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## SOLUTIONS OF SELF-DIFFERENTIAL FUNCTIONAL EQUATIONS

### Abstract

The system of functional differential equations (1) has a continuously differentiable solution for every value of the parameter  $a$ . The boundary values and  $a$  are related with  $d(2 - a) = c(2 + a)$ . When  $a \in S$  where

$$S = \{2^{2n+1} : n = 1, 2, 3, \dots\},$$

the system (1) has infinitely many solutions with boundary values  $c = 0$  and  $d = 0$ . For all other values of  $a$ , the system (1) has a unique solution.

$$\begin{cases} F'(x) = aF(2x) & \text{if } 0 \leq x \leq \frac{1}{2} \\ F'(x) = aF(2 - 2x) & \text{if } \frac{1}{2} \leq x \leq 1 \\ F(0) = c, F(1) = d. \end{cases} \quad (1)$$

### 1 Introduction.

A function  $f : [a, b] \rightarrow \mathbb{R}$  is self-differential if  $[a, b]$  can be subdivided into a finite number of sub-intervals, and on each sub-interval the derivative of  $f$  is equal to  $f$  by the graph transformed by an affine map. The case to be studied here is (1), where  $[0, 1]$  is decomposed into  $\left[0, \frac{1}{2}\right]$  and  $\left[\frac{1}{2}, 1\right]$ , and the affine transformed images of  $F$  are  $aF(2x)$  and  $aF(2 - 2x)$ .

In [4] Fabius showed that the distribution function  $F_2$  of the random variable  $U = \sum_{n=1}^{\infty} 2^{-n}U_n$ , where  $U_1, U_2, \dots$  are independent random variables uniformly distributed on  $[0, 1]$ , is the solution of (1) for a value of the parameter  $a = 2$  and boundary values  $c = 0$  and  $d = 1$ . The function  $F_2$  is infinitely

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Key Words: linear functional differential equations, iteration  
Mathematical Reviews subject classification: 34K06, 26A18  
Received by the editors May 15, 2005  
Communicated by: Zbigniew Nitecki

differentiable and nowhere analytic on  $[0, 1]$ . The derivatives of  $F_2$  of order  $2n$  are also solutions of (1) for  $a = 2^{2n+1}$  and  $c = d = 0$ . In Theorem 2.1 (i), we show that (1) has a unique solution for all other values of  $a$ . We prove the existence of the solution using a modification of the method of the successive approximations on a sub-interval  $\left[0, \frac{1}{4^n}\right]$  and extend it to  $[0, 1]$  with formulas (4.3). This method is used by Kato and McLeod [5] for the solution of the initial value functional differential equation  $y(x) = \alpha y'(\lambda x) + \beta y(x)$ ,  $y(0) = 1$ . Similar initial value functional differential equations have been studied by De Bruijn [1]. A distinctive feature of the solution  $F$  of (1) is that if it is known on any sub-interval, then it can be extended to  $[0, 1]$  using only polynomials. This property of the solutions of boundary value self-differential equations is analogous to the notion of self-similarity for fractals (p. 135, Edgar [3]).

**Definition 1.1.** A differentiable function  $f$  is polynomially divided on the interval  $[0, 1]$  if for every  $N \geq 0$  there exists an integer  $n \geq N$  and polynomials  $\{p_{n,i}(x)\}_{i=1}^{n-1}$  such that either  $f\left(\frac{i}{n} + x\right) = f\left(\frac{i}{n} - x\right) + p_{n,i}(x)$  or  $f\left(\frac{i}{n} + x\right) = f\left(\frac{i-1}{n} + x\right) + p_{n,i}(x)$  for  $i = 1, 2, \dots, n-1$  and  $x \in \left[0, \frac{1}{n}\right]$ .

This definition means that if  $[0, 1]$  is partitioned to  $n$  sub-intervals of equal length, the values of  $f$  on two neighboring intervals differ only by a polynomial. The solutions of (1) are polynomially divided by Lemma 3.1. In Section 2 we find a relation between the boundary values which allows us to decompose equations (1) to the simpler functional differential equations (\*) and (\*\*). The decomposition of the solutions is different, depending on whether or not  $a$  belongs to the set  $S = \{2^{n+1} : n = 1, 2, 3, \dots\}$ . The main motivation to study self-differential equations is to generalize the exponential functions which have derivatives constant multiples of themselves. It is an interesting question to find self-differential equations which have practical applications.

This work is part of the Ph.D. thesis of the first author, written under the direction of the second author.

## 2 Basic Properties.

When  $a = 0$ , equations (1) have a solution  $F(x) = c$  with boundary values  $c = d$ . For boundary conditions  $c = d = 0$ , equations (1) have a solution  $F(x) = 0$  for all values of the parameter  $a$ . The Fabius function  $dF_2(x)$  is a solution of (1) for  $a = 2$ ,  $c = 0$  and every value of  $F(1) = d$ . Other solutions are obtained from the derivatives of  $F_2(x)$  of even order.

**Lemma 2.1.** *The function  $F_2^{(2n)}$  is a solution of (1) with  $a = 2^{2n+1}$  and boundary values  $c = d = 0$ , for all  $n = 1, 2, \dots$*

PROOF. The Fabius function  $F_2$  satisfies

$$\begin{cases} F_2'(x) = 2F_2(2x) & \text{if } 0 \leq x \leq \frac{1}{2}, \\ F_2'(x) = 2F_2(2-2x) & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

By differentiating  $2n$  times these equations, we obtain

$$\begin{cases} F_2^{(2n+1)}(x) = 2^{2n+1}F_2^{(2n)}(2x) & \text{if } 0 \leq x \leq \frac{1}{2}, \\ F_2^{(2n+1)}(x) = 2^{2n+1}F_2^{(2n)}(2-2x) & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}$$

From the first formula and  $x = 0$  we obtain

$$F_2^{(2n+1)}(0) = F_2^{(2n)}(0) = \dots = F_2'(0) = F_2(0) = 0,$$

and by the second formula and  $x = 1$  we have that

$$F_2^{(2n+1)}(1) = 2^{2n+1}F_2^{(2n)}(0) = 0.$$

Therefore  $F_2^{(2n)}$  satisfies (1) with parameter  $a = 2^{2n+1}$  and boundary values  $c = d = 0$ .  $\square$

The graphs of the solutions of (1) for  $a = 2, 8$  and  $32$  are given in Fig. 1. Every constant multiple function of  $F_2^{(2n)}(x)$  is also a solution of (1). Therefore (1) has infinitely many solutions for every  $a \in S$ . In Section 3 we show that equations (1) have a unique solution for all other values of  $a \notin S$ . Now we want to find a relation between the boundary values and the parameter  $a$ . Suppose that the function  $F(x)$  is a solution of the system of equations (1).

**Proposition 2.1.** *The solution  $F$  of (1) satisfies*

$$F\left(\frac{1}{2} + x\right) + F\left(\frac{1}{2} - x\right) = c + d \quad (2.1)$$

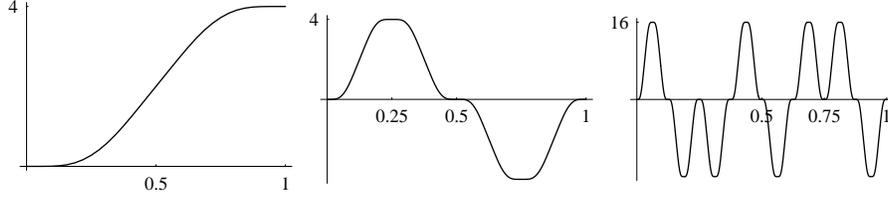


Fig. 1: The solutions  $4F_2$ ,  $F_2''$  and  $F_2^{(4)}$  of (1) for  $a = 2, 8$  and  $a = 32$

PROOF.

$$\text{Let } G(x) = F\left(\frac{1}{2} + x\right) + F\left(\frac{1}{2} - x\right) \text{ for } 0 \leq x \leq \frac{1}{2}.$$

$$\text{Then } G'(x) = F'\left(\frac{1}{2} + x\right) + F'\left(\frac{1}{2} - x\right)$$

$$G'(x) = aF\left(2 - 2\left(\frac{1}{2} + x\right)\right) + aF\left(2\left(\frac{1}{2} - x\right)\right)$$

$$G'(x) = aF(1 - 2x) - aF(1 - 2x) = 0.$$

Therefore,

$$G(x) = G\left(\frac{1}{2}\right) = F(0) + F(1) = c + d. \quad \square$$

$$\text{From (2.1) and } x = \frac{1}{2}, \text{ we obtain } F\left(\frac{1}{2}\right) = \frac{c + d}{2}.$$

