible representations of $\mathrm{H}(k_v)$ or the automorphic representations of $\mathrm{H}(\mathbb{A})$ in terms of admissible homomorphisms of the local Weil-Deligne group W'_{k_v} or the conjectural global Langlands group $\mathcal{L}_k$ into the Langlands L-group ${}^L\mathrm{H}$.

We will begin with the Local Langlands Conjecture. So let v be a place of k, k_v the corresponding completion, and W'_{k_v} the associated Weil-Deligne group [92]. If k_v is archimedean, we simply take $W'_{k_v} = W_{k_v}$ to be the Weil group. Following Borel [3, 6] we let $\Phi(\mathrm{H}_v)$ denote the set of admissible homomorphisms $\phi_v : W'_{k_v} \to {}^L\mathrm{H}$ modulo inner automorphisms. In the case $\mathrm{H} = \mathrm{GL}_n$ these are simply the Frobenius semi-simple complex representations of the Weil-Deligne group and for other H they are an appropriate generalization. Let $\mathcal{A}(\mathrm{H}_v) = \mathcal{A}(\mathrm{H}(k_v))$ denote the set of equivalence classes of irreducible admissible complex representations of $\mathrm{H}(k_v)$.

Local Langlands Conjecture: There is a surjective map $\mathcal{A}(\mathrm{H}_v) \to \Phi(\mathrm{H}_v)$ with finite fibres which partitions $\mathcal{A}(\mathrm{H}_v)$ into disjoint finite sets $\mathcal{A}_{\phi_v} = \mathcal{A}_{\phi_v}(\mathrm{H}_v)$ satisfying certain representation-theoretic desiderata.

For the precise nature of the desiderata we refer the reader to Borel [3] or [6]. Since they will play no role in our discussion we will refrain from listing them. The sets $\mathcal{A}_{\phi_v}$ for $\phi_v \in \Phi(\mathrm{H}_v)$ are called local L-packets.

The following general results are known towards this Conjecture for H:

1. If $\mathrm{H} = \mathrm{GL}_n$ then the Local Langlands Conjecture for GL_n in characteristic zero has been completely established by Harris–Taylor [32] and then Henniart [35]. In this case the correspondence is *bijective* and the desiderata are expressed in terms of the matching of L-factors and ε-factors of pairs.

2. If the local field k_v is archimedean, i.e., $k_v = \mathbb{R}$ or $\mathbb{C}$, then it was completely established by Langlands [62].

3. If k_v is non-archimedean one knows how to parameterize the unramified representations of $\mathrm{H}(k_v)$ via the unramified admissible homomorphisms [3]. This is a rephrasing in this language of the Satake classification.

4. If k_v is non-archimedean then Kazhdan and Lusztig have shown how to parameterize those representations of $\mathrm{H}(k_v)$ having an Iwahori fixed vector in terms of admissible homomorphisms of the Weil-Deligne group [50].

Thinking of the Local Langlands Conjecture as providing an arithmetic parameterization of the irreducible admissible representations of $\mathrm{H}(k_v)$, one can define local L-functions associated to arbitrary $\pi_v \in \mathcal{A}(\mathrm{H}_v)$. One needs a second parameter, namely a continuous complex representation $r : {}^L\mathrm{H} \to \mathrm{GL}_n(\mathbb{C})$. Then, for any admissible homomorphism $\phi_v \in \Phi(\mathrm{H}_v)$, the composition $r \circ \phi_v : W'_{k_v} \to \mathrm{GL}_n(\mathbb{C})$ is a continuous complex representation of the Weil-Deligne group and to it we can associate an L-factor $L(s, r\circ\phi_v)$ and ε-factor $\varepsilon(s, r\circ\phi_v, \psi_v)$ for an additive character ψ_v of k [92]. If $\pi_v \in \mathcal{A}_{\phi_v}$ is in the L-packet defined by the admissible homomorphism ϕ_v then we set

$$L(s, \pi_v, r) = L(s, r \circ \phi_v) \qquad \text{and} \qquad \varepsilon(s, \pi_v, r, \psi_v) = \varepsilon(s, r \circ \phi_v, \psi_v).$$

According to this definition, one cannot distinguish between the representations π_v lying in a given L-packet $\mathcal{A}_{\phi_v}$ in terms of their L-functions and ε-factors, hence the terminology. At present these L-functions are well-defined only for those π_v for which the parameterization is known, for example if π_v is unramified.

One would ideally like a statement of a Global Langlands Conjecture or parameterization which is analogous to the local one, but at present there is no natural global version of the Weil-Deligne group in characteristic zero. One can give such a formulation in terms of a conjectural global Langlands group $\mathcal{L}_k$ for a number field k [68]. Not knowing what this should look like, one still expects to have local–global compatibility. If one begins with an irreducible automorphic representation $\pi = \otimes' \pi_v$ of $\mathrm{H}(\mathbb{A})$ then, assuming the Local Langlands Conjecture for each local group $\mathrm{H}(k_v)$, one can attach to π the collection $\{\phi_v\}$ of local parameters $\phi_v = \phi_{\pi_v} : W'_{k_v} \to {}^L\mathrm{H}$ given by the local components π_v. This system of local parameters can often be used as a substitute for a global parameter. This collection of local data is sufficient to define the global L-function and ε-factor attached to π and a representation $r : {}^L\mathrm{H} \to \mathrm{GL}_n(\mathbb{C})$ by

$$L(s, \pi, r) = \prod_v L(s, \pi_v, r) = \prod_v L(s, r \circ \phi_v)$$

and

$$\varepsilon(s, \pi, r) = \prod_v \varepsilon(s, \pi_v, r, \psi_v) = \prod_v \varepsilon(s, r \circ \phi_v, \psi_v)$$

where $\psi = \otimes \psi_v$ is an additive character of $\mathbb{A}$ trivial on k.

6.1.2. Local Functoriality

We are interested in Functoriality from H to $\mathrm{G} = \mathrm{GL}_n$. In general, Functoriality is associated to an L-homomorphism, which in our context is simply a complex analytic homomorphism $u : {}^L\mathrm{H} \to {}^L\mathrm{G} = \mathrm{GL}_n(\mathbb{C})$.

