# Day 2: Linear Equations

Let's start with a linear equation

$$\overbrace{1\underbrace{x}_{}+2y-1z}^{\text{unknowns}}=1$$

$$\underbrace{1x+2y-1z}_{\text{coefficients}}=1$$

Because we're about to run out of letters, so let's call the unknowns $x_1, x_2, x_3$

$$1x_1+2x_2-1x_3=1$$

We can write the eqn like this as

$$(*)\quad a_1x_1+a_2x_2+a_3x_3=b$$

where $a_1=1$, $a_2=2$, $a_3=-1$, $b=1$.

The coeff $a_1, a_2, a_3, b$ are treated as constants not unknowns. We don't try to solve for them.

If we're real nerds, we can write $(*)$ as

$$\sum_{i=1}^{3} a_ix_i=b.$$

We can solve for $x_1$ as

$$x_1=1-2x_2+1x_3.$$

If we know $x_2, x_3$, this determines $x_1$. This lets us write down all solns. We'll talk about this in more generality.

# Day 2: Linear Equations

## Systems

Now, we can look at systems of linear eqns where we have a number of eqns.

Ex $1x_1 + 2x_2 - 4x_3 = 4$

(A) $3x_1 + 4x_2 - 6x_3 = 10$

There are two eqns in three unknowns:

$$\underset{\#\text{ of eqns}}{(2} \times \underset{\#\text{ of unknowns}}{3)} \text{ system}$$

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(B) Or we could have

$$x_1 - 3x_2 = 7$$
$$2x_1 + 3x_2 = 2$$
$$x_1 - x_2 = 1.$$

3 eqns in 2 unknowns $(3 \times 2)$ system

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a soln $(x_1, x_2, x_3)$ satisfies all the eqns in the system

For example in (A):

$$(x_1, x_2, x_3) = (2, 1, 0)$$

since if we sub:

$$1(2) + 2(1) - 4(0) = 4$$
$$3(2) + 4(1) - 6(0) = 10$$

# Day 2: Linear Equations



So is $(x_1, x_2, x_3) = (6, -5, -2)$:

$$1(6) + 2(-5) - 4(-2) = 4$$
$$3(6) + 4(-5) - 6(-2) = 10.$$

There are $\infty$ly many solns. We can write them as

$$x_1 = 2 - 2x_3$$
$$x_2 = 1 + 3x_3$$

for any $x_3$.

This describes a line in 3d:

$$(x_1, x_2, x_3) = (2 - 2x_3, 1 + 3x_3, x_3)$$

$$= (2, 1, 0) + x_3(-2, 3, 1)$$

In general, we can write m eqns in n unknowns

$$a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2$$
$$\vdots \qquad\qquad \vdots \qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_n$$

$(m \times n)$ system

Note There are m rows, one for each eqn

" " n columns, one for each unknown

First row, then column

# Day 2: Linear Eqns

For
$$\overset{a_{11}}{1}x_1 + \overset{a_{12}}{2}x_2 - \overset{a_{13}}{4}x_3 = \overset{b_1}{4}$$
$$\underset{a_{21}}{3}x_1 + \underset{a_{22}}{4}x_2 - \underset{a_{23}}{6}x_3 = \underset{b_2}{10}$$

We write coeffs as

$a_{ij}$ = the coeff of $x_j$ in the $i^{th}$ eqn.

We can write the system w/o the $x_i$'s

The <u>coeff matrix</u> of the system is

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

It is an $(m \times n)$-matrix

\# of rows

\# of columns

If we want to include b's, we can write <u>augmented matrix</u>

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

this bar helps us keep things straight.

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Ex
$$\begin{aligned} 1x_1 + 2x_2 - 4x_3 &= 4 \\ 3x_1 + 4x_2 - 6x_3 &= 10 \end{aligned}$$

has coeff matrix $A = \begin{bmatrix} 1 & 2 & -4 \\ 3 & 4 & -6 \end{bmatrix}$

and augmented matrix $\left[\begin{array}{ccc|c} 1 & 2 & -4 & 4 \\ 3 & 4 & -6 & 10 \end{array}\right]$

We may write $\vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$ and the augmented

matrix $[A|\vec{b}]$

We'll solve linear system by simplifying them. We have to be sure they have the same set of solns.