**Lemma 2.2.** *The boundary values of equations (1) satisfy  $d(2-a) = c(2+a)$ .*

PROOF. We evaluate  $\int_0^1 F(x) dx$  in two ways:

$$\int_0^1 F(x) dx = \int_0^{\frac{1}{2}} F(x) dx + \int_{\frac{1}{2}}^1 F(x) dx = \int_0^{\frac{1}{2}} F(x) dx + \int_0^{\frac{1}{2}} F(1-u) du$$

and

$$\int_0^1 F(x) dx = \int_0^{\frac{1}{2}} [F(u) + F(1-u)] du.$$

By Proposition 2.1 we have that  $F(u) + F(1-u) = c + d$ . Then

$$\int_0^1 F(x) dx = \int_0^{\frac{1}{2}} [c + d] dx = \frac{c + d}{2}. \quad (2.2)$$

Let  $x = 2y$ .

$$\begin{aligned} \int_0^1 F(x) dx &= 2 \int_0^{\frac{1}{2}} F(2y) dy = \frac{2}{a} \int_0^{\frac{1}{2}} F'(y) dy \\ &= \frac{2}{a} \left( F\left(\frac{1}{2}\right) - F(0) \right) = \frac{2}{a} \left( \frac{c + d}{2} - c \right) = \frac{d - c}{a} \end{aligned} \quad (2.3)$$

By (2.2) and (2.3) we obtain  $\frac{d - c}{a} = \frac{c + d}{2}$ . Therefore,  $c(a + 2) = d(2 - a)$ .  $\square$

In Lemma 2.2 we showed that if equations (1) have a solution, then the boundary values satisfy  $c(a + 2) = d(2 - a)$ . In this way we have the following four possibilities for the values of the parameters  $c$  and  $d$  depending on the values of  $a$ :

- $a \neq \pm 2$ . The value of  $d$  is determined from  $c$  with  $d = \frac{(a + 2)c}{2 - a}$ .
- $a = 2$ . Then  $c = 0$  and  $d$  is an arbitrary real number.
- $a = -2$ . Then  $d = 0$  and  $c$  is an arbitrary real number. These relations of the boundary values lead to the following system of functional differential equations.

$$\begin{cases} f'(x) = af(2x) & \text{if } 0 \leq x \leq \frac{1}{2} \\ f'(x) = af(2 - 2x) & \text{if } \frac{1}{2} \leq x \leq 1 \\ f(0) = 2 - a, f(1) = 2 + a. \end{cases} \quad (*)$$

In Theorem 2.1 we show that (\*) has a unique solution when  $a \notin S$ .

- $c = d = 0$  and  $a$  is an arbitrary real number. With these values of the parameters equations (1) become

$$\begin{cases} f'(x) = af(2x) & \text{if } 0 \leq x \leq \frac{1}{2} \\ f'(x) = af(2-2x) & \text{if } \frac{1}{2} \leq x \leq 1 \\ f(0) = 0, f(1) = 0. \end{cases} \quad (**)$$

In Lemma 2.2 we have already found infinitely many nonzero solutions of (\*\*) for every  $a \in S$ . The solution of (1) is either a constant multiple of a solution of (\*) or it is a solution of (\*\*) if  $c = d = 0$ . These results are summarized in Theorem 2.1 and Corollary 2.1.

**Theorem 2.1.** (i) Equations (\*) have a unique  $C^1$  solution for all  $a \notin S$  and have no solution if  $a \in S$ .

(ii) Equations (\*\*) have infinitely many  $C^1$  solutions  $\{rF_2^{(2^n)} \mid r \in \mathbb{R}\}$  for every  $a = 2^{2^{n+1}} \in S$  and have no nonzero solutions if  $a \notin S$ .

The proof of Theorem 2.1 is divided into lemmas. In Section 3 we show that a necessary condition for a solution of equations (\*) is that  $a \notin S$  and we prove that all solutions of (\*\*) with parameter  $a = 2^{2^{n+1}}$  are constant multiples of  $F_2^{(2^n)}$ . In Section 4 we construct the solution of (\*) using a modification of the method of the successive approximations. The solution of equations (1) is derived from the solution of (\*) in the following way.

**Corollary 2.1.** Let  $a \notin S$  and  $f(x)$  be the unique solution of equations (\*). If  $d(2-a) = c(a+2)$ , then equations (1) have a unique  $C^1$  solution  $F(x)$  where  $F(x) = \frac{cf(x)}{2-a}$  if  $a \neq 2$ . When  $a = 2$  and  $c = 0$  the solution is  $F(x) = \frac{d}{4}f(x)$ .

### 3 Necessary Conditions.

In Section 2 we found infinitely many solutions of (\*\*) for every  $a \in S$ . In this Section we prove that these are the only solutions of (\*\*) and that (\*) has no solution if  $a \in S$ . Let  $f(x)$  be a solution of (\*) or (\*\*) and denote  $b_k = f\left(\frac{1}{2^k}\right)$ .

In Proposition 2.1 we showed that  $f\left(\frac{1}{2} + x\right) + f\left(\frac{1}{2} - x\right) = c + d$ . Now we show that similar property holds at each point  $\frac{1}{2^n}$ .

**Lemma 3.1.** The function  $f$  satisfies

$$f\left(\frac{1}{2^n} + x\right) + (-1)^{n+1}f\left(\frac{1}{2^n} - x\right) = p_n(x) \quad (3.1)$$

for  $0 \leq x \leq \frac{1}{2^n}$ . The polynomials  $p_n$  are defined recursively by

$$\begin{cases} p'_{2l-1}(x) = ap_{2l-2}(2x) \\ p_{2l-1}(0) = 2b_{2l-1} \end{cases} \quad (3.2)$$

and

$$\begin{cases} p'_{2l}(x) = ap_{2l-1}(2x) \\ p_{2l}(0) = 0 \end{cases} \quad (3.3)$$

where

$$p_1(x) = \begin{cases} 4 & \text{if } f \text{ is a solution of } (*) \\ 0 & \text{if } f \text{ is a solution of } (**). \end{cases}$$

PROOF. We prove Lemma 3.1 by induction on  $n$ . When  $n = 1$  equation (3.1) is satisfied by Proposition 2.1. Suppose that

$$f\left(\frac{1}{2^n} + x\right) + (-1)^{n+1}f\left(\frac{1}{2^n} - x\right) = p_n(x).$$

Put  $x = 2t$  to obtain

$$f\left(\frac{1}{2^n} + 2t\right) + (-1)^{n+1}f\left(\frac{1}{2^n} - 2t\right) = p_n(2t).$$

From the second equations of (\*) and (\*\*) with  $x = \frac{1}{2^n} + 2t$  and  $x = \frac{1}{2^n} - 2t$  we have that

$$f\left(\frac{1}{2^n} - 2t\right) = \frac{1}{a}f'\left(\frac{1}{2^{n+1}} - t\right)$$

and

$$f\left(\frac{1}{2^n} + 2t\right) = \frac{1}{a}f'\left(\frac{1}{2^{n+1}} + t\right).$$

Then

$$f'\left(\frac{1}{2^{n+1}} + t\right) + (-1)^{n+1}f'\left(\frac{1}{2^{n+1}} - t\right) = ap_n(2t).$$

By integrating the above equation from 0 to  $x$  we obtain

$$\begin{aligned} \int_0^x \left[ f'\left(\frac{1}{2^{n+1}} + t\right) + (-1)^{n+1}f'\left(\frac{1}{2^{n+1}} - t\right) \right] dt &= \int_0^x ap_n(2t) dt \\ \left[ f\left(\frac{1}{2^{n+1}} + t\right) + (-1)^{n+2}f\left(\frac{1}{2^{n+1}} - t\right) \right]_0^x &= a \int_0^x p_n(2t) dt \\ f\left(\frac{1}{2^{n+1}} + x\right) + (-1)^{n+2}f\left(\frac{1}{2^{n+1}} - x\right) - \gamma f\left(\frac{1}{2^{n+1}}\right) &= a \int_0^x p_n(2t) dt \end{aligned}$$

where  $\gamma = 1 + (-1)^{n+2}$ .