Local Functoriality is very natural if one assumes the Local Langlands Conjecture for $\mathrm{H}(k_v)$. In this case, if π_v is an irreducible admissible representation of $\mathrm{H}(k_v)$ then here is associated a parameter or admissible homomorphism $\phi_v : W'_{k_v} \to {}^L\mathrm{H}$. If we compose this with the L-homomorphism u we obtain a parameter for $\mathrm{GL}_n(k_v)$, namely $\Phi_v = u \circ \phi_v : W'_{k_v} \to {}^L\mathrm{G} = \mathrm{GL}_n(\mathbb{C})$. Since the local Langlands correspondence for GL_n is bijective, this parameter determines a unique irreducible admissible representation Π_v of $\mathrm{GL}_n(k_v)$:

$$\begin{tikzcd}
 & & {}^LH \arrow[rr, "u"] & & {}^LG & & \\
\pi_v \arrow[r, mapsto] & \phi_v & & & & \Phi_v \arrow[r, mapsto] & \Pi_v . \\
 & & & W'_{k_v} \arrow[uul] \arrow[uur] & & &
\end{tikzcd}$$

The representation Π_v is called the *local Langlands lift* or *transfer* of π_v associated to the L-homomorphism u. Note that in terms of local L-functions, we have the equalities

$$L(s, \pi_v, u) = L(s, u \circ \phi_v) = L(s, \Phi_v) = L(s, \Pi_v)$$

and

$$\varepsilon(s, \pi_v, u, \psi_v) = \varepsilon(s, u \circ \phi_v, \psi_v) = \varepsilon(s, \Phi_v, \psi_v) = \varepsilon(s, \Pi_v, \psi_v)$$

as well as equalities for the twisted versions with representations π'_v of $\mathrm{GL}_m(k_v)$.

6.1.3. Global Functoriality

We retain our L-homomorphism $u : {}^L\mathrm{H} \to {}^L\mathrm{G} = \mathrm{GL}_n(\mathbb{C})$. If we begin with a cuspidal representation $\pi = \otimes' \pi_v$ of $\mathrm{H}(\mathbb{A})$ then the global Principle of Functoriality states that to π and u should be associated an automorphic representation Π of $\mathrm{GL}_n(\mathbb{A})$ for which $L(s, \pi, u) = L(s, \Pi)$ among other things.

If we assume the Local Langlands Conjecture for each local group $\mathrm{H}(k_v)$ then this global Functoriality is easy to formulate. We first take π and decompose it into its local components $\pi = \otimes' \pi_v$. For each local representation π_v we apply our local lifting diagram:

$$\begin{tikzcd}
& {}^LH \arrow[rr, "u"] & & {}^LG & \\
\pi_v \arrow[r, mapsto] & \phi_v & & \Phi_v \arrow[r, mapsto] & \Pi_v . \\
& & W'_{k_v} \arrow[uul] \arrow[uur] & &
\end{tikzcd}$$

Then piecing the local representations Π_v together we obtain an irreducible admissible representation $\Pi = \otimes' \Pi_v$ of $\mathrm{GL}_n(\mathbb{A})$. Langlands' Principle of Functoriality then says that Π should be automorphic. Then Π would be *the global Langlands lift* or *transfer* of π associated to the L-homomorphism u. Note that in this case we would have the equality of L- and ε-factors

$$L(s,\pi,u) = \prod_v L(s,\pi_v,u) = \prod_v L(s,\Pi_v) = L(s,\Pi)$$

and

$$\varepsilon(s,\pi,u) = \prod_v \varepsilon(s,\pi_v,u,\psi_v) = \prod_v \varepsilon(s,\Pi_v,\psi_v) = \varepsilon(s,\Pi)$$

as well as equalities for the twisted versions with representations π' of $\mathrm{GL}_m(\mathbb{A})$

$$L(s,\pi\times\pi',u\otimes\iota) = L(s,\Pi\times\pi')$$

for cuspidal automorphic representations of $\mathrm{GL}_m(\mathbb{A})$, where $\iota : \mathrm{GL}_m(\mathbb{C}) \to \mathrm{GL}_m(\mathbb{C})$ is the identity map viewed as an L-homomorphism, and the related equalities of ε–factors.

In general, as noted above, there will be a finite set S of finite places of k for which we do not know the local Langlands conjecture for $\mathrm{H}(k_v)$. So for any finite set of finite places S we will call an automorphic representation Π of $\mathrm{GL}_n(\mathbb{A})$ *a global Langlands lift* of π if for every $v \notin S$ we have that Π_v is the local Langlands lift of π_v. In particular this will imply an equality of partial L–functions

$$L^S(s,\pi,u) = L^S(s,\Pi)$$

as well as the related equalities of ε–factors and twisted versions.

6.2. Functoriality and the Converse Theorem

It should now be clear how one can apply the Converse Theorem to establish liftings or transfers from split connected reductive groups H to an appropriate GL_N associated to an L-homomorphism $u : {}^L\mathrm{H} \to \mathrm{GL}_N(\mathbb{C})$. There are essentially three steps.

1. *Construction of a candidate lift.* We begin with a cuspidal automorphic representation $\pi = \otimes' \pi_v$ of $\mathrm{H}(\mathbb{A})$. Assume that for each place v we can construct an appropriate local lift $\pi_v \mapsto \Pi_v$ associating to π_v an irreducible admissible representation of $\mathrm{GL}_N(k_v)$. If the local Langlands conjecture is known for $\mathrm{H}(k_v)$ we take Π_v to be the local Langlands lift of π_v as defined above. At the remaining places, if any, the existence of a local lift is a problem that will be addressed below. Putting these local lifts together we obtain a candidate lift $\Pi = \otimes'\Pi_v$ which is an irreducible admissible representation of $\mathrm{GL}_N(\mathbb{A})$ as in the statement of the Converse Theorems. One must take care that for each $\pi' \in \mathcal{T}$ for an appropriate twisting set $\mathcal{T}$ we have equalities

$$L(s, \pi \times \pi', u \otimes \iota) = L(s, \Pi \times \pi')$$

and

$$\varepsilon(s, \pi \times \pi', u \otimes \iota) = \varepsilon(s, \Pi \times \pi').$$

2. *Control the analytic properties of the twisted L-functions for* H. In our examples this will be done using the Langlands–Shahidi method as explained in Shahidi's article in this volume [84]. To apply the Langlands–Shahidi method we must at present assume that k is a number field, as we have, and that the cuspidal representation π is *globally generic* [84]. Then we need to know that for an appropriate twisting set $\mathcal{T}$ the twisted L-functions $L(s, \pi \times \pi', u \otimes \iota)$ are *nice*, ie, for π' in an appropriate twisting set $\mathcal{T}$ we need

1. $L(s, \pi \times \pi', u \otimes \iota)$ and $L(s, \widetilde{\pi} \times \widetilde{\pi}', u \otimes \iota)$ have analytic continuations to entire functions of s,
2. these entire continuations are bounded in vertical strips of finite width, nd
3. they satisfy the standard functional equation

$$L(s, \pi \times \pi', u \otimes \iota) = \varepsilon(s, \pi \times \pi', u \otimes \iota) L(1 - s, \widetilde{\pi} \times \widetilde{\pi}', u \otimes \iota).$$

The functional equation (3) is known in wide generality [80,84]. The boundedness in vertical strips (2) is likewise known [25, 84]. After a moments thought one realizes that the entirety (1) will not be true in general, since certain cuspidal π of $\mathrm{H}(\mathbb{A})$ are expected to lift to non-cuspidal Π on $\mathrm{GL}_N(\mathbb{A})$ and hence the twisted L-functions $L(s, \Pi \times \pi')$ need not be entire. This is a difficulty that we will also address below.

