Day 3: Linear Equations 2

Review row operations

Let's solve

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 1 \\ x_1 + x_2 + 2x_3 &= 2 \\ -2x_1 + x_2 \qquad &= 4 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 1 & 1 & 2 & 2 \\ -2 & 1 & 0 & 4 \end{array}\right]$$

Let's eliminate x_1 from 2^{nd} row

$$R_2 + (-1)R_1: \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & 1 \\ -2 & 1 & 0 & 4 \end{array}\right]$$

$$R_3 + 2R_1: \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & -1 & 3 & 1 \\ 0 & 5 & -2 & 6 \end{array}\right]$$

$$(-1)R_2: \left[\begin{array}{ccc|c} 1 & 2 & -1 & 1 \\ 0 & 1 & -3 & -1 \\ 0 & 5 & -2 & 6 \end{array}\right]$$

Let's eliminate x_2 from first eqn

$$R_1 + (-2)R_2: \left[\begin{array}{ccc|c} 1 & 0 & 5 & 3 \\ 0 & 1 & -3 & -1 \\ 0 & 5 & -2 & 6 \end{array}\right]$$

$$R_3 + (-5)R_2: \left[\begin{array}{ccc|c} 1 & 0 & 5 & 3 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 13 & 11 \end{array}\right]$$

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$\frac{1}{13}R_3$: $\left[\begin{array}{ccc|c} 1 & 0 & 5 & 3 \\ 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 11/13 \end{array}\right]$

$R_2 + 3R_3$: $\left[\begin{array}{ccc|c} 1 & 0 & 5 & 3 \\ 0 & 1 & 0 & 20/13 \\ 0 & 0 & 1 & 11/13 \end{array}\right]$

$R_1 + (-5)R_3$: $\left[\begin{array}{ccc|c} 1 & 0 & 0 & -16/13 \\ 0 & 1 & 0 & 20/13 \\ 0 & 0 & 1 & 11/13 \end{array}\right]$

so $x_1 = -16/13$ $x_2 = 20/13$, $x_3 = 11/13$

So we wants to put matrices in a simple form by applying row operations.

Aside: Q What if we get this matrix by applying row operations?

$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 2 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 0 & 1 \end{array}\right] \rightsquigarrow \begin{array}{l} x_1 + 5x_3 = 2 \\ x_2 + 3x_3 = 4 \\ 0 = 1 \end{array}$

It's impossible to satisfy the last eqn.
So the system had no solns to begin w/.
Inconsistent!

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One simple form for matrices is Echelon form

Def An $(m \times n)$-matrix B is in <u>echelon form</u> if

1) All rows that consist entirely of zeroes are grouped together at the bottom of the matrix
2) In every nonzero row, the first nonzero entry (going from left to right) is 1. This entry is called a pivot.
3) The pivots move from left to right as we move down the matrix.

<u>Ex</u> $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 7 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 1 & 2 & 7 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & \pi & 16 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

are in echelon form.

$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 7 \\ 0 & 1 & -3 \end{bmatrix}$ is not in echelon form since the pivot doesn't move from left to right.

For augmented matrices don't forget the entries to the right of the bar.

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Another "simple form"

Def A matrix is in reduced echelon form (REF) if it is in row echelon form and there are zeroes above and below every pivot.

Ex
$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 16 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 3 & 4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ are all in REF

We could ~~write~~ get

$$\begin{bmatrix} 1 & * & 0 & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where $*$'s are some entries that I haven't told you.

Non-ex $\begin{bmatrix} 0 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ is in echelon form, not reduced echelon form

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<u>Thm</u> Any matrix A can be put in ~~ ~~ reduced echelon form by row operations. Moreover, reduced echelon form of a matrix A is unique.

[Explain what this means]

Solving Systems in Reduced Echelon Form

<u>Ex</u> $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 4 \end{array}\right]$ corresponds to $\begin{array}{l} x_1 = 3 \\ x_2 = 1 \\ x_3 = 4. \end{array}$

That's easy!

<u>Ex</u> $\left[\begin{array}{ccc|c} 1 & 0 & 3 & 5 \\ 0 & 1 & 4 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$ corresponds to $\begin{array}{l} x_1 + 3x_3 = 5 \\ x_2 + 4x_3 = 2 \end{array}$

<u>Note</u> Columns 1 and 2 correspond to x_1 and x_2. They both have pivots in them. The 3rd column doesn't have a pivot in it.

Solve for x_1, x_2:

$$x_1 = 5 - 3x_3$$
$$x_2 = 2 - 4x_3.$$

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x_3 is unconstrained: it can be anything.
It is called a free variable. It determines x_1 and x_2 (which are called pivot variables)

There is one soln for each choice of x_3. ∞ly many solns. $\boxed{\text{Write some!}}$

Ex $\left[\begin{array}{cccc|c} 1 & 3 & 0 & -7 & 3 \\ 0 & 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$ is in reduced echelon form

Columns 1 and 3 have pivots in them. Pivot columns!
Column 2 and 4 do not. Free columns!

$$x_1 + 3x_2 - 7x_4 = 3$$
$$x_3 + 2x_4 = 4$$

Solve for x_1, x_3:

$$x_1 = 3 - 3x_2 + 7x_4$$
$$x_3 = 4 - 2x_4$$

x_2 and x_4 can be anything (free variables)
They determine x_1 and x_3 (pivot variables).
∞ly many solns. One for each choice of x_2 and x_4.
What shape do they form?

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Aside:
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3-3x_2+7x_4 \\ x_2 \\ 4-2x_4 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 4 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 7 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

It's a plane in 4d space. Two degrees of freedom!

Ex
$$\left[\begin{array}{cccc|c} 1 & 3 & 0 & 7 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$ pivots circled.

$$x_1 + 3x_2 + 7x_4 = 0$$
$$x_3 + 2x_4 = 0$$
$$\cancel{x_4}$$
$$0 = 1.$$

No solns. Inconsistent!

Ex
$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 4 \end{array}\right]$$

$x_1 = 3$, $x_2 = 4$

What can we say about x_3? It can be anything! It's free! 1 degree of freedom.

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Someone might ask you to write all 2×3 matrices in reduced echelon form.

$\begin{bmatrix} 1 & 0 & ? \\ 0 & 1 & ? \end{bmatrix}$ "?" refers to entries that we don't know. They can be any number, even zero.

2 pivots: $\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}$, $\begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

1 pivot: $\begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

0 pivots: $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

<u>Q</u> If I have a 4×5 matrix in reduced echelon form, how many pivots could I have?

Least # of pivots? 0

Most?

$$\begin{array}{c|ccccc} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & & & & & \\ 2 & & & & & \\ 3 & & & & & \\ 4 & & & & & \end{array}$$

There's at most one pivot in every row and at most one in every column. So 4. <u>A</u> Any # between 0 and 4.