**Case 1**  $n$  is even. Then  $\gamma = 0$ , and when  $n = 2l$ , we have that

$$\begin{aligned} f\left(\frac{1}{2^{2l+1}} + x\right) + f\left(\frac{1}{2^{2l+1}} - x\right) - 2f\left(\frac{1}{2^{2l+1}}\right) &= a \int_0^x p_{2l}(2t) dt \\ f\left(\frac{1}{2^{2l+1}} + x\right) + f\left(\frac{1}{2^{2l+1}} - x\right) &= 2b_{2l+1} + a \int_0^x p_{2l}(2t) dt. \end{aligned}$$

Therefore,  $f$  satisfies (3.1) with  $n = 2l + 1$  and

$$p_{2l+1}(x) = 2b_{2l+1} + a \int_0^x p_{2l}(2t) dt. \quad (3.4)$$

**Case 2**  $n$  is odd. Then  $\gamma = 0$ , and when  $n = 2l - 1$  we have that

$$\begin{aligned} f\left(\frac{1}{2^{2l}} + x\right) - f\left(\frac{1}{2^{2l}} - x\right) &= a \int_0^x p_{2l-1}(2t) dt \\ p_{2l}(x) &= a \int_0^x p_{2l-1}(2t) dt. \end{aligned} \quad (3.5)$$

We obtain equations (3.2) and (3.3) from (3.4) and (3.5) by differentiation with respect to  $x$ .  $\square$

The solution  $f$  of (1) is polynomially divided because it satisfies (3.1). If  $f$  is known on the interval  $\left[0, \frac{1}{2^m}\right]$ , then we can extend it to  $\left[0, \frac{1}{2^{m-1}}\right]$  with formula (3.1) and  $n = m$ . By using the same procedure  $m - 1$  times with  $n = m - 1, m - 2, \dots, 1$  we can reconstruct  $f$  on the interval  $[0, 1]$ . Even more, if the function  $f$  is known on an arbitrary sub-interval  $(a, b)$  of  $[0, 1]$  where  $a < \frac{s}{2^m} < \frac{s+1}{2^m} < b$ , for some integers  $s$  and  $m$ , then using the reverse procedure we can find the values of  $f$  on  $\left[0, \frac{1}{2^m}\right]$  and then extend  $f$  to the interval  $[0, 1]$ . In this way, we can reconstruct the solution of (1) from any sub-interval of  $[0, 1]$  using only the polynomials  $p_n$ . Now we express the coefficients of  $p_{2n}(x)$  with  $b_1, b_3, \dots, b_{2n-1}$ . From (3.1) and  $x = \frac{1}{2^n}$  we can find the values of  $p_n$  and  $x = \frac{1}{2^n}$ .

**Corollary 3.1.** *Suppose that  $f$  is a solution of  $(*)$ . Then*

$$p_{2n} \left( \frac{1}{2^{2n}} \right) = b_{2n-1} - 2 + a \text{ and } p_{2n+1} \left( \frac{1}{2^{2n+1}} \right) = b_{2n} + 2 - a.$$

**Corollary 3.2.** *Suppose that  $f$  is a solution of  $(**)$ . Then*

$$p_n \left( \frac{1}{2^n} \right) = b_{n-1}.$$

In the next lemma we find a formula for the polynomials  $p_{2n}$ .

**Lemma 3.2.** *The polynomials  $p_n$  with even index are given by*

$$p_{2n}(x) = \sum_{k=1}^n \frac{a^{2k-1} 2^{2k^2-3k+2} b_{2n-2k+1} x^{2k-1}}{(2k-1)!}. \quad (3.6)$$

PROOF. By Lemma 3.1,  $p_{2n}$  is a polynomial of degree  $2n-1$ . The coefficients may be computed by Taylor's formula. Compute successive derivatives using Lemma 3.1:

$$\begin{aligned} p'_{2n}(x) &= ap_{2n-1}(2x) \\ p''_{2n}(x) &= 2ap'_{2n-1}(2x) = 2a^2 p_{2n-2}(4x) \\ p'''_{2n}(x) &= 8a^2 p'_{2n-2}(4x) = 22^2 a^3 p_{2n-3}(8x) = 2^3 a^3 p_{2n-3}(8x) \\ p^{(4)}_{2n}(x) &= 64a^3 p'_{2n-3}(8x) = 22^2 2^3 a^4 p_{2n-4}(16x) = 2^6 a^4 p_{2n-4}(16x), \end{aligned}$$

and in general, by induction

$$p_{2n}^{(l)}(x) = 2^{1+2+3+\dots+(l-1)} a^l p_{2n-l}(2^l x) = 2^{\frac{l(l-1)}{2}} a^l p_{2n-l}(2^l x).$$

But  $p_{2n-2k}(0) = 0$  and  $p_{2n-2k+1}(0) = 2b_{2n-2k+1}$ ; so

$$p_{2n}(x) = \sum_{k=1}^{2n-1} \frac{p_{2n}^{(k)}(0) x^k}{k!} = \sum_{k=1}^n \frac{a^{2k-1} 2^{2k^2-3k+2} b_{2n-2k+1} x^{2k-1}}{(2k-1)!}. \quad \square$$

In Lemma 2.1, we found infinitely many solutions

$$f(x) = rF_2^{(2n)}(x)$$

of equations  $(**)$  for  $a = 2^{2n+1} \in S$  and  $r \in \mathbb{R}$ . In the next corollary, we show that  $a \in S$  is a necessary condition for a solution of  $(**)$ .

**Corollary 3.3.** *Let  $f$  be a nonzero solution of (\*\*) such that  $b_1 = \dots = b_{2n-1} = 0$  and  $b_{2n+1} \neq 0$ . Then  $a = 2^{2n+1}$ .*

PROOF. By Lemma 3.2,

$$p_{2n+2}(x) = \sum_{k=1}^{n+1} \frac{a^{2k-1} 2^{2k^2-3k+2} b_{2n-2k+3} x^{2k-1}}{(2k-1)!} = 2ab_{2n+1}x.$$

Then

$$f\left(\frac{1}{2^{2n+2}} + x\right) - f\left(\frac{1}{2^{2n+2}} - x\right) = 2ab_{2n+1}x.$$

Put  $x = \frac{1}{2^{2n+2}}$  to obtain

$$f\left(\frac{1}{2^{2n+1}}\right) - f(0) = \frac{ab_{2n+1}}{2^{2n+1}} \text{ and } b_{2n+1} = \frac{ab_{2n+1}}{2^{2n+1}}.$$

Therefore,  $a = 2^{2n+1}$  as required.  $\square$

Now we use Lemma 3.2 to find a formula which relates the numbers  $b_1, b_3, \dots, b_{2n-1}$ .

**Lemma 3.3.** *The numbers  $\{b_{2n-1}\}_{n=0}^{\infty}$  satisfy*

$$b_{2n-1} (2^{2n-1} - a) = 2^{2n-1} \left( 2 - a + \sum_{k=2}^n \frac{a^{2k-1} 2^{2k^2-3k+2-2n(2k-1)} b_{2n-2k+1}}{(2k-1)!} \right). \quad (3.7)$$

PROOF. By Corollary 3.1,

$$p_{2n}\left(\frac{1}{2^{2n}}\right) = b_{2n-1} - 2 + a.$$

From (3.6) and  $x = \frac{1}{2^{2n}}$  we have that

$$p_{2n}\left(\frac{1}{2^{2n}}\right) = \sum_{k=1}^n \frac{a^{2k-1} 2^{2k^2-3k+2-2n(2k-1)} b_{2n-2k+1}}{(2k-1)!}.$$

Then

$$b_{2n-1} \left(1 - \frac{a}{2^{2n-1}}\right) = 2 - a + \sum_{k=2}^n \frac{a^{2k-1} 2^{2k^2-3k+2-2n(2k-1)} b_{2n-2k+1}}{(2k-1)!}. \quad \square$$

So far we have found formulas to compute the polynomials  $p_{2n}(x)$  and the numbers  $b_{2n-1}$ .