3. *Application of the appropriate Converse Theorem.* In all the examples in which we have been able to carry out this program, the Converse Theorem that is used is either Theorem 5.1 or 5.2 in conjunction with Observation 5.1 of Section 5.6. The use of Observation 5.1 with η a sufficiently highly ramified idele class character will be used to solve the local and global difficulties remaining in steps 1 and 2. Once we apply the Converse Theorem we can conclude that there is a automorphic representation $\Pi' = \otimes'\Pi'_v$ of $\mathrm{GL}_N(\mathbb{A})$ such that $\Pi'_v = \Pi_v$ is the local Langlands lift of π_v for all v outside a finite set of places S, i.e, Π' is a global Langlands lift of π with respect to the L-homomorphism u. Thus the global Langlands Functoriality from H to GL_n associated to the L-homomorphism u is established.

6.3. Statement of Results

We left two problems open in the sketch above: (i) the lack of knowledge of the Local Langlands Conjecture at certain places for $\mathrm{H}(k_v)$, and hence the lack of a natural local lift, and (ii) the possibility of global poles for $L(s, \pi \times \pi', u \otimes \iota)$. We have been able to overcome these difficulties, as will be discussed below, and this method has been applied in the following cases:

H	${}^L\mathrm{H}$	$u : {}^L\mathrm{H} \to {}^L\mathrm{G}$	${}^L\mathrm{G}$	G
SO_{2n+1}	$\mathrm{Sp}_{2n}(\mathbb{C})$	$\hookrightarrow$	$\mathrm{GL}_{2n}(\mathbb{C})$	GL_{2n}
SO_{2n}	$\mathrm{SO}_{2n}(\mathbb{C})$	$\hookrightarrow$	$\mathrm{GL}_{2n}(\mathbb{C})$	GL_{2n}
Sp_{2n}	$\mathrm{SO}_{2n+1}(\mathbb{C})$	$\hookrightarrow$	$\mathrm{GL}_{2n+1}(\mathbb{C})$	GL_{2n+1}
$\mathrm{GL}_2 \times \mathrm{GL}_2$	$\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_2(\mathbb{C})$	$\otimes$	$\mathrm{GL}_4(\mathbb{C})$	GL_4
$\mathrm{GL}_2 \times \mathrm{GL}_3$	$\mathrm{GL}_2(\mathbb{C}) \times \mathrm{GL}_3(\mathbb{C})$	$\otimes$	$\mathrm{GL}_6(\mathbb{C})$	GL_6
GL_4	$\mathrm{GL}_4(\mathbb{C})$	$\wedge^2$	$\mathrm{GL}_6(\mathbb{C})$	GL_6

In this table, the first three maps $u : {}^L\mathrm{H} \to {}^L\mathrm{G}$ are the natural embeddings, the next two are the tensor product maps, and the last is the exterior square map.

Theorem 6.1. Let k be a number field. Let H be a split reductive algebraic group over k from the table above. Let π be a globally generic cuspidal representation of $\mathrm{H}(\mathbb{A})$. Then π has a global Langlands lift Π' to $\mathrm{GL}_N(\mathbb{A})$ associated to the map $u : {}^L\mathrm{H} \to \mathrm{GL}_N(\mathbb{C})$ from the table. More specifically, there is a non-empty finite set of finite places S and an automorphic representation Π' of $\mathrm{GL}_N(\mathbb{A})$ such that for all $v \notin S$ we have Π'_v is the local Langlands lift of π_v with respect to the L-homomorphism u.

The first case of this Theorem to appear was the tensor product lifting from $\mathrm{GL}_2 \times \mathrm{GL}_2$ to GL_4 by Ramakrishnan [69]. His method was slightly different from the one we have outlined here in that he controlled the analytic properties of the twisted L-functions for $\mathrm{H} = \mathrm{GL}_2 \times \mathrm{GL}_2$ by a combination of both the Langlands–Shahidi method and integral representations. In addition he used Theorem 5.2 but made no use of Observation 5.1 or the highly ramified twist. The first case which was completely treated by the method outlined here was the lifting from SO_{2n+1} to GL_{2n} in [7]. Once this method was understood, particularly the global use of Observation 5.1, then other liftings could be obtained whenever one could control the L-functions. The tensor product lifting from $\mathrm{GL}_2 \times \mathrm{GL}_3$ to GL_6 by Kim and Shahidi [55] and the exterior square lift from GL_4 to GL_6 by Kim [53] soon followed. More recently, we have completed the local results necessary to complete the liftings from the other classical groups SO_{2n} and Sp_{2n} [8]. In addition, the Asai lifting from GL_2/K to GL_4/k, where K/k is a quadratic extension, has been analyzed by Ramakrishnan [70] by a variant of this method and by Krishnamurthy [57] using the Langlands-Shahidi method to control the L-functions.

We should point out that in all cases, particularly those of Kim–Shahidi [55] and Kim [53], the Theorem we have stated is the starting point of a more complete analysis of the lifting as well as applications.

6.4. Example: The lifting from SO_{2n+1} to GL_{2n}

In this section we would like to give a more detailed sketch of the proof of Theorem 6.1 in the case of the lifting from SO_{2n+1} to GL_{2n} associated to the embedding of L-groups $\mathrm{Sp}_{2n}(\mathbb{C}) \hookrightarrow \mathrm{GL}_{2n}(\mathbb{C})$. Since the L-functions for SO_{2n+1} associated to this representation of ${}^L\mathrm{H}$ are the standard L-functions, we will omit the L-homomorphism from our notation for the L-functions and ε-factors. The L-functions and ε-factors for H below are those defined by the Langlands-Shahidi method [84]. More details can be found in [7].

Recall that k is taken to be a number field. For definiteness, we will take H to be the split special orthogonal group with respect to the form $\begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$. Let $\pi = \otimes' \pi_v$ be a globally generic cuspidal representation of $\mathrm{H}(\mathbb{A})$.

6.4.1. Construction of a candidate lift

Let S be the finite set of finite places at which the local component π_v of π is ramified.