Def Two systems of linear eqns are <u>equivalent</u> if they have the same solution set.

$$(*) \quad \begin{aligned} &\text{i)}\ 1x_1 + 2x_2 - 4x_3 = 4 \\ &\text{ii)}\ 3x_1 + 4x_2 - 6x_3 = 10 \end{aligned}$$

$\boxed{\text{SAVE}}$

Ex A $(*)$ is equivalent to

$$\begin{aligned} &\text{i)}\ 3x_1 + 4x_2 - 6x_3 = 10 \\ &\text{ii)}\ 1x_1 + 2x_2 - 4x_3 = 4. \end{aligned}$$

We just changed the order of the eqns.

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Ex B (*) is equivalent to

i) $3x_1 + 6x_2 - 12x_3 = 12$

ii) $3x_1 + 4x_2 - 6x_3 = 10$

because we just multiplied eqn (i) by 3.
This doesn't change the solns to (i).

Ex C What if I add $(-2) \times$ (i) to (ii) but keep (i) the same:

(i)' $1x_1 + 2x_2 - 4x_3 = 4$

(ii)' $1x_1 + 0x_2 + 2x_3 = 2$

The system is equivalent because we can get (*) back by adding $2 \times$ (i)' to (ii)'.

In general:

Recall Scalar is just a fancy term for a number.

Thm If one of the following is applied to ~~the~~ a system, then the resulting system is equivalent to the original system

1) Interchange two eqns
2) Multiply an eqn by a nonzero scalar
3) Add a constant multiple of one eqn to another.

# Day 2: Linear Eqns

These are the elementary operations and denoted

1) $E_i \leftrightarrow E_j$ - rows $i$ and $j$ are interchanged
2) $kE_i$ - row $i$ is multiplied by a nonzero scalar $k$
3) $E_i + kE_j$ - $k \times j^{th}$ eqn is added to $i^{th}$ eqn

Ex A is $E_1 \leftrightarrow E_2$

Ex B is $3E_1$

Ex C is $E_2 + (-2)E_1$.

Let's use elementary operations to simplify (*)

$$E_2 + (-3)E_1: \quad \begin{array}{ll} \text{i)} & 1x_1 + 2x_2 - 4x_3 = 4 \\ \text{ii)} & \quad -2x_2 + 6x_3 = -2 \end{array}$$

$$E_1 + 1E_2: \quad \begin{array}{ll} \text{i)} & 1x_1 \qquad -2x_3 = 2 \\ \text{ii)} & \quad -2x_2 + 6x_3 = -2 \end{array}$$

$$-\tfrac{1}{2}E_2: \quad \begin{array}{ll} \text{i)} & x_1 \qquad -2x_3 = 2 \\ \text{ii)} & \quad x_2 - 3x_3 = 1. \end{array}$$

Now, we can write $x_1$ and $x_2$ in terms of $x_3$:

$$x_1 = 2 + 2x_3$$
$$x_2 = 1 + 3x_3.$$

# Day 2: Linear Eqns



To describe the set of solutions, introduce a new variable $t$ and write (so $t$ takes the place of $x_3$)

$$x_1 = 2 + 2t$$
$$x_2 = 1 + 3t$$
$$x_3 = t$$

or

$$(x_1, x_2, x_3) = (2+2t, 1+3t, t)$$

$$= (2,1,0) + t\,(2,3,1)$$

We were able to solve this because we simplified so that $x_1$ appeared in exactly one eqn and

$x_2$ " " " " .

Solved for $x_1$ and $x_2$ in terms of $x_3$.

Instead of working w/ the system, we can work w/ the augmented matrix.

**Def** The elementary row operations are

1) Interchange $i$ and $j$ ($R_i \leftrightarrow R_j$)
2) Multiply row $i$ by a nonzero scalar $k$ ($kR_i$)
3) For a constant $k$, add $k \times$ row $j$ to row $i$

$$R_i + kR_j$$

# Day 2 : Linear Eqn



"Algorithm" for solving a linear system:

1) Write the augmented matrix for the system
2) Use the row ops to find a "simpler" system
3) Solve the simpler system.

I'll explain simpler next class.

Here are some simpler systems

A) $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 6 \end{array}\right]$

$x_1 = 2$
$x_2 = 1$
$x_3 = 6$

B) $\left[\begin{array}{ccc|c} 1 & 0 & 2 & 4 \\ 0 & 1 & -3 & 5 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$x_1 + 2x_3 = 4$
$x_2 - 3x_3 = 5$

$x_1 = 4 - 2x_3$
$x_2 = 5 + 3x_3.$

If we choose $x_3$, it determines $x_1, x_2$.