**Corollary 3.4.** *Let  $f$  be a solution of (\*) or (\*\*). The numbers  $b_{2n}$  and the coefficients of  $p_{2n+1}$  are obtained from  $\{b_{2k-1}\}_{k=1}^n$  by*

$$p_{2n+1}(x) = \sum_{k=0}^n \frac{a^{2k} 2^{2k^2-k+1} b_{2n-2k+1} x^{2k}}{(2k)!}$$

and

$$b_{2n} = \begin{cases} \sum_{k=0}^n \frac{a^{2k} 2^{2k^2-k+1-2k(2n+1)} b_{2n-2k+1}}{(2k)!} + a - 2 & \text{if } f \text{ satisfies (*)} \\ \sum_{k=0}^n \frac{a^{2k} 2^{2k^2-k+1-2k(2n+1)} b_{2n-2k+1}}{(2k)!} & \text{if } f \text{ satisfies (**)} \end{cases} \quad (3.8)$$

PROOF. We have that

$$p_{2n+1}(x) = 2b_{2n+1} + a \int_0^x p_{2n}(2t) dt.$$

From (3.6) and  $x = 2t$ ,

$$p_{2n}(2t) = \sum_{k=1}^n \frac{a^{2k-1} 2^{2k^2-k+1} b_{2n-2k+1} t^{2k-1}}{(2k-1)!}.$$

Then

$$\begin{aligned} p_{2n+1}(x) &= 2b_{2n+1} + a \sum_{k=1}^n \int_0^x \frac{a^{2k-1} 2^{2k^2-k+1} b_{2n-2k+1} t^{2k-1}}{(2k-1)!} dt \\ &= \sum_{k=0}^n \frac{a^{2k} 2^{2k^2-k+1} b_{2n-2k+1} x^{2k}}{(2k)!}. \end{aligned}$$

If  $f$  is a solution of (\*), then  $p_{2n+1}\left(\frac{1}{2^{2n+1}}\right) = b_{2n} + 2 - a$  and

$$b_{2n} = \sum_{k=0}^n \frac{a^{2k} 2^{2k^2-k+1-2k(2n+1)} b_{2n-2k+1}}{(2k)!} + a - 2.$$

If  $f$  is a solution of (\*\*), then  $p_{2n+1} \left( \frac{1}{2^{2n+1}} \right) = b_{2n}$  and

$$b_{2n} = \sum_{k=0}^n \frac{a^{2k} 2^{2k^2 - k + 1 - 2k(2n+1)} b_{2n-2k+1}}{(2k)!}. \quad \square$$

**Remark 3.1.** If  $a \notin S$  and  $f$  is a solution of (\*), then the values  $b_n = f \left( \frac{1}{2^n} \right)$  are computed with formulas (3.7) and (3.8). The values of  $f$  on the set

$$D = \left\{ \frac{k}{2^n} \mid k = 1, \dots, 2^n, n \in \mathbb{N} \right\}$$

are computed from  $\{b_n\}$  with formula (3.1). The set  $D$  is dense in  $[0, 1]$  and the values of  $f$  are determined for every  $x \in [0, 1]$  from the values of  $f$  on  $D$  with  $f(x) = \lim_{n \rightarrow \infty} f(d_n)$  where  $d_n \in D$  and  $\lim_{n \rightarrow \infty} d_n = x$ . Therefore, (\*) has at most one solution for every  $a \notin S$ . We show that (\*) has a unique solution for all  $a \notin S$  in Section 4.

**Remark 3.2.** If  $a = 2^{2n+1}$  and  $f$  is a solution of (\*\*), then  $b_1 = 0$  and  $b_3, \dots, b_{2n-1}$  are computed with (3.7), but the value of  $b_{2n+1}$  cannot be computed with (3.7). If we choose  $b_{2n+1}$  to be an arbitrary number, then  $b_{2n+3}, b_{2n+5}, \dots$  may also be computed with (3.7). The values of  $b_{2n}$  may be computed from the values of  $b_{2n-1}$  with formula (3.8). Similarly to Remark 3.1, the values of  $f$  on  $D$  may be computed with (3.1). Therefore, (\*\*) has at most one solution for every choice of  $b_{2n+1}$ . This solution is  $\frac{b_{2n+1} F_2^{(2n)}(x)}{F_2^{(2n)} \left( \frac{1}{2^{2n+1}} \right)}$ . Therefore, all solutions of (\*\*) are  $\left\{ r F_2^{(2n)}(x) \mid r \in \mathbb{R} \right\}$ .

The graphs of  $f$  obtained by calculating the values of  $f$  on  $D$  for  $a = -32, -4, 7.9, 8.1, 16, 50$  are given on Fig. 2. Now we show that (\*) has no solution when  $a \in S$ . Let  $a = 2^{2m-1}$ . With this value of  $a$ , equation (3.7)

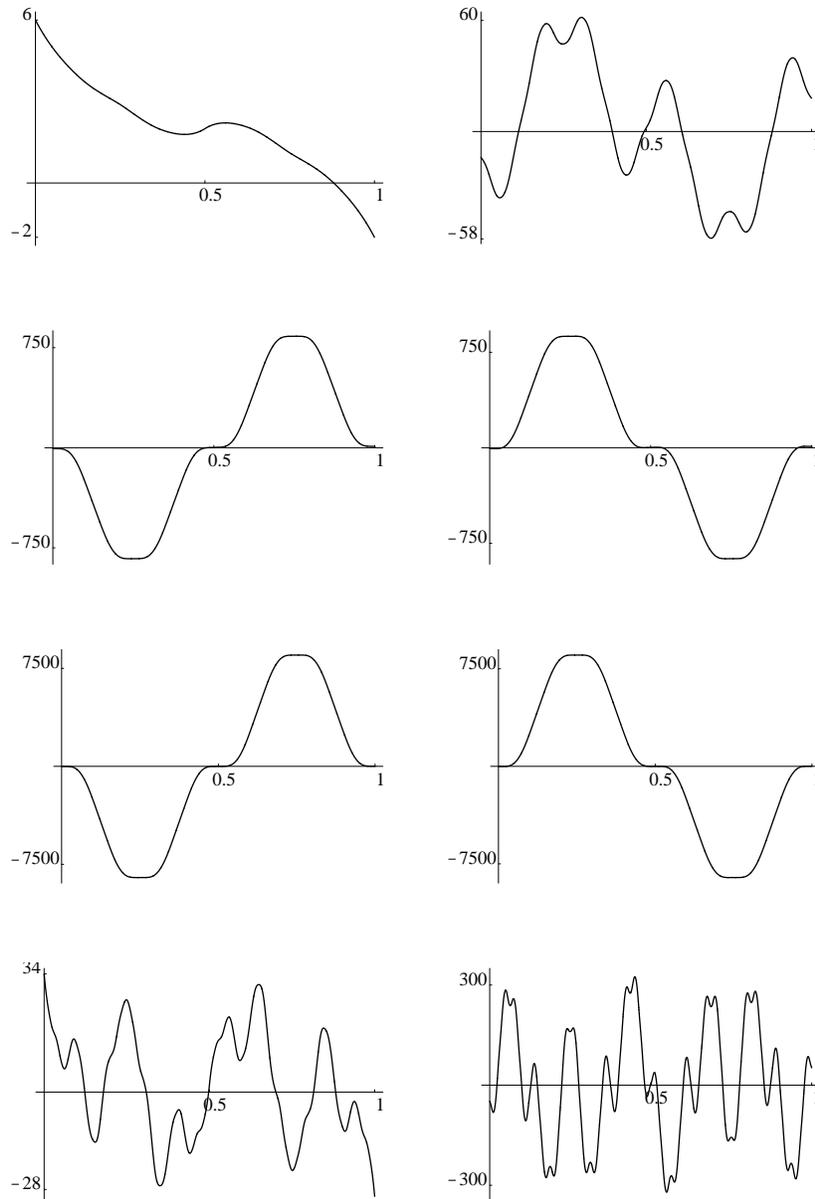


Fig. 2: The solutions of equations (\*) for  $a = -4, 16$  (top),  $a = 7.9, 8.1$  and  $a = 7.99, 8.01$  (middle) and  $a = -32, 50$  (bottom)

becomes

$$\begin{aligned}
b_{2n-1} (2^{2n-1} - 2^{2m-1}) &= 2^{2n-1} \left( 2 - 2^{2m-1} + \right. \\
&\quad \left. \sum_{k=2}^n \frac{2^{(2m-1)(2k-1)} 2^{2k^2-3k+2-2n(2k-1)} b_{2n-2k+1}}{(2k-1)!} \right) \\
b_{2n-1} (1 - 2^{2(m-n)}) &= 2 - 2^{2m-1} + \\
&\quad \sum_{k=2}^n \frac{2^{2k^2-3k+2+(2m-2n-1)(2k-1)} b_{2n-2k+1}}{(2k-1)!}
\end{aligned} \tag{3.9}$$

$$b_{2n-1} = \frac{1}{1 - 4^{m-n}} \left( 2(1 - 4^{m-1}) + \sum_{k=2}^n \frac{2^{2k^2-3k+2+(2m-2n-1)(2k-1)} b_{2n-2k+1}}{(2k-1)!} \right).$$

The numbers  $b_1 = 2, b_3, \dots, b_{2m-3}$  are computed with the above formula. When  $n = m$ , formula (3.9) becomes

$$\begin{aligned}
0 &= 2(1 - 4^{m-1}) + \sum_{k=2}^m \frac{2^{2k^2-5k+3} b_{2m-2k+1}}{(2k-1)!} \\
4^{m-1} - 1 &= \sum_{k=2}^m \frac{2^{2k^2-5k+2} b_{2m-2k+1}}{(2k-1)!}.
\end{aligned} \tag{3.10}$$

Formula (3.10) gives a relation between  $b_1, \dots, b_{2m-3}$  and is a necessary condition for existence of a solution of equations (\*) when  $a = 2^{2m-1}$ . In Lemma 3.5 we show that (3.10) is not satisfied. In the proof of Lemma 3.5 we use Proposition 3.1 and Lemma 3.4.