For $v \notin S$ the Local Langlands Conjecture is known for $\mathrm{H}(k_v)$ and we can associate to π_v its local Langlands lift Π_v from the local lifting diagram:

$$\begin{tikzcd}[column sep=small] & & \mathrm{Sp}_{2n}(\mathbb{C}) \arrow[rr, hook] & & \mathrm{GL}_{2n}(\mathbb{C}) & & \\ \pi_v \arrow[r, mapsto] & \phi_v & & & & \Phi_v \arrow[r, mapsto] & \Pi_v. \\ & & & W'_{k_v} \arrow[ull] \arrow[ur] & & & \end{tikzcd}$$

In these cases we have the following proposition, as is expected from the formalism.

Proposition 6.1. Let $v \notin S$ and let Π_v be the local Langlands lift of π_v as above. Let π'_v be an irreducible admissible generic representation of $\mathrm{GL}_m(k_v)$ with $m < 2n$. Then

$$L(s, \pi_v \times \pi'_v) = L(s, \Pi_v \times \pi'_v) \quad \text{and} \quad \varepsilon(s, \pi_v \times \pi'_v, \psi_v) = \varepsilon(s, \Pi_v \times \pi'_v, \psi_v).$$

Now we come to the places $v \in S$ where we do not have the Local Langlands Conjecture at our disposal. Instead, we will replace it with the following two local facts about representations of $\mathrm{H}(k_v)$. As was the case for linear groups, there is a local γ-factor $\gamma(s, \pi_v \times \pi'_v, \psi_v)$ for representations of $\mathrm{H}(k_v)$, where π_v is our generic representation of $\mathrm{H}(k_v)$ and π'_v is a generic representation of $\mathrm{GL}_m(k_v)$ [80, 84]. It is related to the local L- and ε-factors by

$$\gamma(s, \pi_v \times \pi'_v, \psi_v) = \frac{\epsilon(s, \pi_v \times \pi'_v, \psi_v) L(1 - s, \tilde{\pi}_v \times \tilde{\pi}'_v)}{L(s, \pi_v \times \pi'_v)}.$$

The following two properties of the local γ–factor are crucial to our local lifting. The first is the mutliplicativity of local γ-factors and is known in quite some generality [81, 84].

Proposition 6.2 (Multiplicativity of γ-factors)**.** If π_v is a generic irreducible admissible representation of $\mathrm{H}(k_v)$ and if π'_v is a generic irreducible admissible representation of $\mathrm{GL}_m(k_v)$ with $\pi'_v = \mathrm{Ind}(\pi'_{1,v} \otimes \pi'_{2,v})$, with $\pi_{i,v}$ an irreducible admissible representation of $\mathrm{GL}_{r_i}(k)$, then

$$\gamma(s, \pi_v \times \pi'_v, \psi_v) = \gamma(s, \pi_v \times \pi'_{1,v}, \psi_v)\gamma(s, \pi_v \times \pi'_{2,v}, \psi_v).$$

There is a similar multiplicativity in the first variable, i.e., if the representation π_v of $\mathrm{H}(k_v)$ is a full induced representation.

The second property is the stability of the local γ-factors under highly ramified twists. The knowledge of this property is more limited. It is known in the case we need, namely $\mathrm{H} = \mathrm{SO}_{2n+1}$ [11], and there is progress in establishing it in general [8, 82].

Proposition 6.3 (Stability of γ-factors)**.** If $\pi_{1,v}$ and $\pi_{2,v}$ are two irreducible admissible generic representations of $\mathrm{H}(k_v)$ then for every sufficiently highly ramified character η_v of $k_v^\times$ we have

$$\gamma(s, \pi_{1,v} \times \eta_v, \psi_v) = \gamma(s, \pi_{2,v} \times \eta_v, \psi_v)$$

and

$$L(s, \pi_{1,v} \times \eta_v) = L(s, \pi_{2,v} \times \eta_v) \equiv 1.$$

Hence the local ε-factor stabilizes as well.

To see how these function as a replacement for the Local Langlands Conjecture at the places is S, first recall from Section 3.1.6 that we also know the local multiplicativity and stability of γ-factors for $\mathrm{GL}_{2n}(k_v)$. If we use the multiplicativity of the γ-factor in the first variable then we can actually compute the stable form of the γ-factors $\gamma(s, \pi_v \times \eta_v, \psi_v)$ with π_v a generic representation of $\mathrm{H}(k_v)$ and $\gamma(s, \Pi_v \times \eta_v, \psi_v)$ with Π_v a representation of $\mathrm{GL}_{2n}(k_v)$ by taking $\pi_{2,v}$ or $\Pi_{2,v}$ to be full induced representations in the statement of stability. Multiplicativity in the first variable then reduces both γ-factors to a product of $2n$ one dimensional Artin γ-factors. This then allows for a *comparison* of the stable forms on these two different groups. As a result we find the following proposition.

Proposition 6.4 (Comparison of stable forms)**.** Let π_v be a generic irreducible admissible representation of $\mathrm{H}(k_v)$ and let Π_v be a generic irreducible admissible representation of $\mathrm{GL}_{2n}(k_v)$ having *trivial central character.* Then for every sufficiently ramified character η_v of $k_v^\times$ we have

$$\gamma(s, \pi_v \times \eta_v, \psi_v) = \gamma(s, \Pi_v \times \eta_v, \psi_v)$$

and

$$L(s, \pi_v \times \eta_v) = L(s, \Pi_v \times \eta_v) \equiv 1.$$

Hence the local ε-factor are stably equal as well.

This equality, combined with the multiplicativity of the γ-factors, lets us make the following definition of a local lift. If v is a place in S then we take the local lift of π_v to be any irreducible generic representation Π_v of $\mathrm{GL}_{2n}(k_v)$ having trivial central character. We can then establish the following analogue of Proposition 6.1.

Proposition 6.5. Let $v \in S$ and let Π_v be the local lift of π_v as above, that is, any generic irreducible admissible representation of $\mathrm{GL}_{2n}(k_v)$ having trivial central character. Let π'_v be an irreducible admissible generic representation of $\mathrm{GL}_m(k_v)$ with $m < 2n$ of the form $\pi'_v = \pi'_{0,v} \otimes \eta_v$ with $\pi'_{0,v}$ unramified and η_v a fixed sufficiently highly ramified character of $k_v^\times$. Then

$$L(s, \pi_v \times \pi'_v) = L(s, \Pi_v \times \pi'_v) \quad \text{and} \quad \varepsilon(s, \pi_v \times \pi'_v, \psi_v) = \varepsilon(s, \Pi_v \times \pi'_v, \psi_v).$$