**Proposition 3.1.** *Let  $e_k$  be the power of 3 in  $(2k-1)!$ . Then  $e_k \leq k-1$ .*

PROOF.

$$e_k = \sum_{3^r \leq 2k-1} \left\lfloor \frac{2k-1}{3^r} \right\rfloor \leq \sum_{r=1}^{r_0} \frac{2k-1}{3^r}$$

where  $r_0 = \lfloor \log_3(2k-1) \rfloor$ .

$$e_k \leq \frac{2k-1}{3} \frac{1 - \frac{1}{3^{r_0}}}{1 - \frac{1}{3}} = \frac{2k-1}{2} \left( 1 - \frac{1}{3^{r_0}} \right) < \frac{2k-1}{2} < k.$$

Therefore,  $e_k \leq k-1$ . □

The numbers  $b_1, b_3, \dots, b_{2m-3}$  are rational numbers in lowest terms. Let  $d_{2n-1}$  be the power of 3 in the denominator of  $b_{2n-1}$ . The sequence  $\{d_{2n-1}\}_{n=1}^{m-1}$  is increasing and satisfies the following inequality.

**Lemma 3.4.**

$$d_{2n-1} - d_{2n-3} \geq 2 \quad (3.11)$$

PROOF. We prove (3.11) by induction on  $n$ .  $b_1 = 2$  and  $d_1 = 0$ . By (3.7),

$$\begin{aligned} b_3 &= \frac{1}{1-4^{m-2}} \left( 2(1-4^{m-1}) + \frac{2^{8-6+2+1+(2m-5)3}}{3!} \right) \\ &= \frac{6(1-4^{m-1}) + 2^{6m-11}}{3(1-4^{m-2})}. \end{aligned}$$

Let's denote by  $d(k)$  the power of 3 in  $4^k - 1$ . Then  $d(k) \geq 1$  because

$$4^k - 1 = 3(4^{k-1} + 4^{k-2} + \dots + 1).$$

The numerator of  $b_3$  is not divisible by 3. Therefore  $d_3 = d(m-2) + 1 \geq 2$  and (3.11) is satisfied for  $n = 2$ . Suppose that  $d_{2k-1} - d_{2k-3} \geq 2$  for every  $k = 2, 3, \dots, n-1$ .

$$\begin{aligned} b_{2n-1} &= \frac{1}{1-4^{m-n}} \left( 2(1-4^{m-1}) + \frac{64^{m-n} b_{2n-3}}{3} + \right. \\ &\quad \left. \sum_{k=3}^n 2^{2k^2-3k+2+(2m-2n-1)(2k-1)} \frac{b_{2n-2k+1}}{(2k-1)!} \right). \end{aligned}$$

By the induction assumption,

$$d_{2n-3} - d_{2n-2k+1} = \sum_{l=2}^{k-1} [d_{2n-2l+1} - d_{2n-2l-1}] \geq 2(k-2) = 2k-4$$

$$d_{2n-2k+1} + 2k-4 \leq d_{2n-3}.$$

The power of 3 in the denominator of  $\frac{b_{2n-2k+1}}{(2k-1)!}$  is  $d_{2n-2k+1} + e_k$ . From Proposition 3.1, we have that

$$d_{2n-2k+1} + e_k \leq d_{2n-2k+1} + k-1 \leq d_{2n-2k+1} + 2k-4 \leq d_{2n-3} < d_{2n-3} + 1$$

for  $3 \leq k \leq n$ . The power of 3 in the denominator of  $\frac{b_{2n-3}}{3!}$  is  $d_{2n-3} + 1$ . Therefore, the power of 3 in the denominator of  $\frac{b_{2n-3}}{3!}$  is greater than the power of 3 in the denominator of  $\frac{b_{2n-2k+1}}{(2k-1)!}$  for every  $k = 3, 4, \dots, n$ . Then the power of 3 in the denominator of

$$\frac{64^{m-n}b_{2n-3}}{3} + \sum_{k=2}^n 2^{2k^2-3k+2-(2m-2n-1)(2k-1)} \frac{b_{2n-2k+1}}{(2k-1)!}$$

is exactly  $d_{2n-3} + 1$ , and so the power of 3 in the denominator of

$$\frac{1}{1-4^{m-n}} \left( \frac{64^{m-n}b_{2n-3}}{3} + \sum_{k=3}^n 2^{2k^2-3k+2-(2m-2n-1)(2k-1)} \frac{b_{2n-2k+1}}{(2k-1)!} \right)$$

is equal to  $d_{2n-3} + d(m-n) + 1$ . Then

$$d_{2n-1} = d_{2n-3} + d(m-n) + 1 \geq d_{2n-3} + 2. \quad \square$$

In Lemma 3.5, we prove that the necessary condition (3.10) for a solution of (\*) is not satisfied.

**Lemma 3.5.**

$$4^{m-1} - 1 \neq \sum_{k=2}^n \frac{2^{2k^2-5k+2} b_{2m-2k+1}}{(2k-1)!}.$$

PROOF. We want to show that the denominator of

$$\sum_{k=2}^n 2^{2k^2-5k+2} \frac{b_{2m-2k+1}}{(2k-1)!}$$

is divisible by 3.

$$\sum_{k=2}^n 2^{2k^2-5k+2} \frac{b_{2m-2k+1}}{(2k-1)!} = \frac{b_{2m-3}}{3!} + \sum_{k=3}^n 2^{2k^2-5k+2} \frac{b_{2m-2k+1}}{(2k-1)!}$$

The power of 3 in the denominator of

$$2^{2k^2-5k+2} \frac{b_{2m-2k+1}}{(2k-1)!}$$

is  $d_{2m-2k+1} + e_k$ . By Proposition 3.1 and Lemma 3.4, we have that

$$d_{2m-2k+1} + e_k \leq d_{2m-2k+1} + k - 1 \leq d_{2m-2k+1} + 2k - 4 \leq d_{2m-3} < d_{2m-3} + 1$$

for  $3 \leq k \leq n$ . The power of 3 in the denominator of  $\frac{b_{2m-3}}{3!}$  is  $d_{2m-3} + 1$ . Therefore, the power of 3 in the denominator of

$$\sum_{k=2}^n 2^{2k^2-5k+2} \frac{b_{2m-2k+1}}{(2k-1)!}$$

is  $d_{2m-3} + 1$ . Then the denominator of

$$\sum_{k=2}^n 2^{2k^2-5k+2} \frac{b_{2m-2k+1}}{(2k-1)!}$$

is divisible by 3. Hence,

$$4^{m-1} - 1 \neq \sum_{k=2}^n \frac{2^{2k^2-5k+2} b_{2m-2k+1}}{(2k-1)!}$$

as required.  $\square$

From Lemma 3.5 it follows that equations (\*) have no solution when  $a \in S$  because the necessary condition (3.10) is not satisfied.

#### 4 Successive Approximations to the Solution.

In Section 2 and Section 3, we proved part (ii) of Theorem 2.1. We also showed that if  $a \in S$ , equations (\*) have no solution. In Section 4, we show that equations (\*) have a solution for every  $a \notin S$ . According to the following remark, this solution is unique. Remark 3.1. Let  $f$  be a solution of (\*) and  $x \in \left[0, \frac{1}{2}\right]$ . Then  $f(x) = f(0) + \int_0^x f'(t) dt = 2 - a + \int_0^x af(2t) dt = 2 - a + \frac{a}{2} \int_0^{2x} f(t) dt$ . Let  $n$  be the smallest integer such that  $2|a| < 4^n$ . Now, we use the above equation to define a sequence of functions  $\{h_k\}_{k=0}^\infty$  which approximates  $f$  on the interval  $\left[0, \frac{1}{4^{n-1}}\right]$  by

$$\begin{cases} h_0(x) & = x \\ h_k(x) & = 2 - a + \frac{a}{2} \int_0^{2x} h_{k-1}(t) dt \text{ for } 0 \leq x \leq \frac{2}{4^n} \\ h_k\left(\frac{2}{4^n} + x\right) & = p_{2n-1}(x) - h_k\left(\frac{2}{4^n} - x\right) \text{ for } 0 < x \leq \frac{2}{4^n}. \end{cases} \quad (4.1)$$

**Proposition 4.1.** *The functions  $h_k$  are continuous for all  $k \geq 2$ .*

PROOF. The functions  $h_k$  are continuous on the intervals  $\left[0, \frac{2}{4^n}\right]$  and  $\left(\frac{2}{4^n}, \frac{1}{4^{n-1}}\right]$ .

Now, we show that  $h_k$  is continuous at  $\frac{2}{4^n}$ . From the first equation of (4.1)

and  $x = \frac{2}{4^n}$ , we have that

$$\begin{aligned} h_k\left(\frac{2}{4^n}\right) &= 2 - a + \frac{a}{2} \int_0^{\frac{1}{4^{n-1}}} h_{k-1}(t) dt \\ &= 2 - a + \frac{a}{2} \int_0^{\frac{2}{4^n}} h_{k-1}(t) dt + \frac{a}{2} \int_{\frac{2}{4^n}}^{\frac{1}{4^{n-1}}} h_{k-1}(t) dt \\ &= 2 - a + \frac{a}{2} \int_0^{\frac{2}{4^n}} h_{k-1}(t) dt + \frac{a}{2} \int_0^{\frac{2}{4^n}} h_{k-1}\left(\frac{2}{4^n} + u\right) du \\ &= 2 - a + \frac{a}{2} \int_0^{\frac{2}{4^n}} h_{k-1}(t) dt + \frac{a}{2} \int_0^{\frac{2}{4^n}} \left[ p_{2n-1}(u) - h_{k-1}\left(\frac{2}{4^n} - u\right) \right] du \\ h_k\left(\frac{2}{4^n}\right) &= 2 - a + \frac{a}{2} \int_0^{\frac{2}{4^n}} p_{2n-1}(u) du. \end{aligned}$$

From (3.2), we have that  $ap_{2n-1}(u) = p'_{2n}\left(\frac{u}{2}\right)$ . Then

$$\begin{aligned} h_k\left(\frac{2}{4^n}\right) &= 2 - a + \frac{1}{2} \int_0^{\frac{2}{4^n}} p'_{2n}\left(\frac{u}{2}\right) du = 2 - a + \int_0^{\frac{1}{4^n}} p'_{2n}(u) du \\ h_k\left(\frac{2}{4^n}\right) &= 2 - a + p_{2n}\left(\frac{1}{4^n}\right) - p_{2n}(0) = 2 - a + b_{2n-1} - 2 + a = b_{2n-1}. \end{aligned}$$

From the second equation of (4.1), we obtain

$$\begin{aligned} h_k\left(\frac{2}{4^n} +\right) &= \lim_{x \rightarrow 0^+} f\left(\frac{2}{4^n} + x\right) = \lim_{x \rightarrow 0^+} \left[ p_{2n-1}(x) - h_k\left(\frac{2}{4^n} - x\right) \right] \\ h_k\left(\frac{2}{4^n} +\right) &= p_{2n-1}(0) - h_k\left(\frac{2}{4^n} -\right) = 2b_{2n-1} - b_{2n-1} = b_{2n-1}. \end{aligned}$$

Therefore,

$$h_k\left(\frac{2}{4^n}\right) = h_k\left(\frac{2}{4^n} +\right) = b_{2n-1}.$$

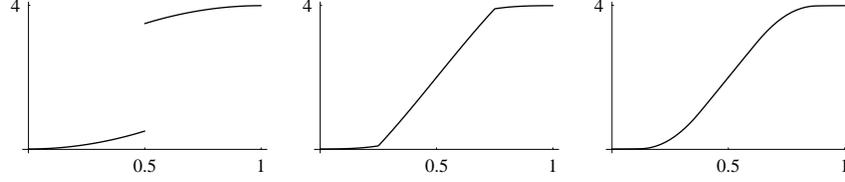


Fig. 3: The approximations  $h_1, h_2$  and  $h_3$  to the solution of (\*) for  $a = 1.99$

The function  $h_k$  is right continuous at  $x = \frac{2}{4^n}$  and so it is continuous on the interval  $\left[0, \frac{1}{4^{n-1}}\right]$ .  $\square$

When  $a = 1.99$ , then  $n = 1$ , because  $2|a| < 4$ . The sequence of functions  $\{h_k\}_{k=0}^{\infty}$  approximates the solution of (\*) on the interval  $[0, 1]$ . The graphs of the first three approximations  $h_1, h_2$  and  $h_3$  are given on Fig. 3. The solution of equations (\*) for  $a = 2$  is  $4F_2$  (Fig. 2). The solutions of (\*) for  $a = 1.99$  and  $a = 2$  differ by less than 0.015. Now we define a system of functional differential equations on the interval  $\left[0, \frac{1}{4^{k-1}}\right]$ . Let  $k$  be a positive integer, and denote by  $Eqns[k]$  the following functional differential equations.

$$\begin{cases} f'(x) & = af(2x) \text{ for } 0 \leq x \leq \frac{2}{4^k} \\ f\left(\frac{2}{4^k} + x\right) & = p_{2k-1}(x) - f\left(\frac{2}{4^k} - x\right) \text{ for } 0 < x \leq \frac{2}{4^k} \\ f(0) & = 2 - a \end{cases} \quad (Eqns[k])$$

**Proposition 4.2.** *Let  $f$  be a solution of  $Eqns[k]$ . Then*

$$f\left(\frac{1}{4^{k-1}}\right) = b_{2k-2} \quad \text{and} \quad f'\left(\frac{1}{4^{k-1}}\right) = ab_{2k-3}.$$

PROOF. From the second equation of  $Eqns[k]$  and  $x = \frac{2}{4^k}$ , we have that

$$f\left(\frac{1}{4^{k-1}}\right) = p_{2k-1}\left(\frac{2}{4^k}\right) - f(0) = p_{2k-1}\left(\frac{2}{4^k}\right) - 2 + a = b_{2k-2}.$$

By differentiating the second equation of  $Eqns[k]$ , we obtain

$$f' \left( \frac{2}{4^k} + x \right) = p'_{2^{k-1}}(x) + f' \left( \frac{2}{4^k} - x \right).$$

Put  $x = \frac{2}{4^k}$  in the above equation,

$$\begin{aligned} f' \left( \frac{1}{4^{k-1}} \right) &= p'_{2^{k-1}} \left( \frac{2}{4^k} \right) + f'(0) = ap_{2^{k-2}} \left( \frac{1}{4^{k-1}} \right) + af(0) \\ f' \left( \frac{1}{4^{k-1}} \right) &= a(b_{2^{k-3}} - 2 + a) + a(2 - a) = ab_{2^{k-3}}. \end{aligned} \quad \square$$

By Proposition 4.2, equations (\*) are the same as  $Eqns[1]$ .