We will sketch the proof of this Proposition on the level of γ-factors. Since $\pi'_{0,v}$ is generic and unramified, it is a full induced representation of the form $\pi'_{0,v} = \mathrm{Ind}(\mu_{1,v} \otimes \cdots \otimes \mu_{m,v})$ with each $\mu_{i,v}(x) = |x|_v^{s_i}$ an unramified character of $k_v^\times$. Then $\pi'_v = \pi'_{0,v} \otimes \eta_v = \mathrm{Ind}(\mu_{1,v}\eta_v \otimes \cdots \otimes \mu_{m,v}\eta_v)$ and we have

$$\begin{aligned}
\gamma(s, \pi_v \times {\pi'}_v, \psi_v) &= \gamma(s, \pi_v \times \mathrm{Ind}(\mu_{1,v}\eta_v \otimes \cdots \otimes \mu_{m,v}\eta_v), \psi_v) \\
&= \prod \gamma(s + s_i, \pi_v \times \eta_v, \psi_v) \text{ (multiplicativity)} \\
&= \prod \gamma(s + s_i, \Pi_v \times \eta_v, \psi_v) \text{ (comparing stable forms)} \\
&= \gamma(s, \Pi_v \times \mathrm{Ind}(\mu_{1,v}\eta_v \otimes \cdots \otimes \mu_{m,v}\eta_v), \psi_v) \text{ (multiplicativity)} \\
&= \gamma(s, \Pi_v \times \pi'_v, \psi_v).
\end{aligned}$$

From this one derives the equality for the $L-$ and $\varepsilon-$factors.

We can now construct our candidate lift. With π and S as above we take Π_v to be the local Langlands lift of π_v for all $v \notin S$ and take Π_v to be any irreducible admissible generic representation of $\mathrm{GL}_{2n}(k_v)$ with trivial central character for $v \in S$. Let $\Pi = \otimes' \Pi_v$. Then Π is an irreducible admissible representation of $\mathrm{GL}_{2n}(\mathbb{A})$. From Proposition 6.1 and 6.5 we may now deduce the following result.

Proposition 6.6. Let π and S be as above and let Π be the candidate lift of π constructed above. Then for any fixed idele class character η for which η_v is sufficiently ramified at the places $v \in S$ so that Proposition 6.4 holds we have

$$L(s, \pi \times \pi') = L(s, \Pi \times \pi') \quad \text{and} \quad \varepsilon(s, \pi \times \pi') = \varepsilon(s, \Pi \times \pi').$$

for all $\pi' \in \mathcal{T}^S(2n-1) \otimes \eta$.

6.4.2. Controlling the analytic properties of the twisted L-functions

The twisted L-functions $L(s, \pi \times \pi')$ are controlled using the Langlands-Shahidi method. We refer to Shahidi's article in these proceedings [84] for a discussion of this method. In our case, these results are obtained by analyzing the Eisenstein series on $\mathrm{SO}_{2(n+m)+1}$ induced from the representation $\widetilde{\pi'}|\det|^s \otimes \pi$ on the maximal parabolic with Levi subgroup $\mathrm{GL}_m \times \mathrm{SO}_{2n+1}$ as well as the Eisenstein series on SO_{2m+1} induced from the representation $\pi'|\det|^{s/2}$ on the maximal parabolic with Levi subgroup GL_m for all $m = 1, \ldots, 2n-1$.

The general functional equation has been understood for many years [80, 84]:

Proposition 6.7. For any cuspidal representation π' of $\mathrm{GL}_m(\mathbb{A})$, $1 \leq m < 2n$, we have the functional equation

$$L(s, \pi \times \pi') = \epsilon(s, \pi \times \pi') L(1 - s, \tilde{\pi} \times \tilde{\pi}').$$

Similarly, the boundedness in vertical strips is true for all π' [25, 84]:

Proposition 6.8. For any cuspidal representation π' of $\mathrm{GL}_m(\mathbb{A})$, $1 \leq m < 2n$, the L-function $L(s, \pi \times \pi')$ is bounded in vertical strips.

As noted above, there is no reason for us to expect the L-functions $L(s,\pi\times\pi')$ to be entire for all cuspidal π'. However, one can analyze the potential poles in terms of the Eisenstein series above. The crucial observation of Kim was the following [7,84].

Proposition 6.9. The relevant Eisenstein series, hence the L-function $L(s,\pi\times\pi')$, can have poles only if the representation π' is essentially self-contragredient, that is, $\widetilde{\pi'}\simeq\pi'\otimes|\det|^t$ for some $t\in\mathbb{C}$.

This condition for poles is a condition on our twisting representation π' and can again be controlled by a ramified twist. If we assume that π' is such that at one finite place v we have $\pi'_v=\pi'_{0,v}\otimes\eta_v$ with $\pi'_{0,v}$ unramified and η_v a character of $k_v^\times$ such that both η_v and η_v^2 are ramified then we can never have $\widetilde{\pi'}\simeq\pi'\otimes|\det|^t$ since this is not possible locally at the place v. Hence $L(s,\pi\times\pi')$ will be entire.

Combining these three results, we have the following statement.

Proposition 6.10. Let π be a globally generic cuspidal representation of $\mathrm{H}(\mathbb{A})$. Let S' be a non-empty set of finite places and suppose that η is an idele class character such that at at least one place $v\in S'$ we have both η_v and η_v^2 are ramified. Then the twisted L-functions $L(s,\pi\times\pi')$ are *nice* for all $\pi'\in\mathcal{T}^{S'}(2n-1)\otimes\eta$.

6.4.3. Application of the Converse Theorem

We are now ready to complete the proof of Theorem 6.1 in the case of $\mathrm{H}=\mathrm{SO}_{2n+1}$. Let π be a globally generic cuspidal representation of $\mathrm{H}(\mathbb{A})$. Let S be the finite set of finite places at which π_v is ramified. Let Π be the candidate lift of π to $\mathrm{GL}_{2n}(\mathbb{A})$ constructed above, that is, Π_v is the local Langlands lift of π_v for $v\notin S$ and Π_v is any irreducible admissible generic representation of $\mathrm{GL}_{2n}(k_v)$ having trivial central character for $v\in S$. If S is non-empty let $S'=S$ and if π is unramified at all finite places take $S'=\{v_0\}$ to contain any chosen finite place. Choose a fixed idele class character η which is sufficiently ramified for all $v\in S'$ such that both Propositions 6.6 and 6.10 are valid. Then for all $\pi'\in\mathcal{T}^{S'}(2n-1)\otimes\eta$ we have

$$L(s,\pi\times\pi')=L(s,\Pi\times\pi')\quad\text{and}\quad\varepsilon(s,\pi\times\pi')=\varepsilon(s,\Pi\times\pi')$$

and the $L(s,\Pi\times\pi')$ are thus *nice*. Then applying Theorem 5.1 and Observation 5.1 we can conclude that there exists a automorphic representation $\Pi'=\otimes'\Pi'_v$ of $\mathrm{GL}_{2n}(\mathbb{A})$ such that $\Pi'_v=\Pi_v$ is the local Langlands lift of π_v for all $v\notin S'$. Hence Π' is a global Langlands lift of π associated to the embedding of L-groups $\mathrm{Sp}_{2n}(\mathbb{C})\hookrightarrow\mathrm{GL}_{2n}(\mathbb{C})$. This is Theorem 6.1 in this case.