**Lemma 4.1.** (i) *The sequence of functions  $h_k(x)$  converges uniformly.*  
(ii) *The limit function  $f_n(x) = \lim_{k \rightarrow \infty} h_k(x)$  is continuously differentiable and satisfies  $Eqns[n]$ .*

PROOF. (i) Let  $M_k = \sup_{x \in [0, \frac{1}{4^{k-1}}]} |h_k(x) - h_{k-1}(x)|$ . From the second equation of (4.1), we have that

$$M_k = \sup_{x \in [0, \frac{2}{4^k}]} |h_k(x) - h_{k-1}(x)|.$$

From the first equation of (4.1),

$$\begin{aligned} h_k(x) - h_{k-1}(x) &= \frac{a}{2} \int_0^{2x} h_{k-1}(u) du - \frac{a}{2} \int_0^{2x} h_{k-2}(u) du \\ |h_k(x) - h_{k-1}(x)| &\leq \frac{a}{2} \int_0^{2x} |h_{k-1}(u) - h_{k-2}(u)| du. \end{aligned}$$

Therefore,

$$M_k \leq \frac{a}{2} \int_0^{2x} M_{k-1} du = \frac{|a|}{2} 2x M_{k-1} \leq \frac{2|a|}{4^n} M_{k-1}.$$

Let  $t = \frac{2|a|}{4^n}$ . Then  $0 < t < 1$ , because  $n$  is chosen such that  $2|a| < 4^n$ . Therefore,  $M_k \leq tM_{k-1}$ . By induction, we obtain  $M_k \leq t^{k-1}M_1$ . Then

$$\begin{aligned} \sup_{x \in [0, \frac{1}{4^{n-1}}]} |h_s(x) - h_r(x)| &\leq \sum_{k=r+1}^s \sup_{x \in [0, \frac{1}{4^{n-1}}]} |h_k(x) - h_{k-1}(x)| \\ &\leq \sum_{k=r+1}^s M_k \leq M_1 \sum_{k=r+1}^s t^{k-1} \leq M_1 \sum_{k=r}^{\infty} t^k \leq \frac{M_1 t^r}{1-t} \end{aligned}$$

for  $s > r$ . Therefore,  $h_k$  is a Cauchy sequence, and so, it converges uniformly.

(ii) The function  $f_n$  is continuous because it is a uniform limit of continuous functions. In the proof of Proposition 4.1, we showed that  $h_k\left(\frac{2}{4^n}\right) = b_{2n-1}$  for  $k \geq 2$ . Then  $f_n\left(\frac{2}{4^n}\right) = b_{2n-1}$ . By letting  $k \rightarrow \infty$  in (4.1), we obtain

$$\begin{cases} f_n(x) = 2 - a + \frac{a}{2} \int_0^{2x} f_n(t) dt & \text{for } 0 \leq x \leq \frac{2}{4^n} \\ f_n\left(\frac{2}{4^n} + x\right) = p_{2n-1}(x) - f_n\left(\frac{2}{4^n} - x\right) & \text{for } 0 < x \leq \frac{2}{4^n}. \end{cases} \quad (4.2)$$

Form the first and second equations of (4.2), the function  $f_n$  is continuously differentiable on the intervals  $\left[0, \frac{2}{4^n}\right]$  and  $\left(\frac{2}{4^n}, \frac{1}{4^{n-1}}\right]$ . Now, we show that  $f_n$  is differentiable at  $x = \frac{2}{4^n}$ . Let  $f'(x-)$  and  $f'(x+)$  denote

$$f'(x+) = \lim_{t \rightarrow x^+} \frac{f(t) - f(x)}{t - x} \quad \text{and} \quad f'(x-) = \lim_{t \rightarrow x^-} \frac{f(t) - f(x)}{t - x}.$$

Then

$$\begin{aligned} f'_n\left(\frac{2}{4^n} +\right) &= \lim_{x \rightarrow 0^+} \frac{f_n\left(\frac{2}{4^n} + x\right) - f_n\left(\frac{2}{4^n}\right)}{x} \\ &= \lim_{x \rightarrow 0^+} \frac{p_{2n-1}(x) - f_n\left(\frac{2}{4^n} - x\right) - f_n\left(\frac{2}{4^n}\right)}{x} \\ &= \lim_{x \rightarrow 0^+} \frac{p_{2n-1}(x) - p_{2n-1}(0) - f_n\left(\frac{2}{4^n} - x\right) + f_n\left(\frac{2}{4^n}\right)}{x} \end{aligned}$$

$$\begin{aligned}
& \left( \text{because } p_{2n-1}(0) - f_n\left(\frac{2}{4^n}\right) = 2b_{2n-1} - b_{2n-1} = f_n\left(\frac{2}{4^n}\right) \right) \\
&= \lim_{x \rightarrow 0^+} \frac{p_{2n-1}(x) - p_{2n-1}(0)}{x} + \lim_{x \rightarrow 0^+} \frac{f_n\left(\frac{2}{4^n} - x\right) - f_n\left(\frac{2}{4^n}\right)}{-x} \\
&= p'_{2n-1}(0) + f'_n\left(\frac{2}{4^n} -\right) = ap_{2n-2}(0) + f'_n\left(\frac{2}{4^n} -\right) \\
&= f'_n\left(\frac{2}{4^n} -\right)
\end{aligned}$$

Therefore,  $f_n$  is differentiable at  $x = \frac{2}{4^n}$ . From the second equation of (4.3), we have that

$$f'_n\left(\frac{2}{4^n} + x\right) = p'_{2n-1}(x) + f'_n\left(\frac{2}{4^n} - x\right).$$

Then

$$\lim_{x \rightarrow 0^+} f'_n\left(\frac{2}{4^n} + x\right) = p'_{2n-1}(0) + \lim_{x \rightarrow 0^+} f'_n\left(\frac{2}{4^n} - x\right) = f'_n\left(\frac{2}{4^n} -\right).$$

Therefore,  $f_n$  is continuously differentiable at  $x = \frac{2}{4^n}$ , and so  $f_n$  is continuously differentiable at each point of the interval  $\left[0, \frac{1}{4^{n-1}}\right]$ . From the first equation of (4.2), we have that  $f_n(0) = 2 - a$  and  $f'_n(x) = af_n(2x)$  for  $0 \leq x \leq \frac{2}{4^n}$ . Therefore,  $f_n$  satisfies Eqns[n] as required.  $\square$

**Lemma 4.2.** *Suppose that  $g_1(x)$  is a continuously differentiable function which satisfies Eqns[k + 1]. Let  $g_2(x)$  be an extension of  $g_1(x)$  defined by*

$$\begin{cases} g_2(x) &= g_1(x) & \text{if } 0 \leq x \leq \frac{1}{4^k} \\ g_2\left(\frac{1}{4^k} + x\right) &= p_{2k}(x) + g_2\left(\frac{1}{4^k} - x\right) & \text{if } 0 < x \leq \frac{1}{4^k} \\ g_2\left(\frac{2}{4^k} + x\right) &= p_{2k-1}(x) - g_2\left(\frac{2}{4^k} - x\right) & \text{if } 0 < x \leq \frac{2}{4^k} \end{cases} \quad (4.3)$$

Then  $g_2(x)$  is continuously differentiable on  $\left[0, \frac{1}{4^{k-1}}\right]$ , and satisfies Eqns[k].

PROOF. The function  $g_2$  is continuously differentiable on the intervals  $\left[0, \frac{1}{4^k}\right]$ ,  $\left(\frac{1}{4^k}, \frac{2}{4^k}\right]$  and  $\left(\frac{2}{4^k}, \frac{1}{4^{k-1}}\right]$ . Now, we show that  $g_2$  is differentiable at  $x = \frac{1}{4^k}$ .

$$\begin{aligned}
g_2' \left( \frac{1}{4^k} + \right) &= \lim_{x \rightarrow 0^+} \frac{g_2 \left( \frac{1}{4^k} + x \right) - g_2 \left( \frac{1}{4^k} \right)}{x} \\
&= \lim_{x \rightarrow 0^+} \frac{p_{2k}(x) + g_2 \left( \frac{1}{4^k} - x \right) - g_2 \left( \frac{1}{4^k} \right)}{x} \\
&= \lim_{x \rightarrow 0^+} \frac{p_{2k}(x) - p_{2k}(0)}{x} - \lim_{x \rightarrow 0^+} \frac{g_2 \left( \frac{1}{4^k} - x \right) - g_2 \left( \frac{1}{4^k} \right)}{-x} \\
&= p_{2k}'(0) - g_2' \left( \frac{1}{4^k} - \right)
\end{aligned}$$

By Proposition 4.2,  $g_2' \left( \frac{1}{4^k} - \right) = ab_{2k-1}$ . Then

$$g_2' \left( \frac{1}{4^k} + \right) = ap_{2k-1}(0) - ab_{2k-1} = 2ab_{2k-1} - ab_{2k-1} = g_2' \left( \frac{1}{4^k} - \right).$$

Therefore,  $g_2$  is differentiable at  $x = \frac{1}{4^k}$ , and  $g_2' \left( \frac{1}{4^k} \right) = ab_{2k-1}$ . Now we show that  $g_2'$  is continuous at  $x = \frac{1}{4^k}$ . It is enough to show that

$$\lim_{x \rightarrow 0^+} g_2' \left( \frac{2}{4^k} + x \right) = g_2' \left( \frac{2}{4^k} \right)$$

because  $g_2'$  is continuous on  $\left[0, \frac{1}{4^k}\right]$ . From the second equation of (4.3), we have that

$$g_2' \left( \frac{1}{4^k} + x \right) = p_{2k}'(x) - g_2' \left( \frac{1}{4^k} - x \right).$$

Then

$$\begin{aligned} \lim_{x \rightarrow 0^+} g'_2 \left( \frac{1}{4^k} + x \right) &= \lim_{x \rightarrow 0^+} \left[ p'_{2k}(x) - g'_2 \left( \frac{1}{4^k} - x \right) \right] \\ &= p'_{2k}(0) - g'_2 \left( \frac{1}{4^k} - \right) = 2ab_{2k-1} - ab_{2k-1} \\ &= ab_{2k-1} = g'_2 \left( \frac{1}{4^k} \right). \end{aligned}$$

Therefore,  $g_2$  is continuously differentiable at  $x = \frac{1}{4^k}$ . The proof that  $g_2$  is continuously differentiable at  $x = \frac{2}{4^k}$  is similar. Now we show that  $g_2$  satisfies the conditions of  $Eqns[k]$ . By the definition of  $g_2$ , we have that  $g_2(0) = 0$  and

$$g_2 \left( \frac{2}{4^k} + x \right) = p_{2k-1}(x) - g_2 \left( \frac{2}{4^k} - x \right)$$

for  $0 < x \leq \frac{2}{4^k}$ . Also,  $g'_2(x) = ag_2(2x)$  for  $0 < x \leq \frac{2}{4^{k+1}}$  because  $g_2(x) = g_1(x)$  on the interval  $\left[0, \frac{1}{4^k}\right]$  and  $g_1$  satisfies  $Eqns[k+1]$ . By the third equation of  $Eqns[k+1]$ , we have that

$$g_2 \left( \frac{2}{4^{k+1}} + x \right) = p_{2k+1}(x) - g_2 \left( \frac{2}{4^{k+1}} - x \right)$$

for  $0 < x \leq \frac{2}{4^{k+1}}$ . By differentiating the above equation, we obtain

$$\begin{aligned} g'_2 \left( \frac{2}{4^{k+1}} + x \right) &= p'_{2k+1}(x) + g'_2 \left( \frac{2}{4^{k+1}} - x \right) \\ g'_2 \left( \frac{2}{4^{k+1}} + x \right) &= ap_{2k}(2x) + ag_2 \left( \frac{1}{4^k} - 2x \right) = ag_2 \left( \frac{1}{4^k} + 2x \right). \end{aligned}$$

By the second equation of (4.3),

$$g_2 \left( \frac{1}{4^k} + x \right) = p_{2k}(x) + g_2 \left( \frac{1}{4^k} - x \right)$$

for  $0 < x \leq \frac{1}{4^k}$ . Then

$$\begin{aligned} g'_2 \left( \frac{1}{4^k} + x \right) &= p'_{2k}(x) - g'_2 \left( \frac{1}{4^k} - x \right) \\ g'_2 \left( \frac{1}{4^k} + x \right) &= ap_{2k-1}(2x) - ag_2 \left( \frac{2}{4^k} - 2x \right) = ag_2 \left( \frac{2}{4^k} + 2x \right). \end{aligned}$$

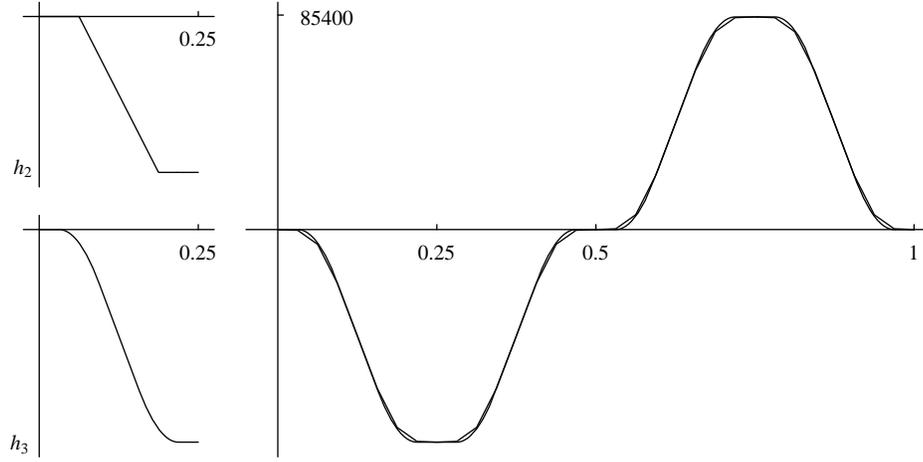


Fig. 4: The graphs of  $h_2$  and  $h_3$  (left) and  $\bar{f}_3$  and  $f$  on the same axis (right) for  $a = 7.999$

Therefore,  $g'_2(x) = ag_2(2x)$  for all  $0 < x \leq \frac{2}{4^k}$ . Then  $g_2$  is a continuously differentiable function on  $\left[0, \frac{1}{4^{k-1}}\right]$  which satisfies  $Eqns[k]$ .  $\square$

Let  $n = \lfloor \log_4 2a \rfloor + 1$ . In Lemma 4.1, we showed that  $f_n$  satisfies  $Eqns[n]$ . Let  $\{\tilde{f}_k\}_{k=1}^n$  be a sequence of functions where  $\tilde{f}_n = f_n$ , and  $\tilde{f}_k$  is the extension of  $\tilde{f}_{k+1}$  to the interval  $\left[0, \frac{1}{4^{k-1}}\right]$  with formulas (4.3) for  $k = 1, 2, \dots, n - 1$ . By Lemma 4.2, the functions  $\tilde{f}_k$  satisfy  $Eqns[k]$  for all  $k = 1, 2, \dots, n$ .

**Corollary 4.1.** *The function  $\tilde{f}_1$  is continuously differentiable and satisfies equations (\*).*

PROOF. The function  $\tilde{f}_1$  satisfies  $Eqns[1]$  and by Proposition 4.2:  $\tilde{f}_1(1) = 2 + a$ . Therefore  $\tilde{f}_1$  satisfies (\*).  $\square$

When  $a = 7.999$ , the sequence  $\{h_k\}_{k=0}^\infty$  defined with (4.1) and  $h_0(x) = x$  converges to the solution of (\*) on the interval  $\left[0, \frac{1}{4}\right]$ . The graphs of  $h_2$  and  $h_3$  are given on Figure 4 (left). Let  $\bar{f}_3$  be the extension of  $h_3$  with formulas (4.3) where  $g_1 = h_3$  and  $g_2 = \bar{f}_3$ . The function  $\bar{f}_3$  is an approximation to

the solution  $f$  of (\*) on  $[0, 1]$ . Although the values of  $f$  are in the interval  $[-85400, 85400]$ , the graph of the third approximation  $f_3$  already resembles the graph of  $f$  (Fig. 4). When  $a = 2$ , the solution of (\*) is  $4F_2$  and the Fabius function  $F_2$  is infinitely differentiable and nowhere analytic in  $[0, 1]$ . Now we show that this is the only infinitely differentiable solution of (\*).

**Corollary 4.2.** *Let  $f$  be the solution of (\*) where  $a \neq 2$  and  $a \notin S$ . Then  $f''(x)$  is discontinuous at  $\frac{1}{2}$ .*

PROOF. By differentiating the first two equations of (\*) we obtain

$$\begin{cases} f''(x) = 2a^2 f(4x) & \text{if } 0 \leq x \leq \frac{1}{4}, \\ f''(x) = 2a^2 f(2 - 4x) & \text{if } \frac{1}{4} < x \leq \frac{1}{2}, \\ f''(x) = -2a^2 f(4x - 2) & \text{if } \frac{1}{2} < x \leq \frac{3}{4}, \\ f''(x) = -2a^2 f(4 - 4x) & \text{if } \frac{3}{4} < x \leq 1. \end{cases} \quad (4.4)$$

From the second and the third equations of (4.4) and  $x = \frac{1}{2}$ , we obtain

$$f''\left(\frac{1}{2}-\right) = 2a^2(2 - a)$$

and

$$f''\left(\frac{1}{2}+\right) = -2a^2(2 - a).$$

Therefore,  $f''$  is discontinuous at  $x = \frac{1}{2}$ . □

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