6.5. Liftings from the other classical groups

The liftings of globally generic cuspidal representations from SO_{2n} to GL_{2n} and Sp_{2n} to GL_{2n+1} follows the same outline as above. At the time of writing [7] the stability of the local γ-factors was known only for $\mathrm{H}=\mathrm{SO}_{2n+1}$. Since then, Shahidi has established formulas for his local coefficients, and hence his local γ-factors, which represent them as Mellin transforms of suitable Bessel functions [82]. Having such a representation was crucial for the proof for stability of γ-factors in the SO_{2n+1} case [11]. Combining the formulas of [82] with the analysis of [11] now gives the stability in these cases. Having this stability in hand, the proof now follows the method above. The complete proof in these cases will appear in [8].

6.6. Complements

As we noted above, Theorem 6.1 is the beginning point for a more complete analysis of the these liftings. We would like to mention two examples of this.

In the case of the lifting from SO_{2n+1} to GL_{2n}, combining this lift with their descent theory Ginzburg, Rallis and Soudry were able to completely characterize the image locally and globally [29] and thus show that the local components Π'_v at those $v \in S'$ are completely determined by the global lift, so there is no ambiguity at these places. This is true for the liftings from the other classical groups as well [8, 88]. Once one knows that these lifts are rigid, then one can begin to define and analyze the local lift for ramified representations by setting the local lift of π_v to be the Π_v determined by the global lift. This is the content of the papers of Jiang and Soudry [48, 49] for the case of $\mathrm{H} = \mathrm{SO}_{2n+1}$. In essence they show that this local lift satisfies the relations on L-functions that one expects from Functoriality and then deduce the *Local Langlands Conjecture for generic representation* SO_{2n+1} from that for GL_{2n}. We refer to their papers for more detail and precise statements. Related results and further applications can be found in the papers of Kim [51, 52].

In the case of the tensor lifting $\mathrm{GL}_2 \times \mathrm{GL}_3$ to GL_6, Kim and Shahidi also showed that in fact this lift is completely determined at the places $v \in S$ and in fact is the local Langlands lift at those places as well [55, 84]. They also characterize when the image is cuspidal, etc. Kim is able to do the same for his exterior square lift from GL_4 to GL_6, except possibly for places lying above 2 and 3 [53, 84]. In addition, combining these two lifts, they are able to deduce and analyze the symmetric cube and fourth power lifts for GL_2 [53, 55, 56]:

H	${}^L\mathrm{H}$	$u : {}^L\mathrm{H} \to {}^L\mathrm{G}$	${}^L\mathrm{G}$	G
GL_2	$\mathrm{GL}_2(\mathbb{C})$	Sym^3	$\mathrm{GL}_4(\mathbb{C})$	GL_4
GL_2	$\mathrm{GL}_2(\mathbb{C})$	Sym^4	$\mathrm{GL}_5(\mathbb{C})$	GL_5

From these they were able to deduce the estimates towards the Generalized Ramanujan Conjecture for GL_2 mentioned in Section 4.5. For more details, we refer the reader to the original papers [53, 55, 56] as well as Shahidi's article in these proceedings [84].

6.7. Concluding remarks

What further cases of Functoriality can we expect from this method? The table in Section 6.3 gives all of the cases of split H which are attainable. Given that the Converse Theorem requires the control of a large family of twisted L-functions, this table covers all cases where the Langlands-Shahidi method is able to supply that control [60, 79]. There are cases of quasi-split H and similitude groups such as $GSpin$ that should also be attainable and Shahidi, Kim, and their students are currently pursuing these.

If we stay within the general Langlands-Shahidi philosophy of controlling analytic properties of L-functions through analyzing the Fourier coefficients of Eisenstein series there are two possible extensions of the method. The first possibility for extending the method would be to relax the requirement of π being globally generic. This idea has been initially investigated by Friedberg and Goldberg [20]. Another possible extension would be to adapt the method to include Eisenstein series on loop groups [21, 22]. Many more twisted L-functions should be attainable from Eisenstein series on these groups since more combinations of groups occur in the Levi decomposition of parabolic subgroups in this context.

Another possibility would be to try to control the twisted L-functions involved by the method of integral representations, as was pursued in this set of notes for GL_n. While there are many integral representations in the literature, with the exception of GL_n there are few if any cases where a complete analysis of the L-functions has been worked out. The version of the Converse Theorem given in Theorem 5.3 was originally put forth for use with integral representations.

Of course, possibly the most natural way to extend the method would be to reduce the amount of twisting needed in the Converse Theorem. For this reason, the pursuit of Conjecture 5.2 becomes very appealing.

BIBLIOGRAPHY

1. J. Bernstein and A. Zelevinsky, *Representations of $GL(n, F)$ where F is a non-archimedean local field*. Russian Math. Surveys **31** (1976), 1–68.
2. ______, *Induced representations of reductive $\mathfrak{p}$–adic groups, I.* Ann. scient. Éc. Norm. Sup., 4^e série, **10** (1977), 441–472.
3. A. Borel, *Automorphic L-functions*. Proc. Sympos. Pure Math. **33**, part 2, (1979), 27–61.
4. W. Casselman and J. Shalika, *The unramified principal series of p-adic groups. II. The Whittaker function*. Compositio Math. **41** (1980), 207–231.
5. L. Clozel, *Motifs et formes automorphes: allpications du principle de functorialité*. Automorphic Forms, Shimura Varieties, and L-functions, I (L. Clozel and J. Milne, eds.), Academic Press, Boston, 1990, 77–159.
6. J.W. Cogdell, *Dual groups and Langlands Functoriality*. An Introduction to the Langlands Program (J. Bernstein and S. Gelbart, eds.). Birkhäuser, Boston, 2003, 251–268.
7. J.W. Cogdell, H.H. Kim, I.I. Piatetski-Shapiro, and F. Shahidi, *On lifting from classical groups to* GL_N. Publ. Math. IHES **93** (2001), 5–30.
8. ______, *Functoriality for the classical groups*. Preprint (2003).
9. J.W. Cogdell and I.I. Piatetski-Shapiro, *Converse Theorems for GL_n*. Publ. Math. IHES **79** (1994), 157–214.
10. ______, *Unitarity and functoriality*. Geom. and Funct. Anal. **5** (1995), 164–173.
11. ______, *Stability of gamma factors for $SO(2n+1)$*. manuscripta math. **95** (1998), 437–461.
12. ______, *Converse Theorems for GL_n*, II. J. reine angew. Math. **507** (1999), 165–188.
13. ______, *Converse Theorems for GL_n and their applications to liftings*. Cohomology of Arithmetic Groups, Automorphic Forms, and L-functions (T.N. Venkataramana, ed.), TIFR/Narosa Publishing House, New Delhi, 2001, 1–34.
14. ______, *Converse Theorems, Functoriality, and applications to number theory*. Proc. ICM 2002, Beijing **II** (2002), 119–128.
15. ______, *Derivatives and L-functions for GL_n*. The Heritage of B. Moishezon (M. Teicher, ed.), IMCP, to appear .

16. ______, *Remarks on Rankin-Selberg convolutions.* Contributions to Automorphic Forms, Geometry and Number Theory (Shalikafest 2002) (H. Hida, D. Ramakrishnan, and F. Shahidi, eds.), Johns Hopkins University Press, Baltimore, to appear.
17. ______, L-functions for GL_n, in progress.
18. K. Doi and H. Naganuma, *On the functional equation of certain Dirichlet series.* Invent. math. **9** (1969), 1–14.
19. D. Flath, *Decomposition of representations into tensor products.* Proc. Sympos. Pure Math. **33**, part 1, (1979), 179–183.
20. S. Friedberg and D. Goldberg, *On local coefficients for non-generic representations of some classical groups.* Compositio Math. **116** (1999), 133–166.
21. H. Garland, *Eisenstein series on arithmetic quotients of loop groups.* Math. Research Letters **6** (1999), 723–733.
22. ______, *Certain Eisenstein series on loop groups: Convergence and the constant term.* Preprint (2002).
23. S. Gelbart and H. Jacquet, *A relation between automorphic representations of* $\mathrm{GL}(2)$ *and* $\mathrm{GL}(3)$. Ann. Sci. École Norm. Sup. (4) **11** (1978), 471–542.
24. S. Gelbart and F. Shahidi, Analytic Properties of Automorphic L-functions, Academic Press, San Diego, 1988.
25. ______, *Boundedness of automorphic L-functions in vertical strips.* J. Amer. Math. Soc. **14** (2001), 79–107.
26. I.M. Gelfand, M.I. Graev, and I.I. Piatetski-Shapiro, Representation Theory and Automorphic Functions, Academic Press, San Diego, 1990.
27. I.M. Gelfand and D.A. Kazhdan, *Representations of $GL(n,K)$ where K is a local field.* Lie Groups and Their Representations (I.M. Gelfand, ed.) John Wiley & Sons, New York–Toronto, 1971, 95–118.
28. S.I. Gelfand, *Representations of the general linear group over a finite field.* Lie Groups and Their Representations (I.M. Gelfand, ed.), John Wiley & Sons, New York–Toronto, 1971, 119–132.
29. D. Ginzburg, S. Rallis, and D. Soudry, *Generic automorphic forms on* $\mathrm{SO}(2n+1)$*: functorial lift to* $\mathrm{GL}(2n)$*, endoscopy, and base change.* Internat. Math. Res. Notices **2001**, no. 14, 729–764.
30. R. Godement, Notes on Jacquet-Langlands' Theory. The Institute for Advanced Study, 1970.
31. R. Godement and H. Jacquet, Zeta Functions of Simple Algebras, Springer Lecture Notes in Mathematics, No.260, Springer-Verlag, Berlin, 1972.
32. M. Harris and R. Taylor, The Geometry and Cohomology of Some Simple Shimura Varieties, Annals of Math. Studies No. 151, Princeton University Press, Princeton, 2001.
33. E. Hecke, *Über die Bestimmung Dirichletscher Reihen durch ihre Funktionalgleichung.* Math. Ann. **112** (1936), 664–699.
34. ______, Mathematische Werke, Vandenhoeck & Ruprecht, Göttingen, 1959.
35. G. Henniart, *Une preuve simple des conjectures de Langlands pour* $\mathrm{GL}(n)$ *sur un corps p-adique.* Invent. Math. **139** (2000), 439–455.
36. H. Iwaniec and P. Sarnak, *Perspectives on the analytic theory of L-functions.* GAFA 2000 (Tel Aviv, 1999), Geom. Funct. Anal. **2000**, Special Volume, Part II, 705–741.
37. H. Jacquet, Automorphic Forms on $GL(2)$, II, Springer Lecture Notes in Mathematics No.278 , Springer-Verlag, Berlin, 1972.

38. ______, *Principal L-functions of the linear group.* Proc. Sympos. Pure Math. **33**, part 2, (1979), 63–86.
39. H. Jacquet and R.P. Langlands, Automorphic Forms on $GL(2)$, Springer Lecture Notes in Mathematics No.114, Springer Verlag, Berlin, 1970.
40. H. Jacquet, I.I. Piatetski-Shapiro, and J. Shalika, *Automorphic forms on* $GL(3)$, I & II. Ann. Math. **109** (1979), 169–258.
41. ______, *Relevement cubique non normal.* C. R. Acad. Sci. Paris, Ser. I. Math., **292** (1981), 567–571.
42. ______, *Rankin–Selberg convolutions.* Amer. J. Math. **105** (1983), 367–464.
43. ______, *Conducteur des représentations du groupe linéaire.* Math. Ann. **256** (1981), 199–214.
44. H. Jacquet and J. Shalika, *A non-vanishing theorem for zeta functions of* GL_n. Invent. math. **38** (1976), 1–16.
45. ______, *On Euler products and the classification of automorphic representations*, I & II. Amer. J. Math. **103** (1981), 499–588 & 777–815.
46. ______, *A lemma on highly ramified ϵ-factors.* Math. Ann. **271** (1985), 319–332.
47. ______, *Rankin-Selberg convolutions: Archimedean theory.* Festschrift in Honor of I.I. Piatetski-Shapiro, Part I, Weizmann Science Press, Jerusalem, 1990, 125–207.
48. D. Jiang and D. Soudry, *The local converse theorem for $SO(2n+1)$ and applications.* Ann. of Math. **157** (2003), 743–806.
49. ______, *Generic representations and local Langlands reciprocity law for p-adic SO_{2n+1}.* Contributions to Automorphic Forms, Geometry and Number Theory (Shalikafest 2002) (H. Hida, D. Ramakrishnan, and F. Shahidi, eds.), Johns Hopkins University Press, Baltimore, to appear.
50. D. Kazhdan and G. Lusztig, *Proof of the Deligne-Langlands conjecture for Hecke algebras.* Invent. Math. **87** (1987), 153–215.
51. H.H. Kim, *Residual spectrum of odd orthogonal groups.* IMRN **2001**, No. 17, (2001), 873–906.
52. ______, *Applications of Langlands' functoriality of odd orthogonal groups.* Trans. Amer. Math. Soc. **354** (2002), 2775–2796.
53. ______, *Functoriality for the exterior square of* GL_4 *and the symmetric fourth of* GL_2. J. Amer. Math. Soc. **16** (2003), 139–183.
54. ______, *On local L-functions and normalized intertwining operators.* Canad. J. Math., to appear.
55. H.H. Kim and F. Shahidi, *Functorial products for* $\mathrm{GL}_2 \times \mathrm{GL}_3$ *and the symmetric cube for* GL_2. Annals of Math. **155** (2002), 837–893.
56. ______, *Cuspidality of symmetric powers with applications.* Duke Math. J. **112** (2002), 177–197.
57. M. Krishnamurthy, *The Asai transfer to* GL_4*: Via the Langlands–Shahidi method.* Preprint (2002).
58. S. Kudla, *The local Langlands correspondence: the non-Archimedean case.* Proc. Sympos. Pure Math. **55**, part 2, (1994), 365–391.
59. L. Lafforgue, *Chtoucas de Drinfeld et correspondance de Langlands.* Invent. math. **147** (2002), 147–241.
60. R.P. Langlands, Euler Products, Yale Univ. Press, New Haven, 1971.
61. ______, *On the notion of an automorphic representation.* Proc. Sympos. Pure Math. **33**, part 1, (1979), 203–207.

62. ______, *On the classification of irreducible representations of real algebraic groups.* Representation Theory and Harmonic Analysis on Semisimple Lie Groups, AMS Mathematical Surveys and Monographs, No.31, 1989, 101–170.
63. W. Luo, Z. Rudnick, and P. Sarnak, *On the generalized Ramanujan conjecture for $GL(n)$.* Proc. Symp. Pure Math. **66**, part 2, (1999), 301–310.
64. I.I. Piatetski-Shapiro, *Euler Subgroups.* Lie Groups and Their Representations (I.M. Gelfand, ed.) John Wiley & Sons, New York–Toronto, 1971, 597–620.
65. ______, *Zeta–functions of $GL(n)$.* University of Maryland preprint MD76-80-PS (1976).
66. ______, *Multiplicity one theorems.* Proc. Sympos. Pure Math. **33**, part 1 (1979), 209–212.
67. ______, Complex Representations of $GL(2, K)$ for Finite Fields K, Contemporary Math. Vol.16, AMS, Providence, 1983.
68. D. Ramakrishnan, *Pure motives and automorphic forms.* Proc. Sympos. Pure Math. **55**, part 2, (1991), 411–446.
69. ______, *Modularity of the Rankin-Selberg L-series, and multiplicity one for SL(2).* Annals of Math. **152** (2000), 45–111.
70. ______, *Modularity of solvable Artin representations of $GO(4)$–type.* IMRN **2002**, no. 1, 1–54.
71. R.A. Rankin, *Contributions to the theory of Ramanujan's function $\tau(n)$ and similar arithmetical functions*, I & II. Proc. Cambridge Phil. Soc. **35** (1939), 351–372.
72. B. Riemann, *Über die Anzahl der Primzahlen unter einer gegebenen Grösse.* Monats. Berliner Akad. (1859), 671–680.
73. A. Selberg, *Bemerkungen über eine Dirichletsche Reihe, die mit der Theorie der Modulformen nahe verbunden ist.* Arch. Math. Naturvid. **43** (1940), 47–50.
74. F. Shahidi, *Functional equation satisfied by certain L-functions.* Compositio Math. **37** (1978), 171–207.
75. ______, *On non-vanishing of L-functions.* Bull. Amer. Math. Soc., N.S. **2** (1980), 462–464.
76. ______, *On certain L-functions.* Amer. J. Math. **103** (1981), 297–355.
77. ______, *Fourier transforms of intertwining operators and Plancherel measures for* $\mathrm{GL}(n)$. Amer. J. Math. **106** (1984), 67–111.
78. ______, *Local coefficients as Artin factors for real groups.* Duke Math. J. **52** (1985), 973–1007.
79. ______, *On the Ramanujan conjecture and finiteness of poles for certain L-functions.* Ann. of Math. **127** (1988), 547–584.
80. ______, *A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups.* Ann. of Math. **132** (1990), 273–330.
81. ______, *On multiplicativity of local factors.* Festschrift in honor of I. I. Piatetski-Shapiro, Part II., Israel Math. Conf. Proc. **3** , Weizmann Science Press, Jerusalem, 1990, 279–289.
82. ______, *Local coefficients as Mellin transforms of Bessel functions; Towards a general stability.* IMRN **2002**, no. 39, 2075–2119.
83. ______, *Automorphic L-functions and Functoriality.* Proc. ICM 2002, Beijing **II** (2002), 655–666.
84. ______, *Langlands-Shahidi method.* PCMI Lectures, this volume.

85. J. Shalika, *The multiplicity one theorem for* $\mathrm{GL}(n)$. Ann. Math. **100** (1974), 171–193.
86. G. Shimura, *On modular forms of half integral weight*. Ann. Math. **97** (1973), 440–481.
87. T. Shintani, *On an explicit formula for class-1 "Whittaker functions" on* GL_n *over* $\mathfrak{P}$*-adic fields*. Proc. Japan Acad. **52** (1976), 180–182.
88. D. Soudry, *On Langlands functoriality from classical groups to* GL_n [IHP Lectures, Spring 2000]. Asterisque, to appear.
89. E. Stade, *Mellin transforms of* $GL(n,\mathbb{R})$ *Whittaker functions*. Amer. J. Math. **123** (2001), 121–161.
90. ______, *Archimedean L-factors on* $GL(n) \times GL(n)$ *and generalized Barnes integrals*. Israel J. Math. **127** (2002), 201–219.
91. J. Tate, *Fourier Analysis in Number Fields and Hecke's Zeta-Functions*. Algebraic Number Theory (J.W.S. Cassels and A. Frolich, eds.), Academic Press, London, 1967, 305–347.
92. ______, *Number theoretic background*. Proc. Symp. Pure Math. **33**, part 2, 3–26.
93. R. Taylor, *Galois representations*. Proc. ICM 2002, Beijing **I** (2002), to appear.
94. N. Wallach, Real Reductive Groups, I & II, Academic Press, Boston, 1988 & 1992.
95. A. Weil, *Über die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen*. Math. Ann. **168** (1967), 149–156.
96. ______, Basic Number Theory, Springer–Verlag, Berlin, 1974.
97. A. Zelevinsky, *Induced representations of reductive* $\mathfrak{p}$*–adic groups, II. Irreducible representations of* $GL(n)$. Ann. scient. Éc. Norm. Sup., 4^e série, **13** (1980), 165–210.