Day 5: Homogeneous Systems

A homogeneous linear system is one with 0's to the right of the equal sign.

$$2x_1 - 3x_2 + 5x_3 = 0$$
$$x_1 + 4x_2 - 6x_3 = 0$$

Q Does this system have a solution?

A Yes! $x_1 = 0$
$x_2 = 0$
$x_3 = 0.$

This is called the zero solution or the trivial soln.
It might have more solns.

The general form of a homogeneous system is

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= 0 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= 0 \\ \vdots \qquad\qquad \vdots \qquad\qquad & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= 0 \end{aligned}$$

It always has the trivial soln

$x_1 = 0$
$x_2 = 0$
$\vdots$
$x_n = 0.$

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As an augmented matrix

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & 0 \\ a_{21} & a_{22} & \cdots & a_{2n} & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & 0 \end{array}\right]$$

When you apply row ops, there's still a column of all 0's to the right of the bar. The system is always consistent.

<u>Ex</u> Find all solns to

$$2x_1 + 3x_2 - x_3 = 0$$
$$x_1 - x_2 + 2x_3 = 0$$

$$\left[\begin{array}{ccc|c} 2 & 3 & -1 & 0 \\ 1 & -1 & 2 & 0 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 2 & 3 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2 + (-2)R_1} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 5 & -5 & 0 \end{array}\right] \xrightarrow{\frac{1}{5}R_2} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{R_1 + (1)R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right]$$

As a system, $\begin{aligned} x_1 + x_3 &= 0 \\ x_2 - x_3 &= 0 \end{aligned}$

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So, $x_1 = -x_3$

$x_2 = x_3$

x_3 can be anything and determines x_1 and x_2.

∞ly many solns.

Ex Find all solns to

$$x_1 + 2x_2 - x_3 = 0$$
$$x_2 + 2x_3 = 0$$
$$x_3 = 0$$

Can you see the soln right away?

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_2+(-2)R_3} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_1+R_3} \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1+(-2)R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$x_1 = 0, \; x_2 = 0, \; x_3 = 0$

Recall r: Rank of system $= \#$ of pivots

$\#$ of free variables $= n - r$

Homogeneous systems are always consistent.

$\exists$ $(n-r)$ degrees of freedom. Only one soln if there's a pivot in every column.

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Ex
$$x_1 + 3x_2 + 4x_3 + 7x_4 = 0$$
$$x_3 + x_4 = 0$$

$$\left[\begin{array}{cccc|c} 1 & 3 & 4 & 7 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{R_1 - 4R_2} \left[\begin{array}{cccc|c} 1 & 3 & 0 & 3 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

$\cancel{x_1 + 3x_3 + 3x}$

$$x_1 + 3x_2 + 3x_4 = 0$$
$$x_3 + x_4 = 0$$

x_1 & x_3 are pivot variables

x_2 & x_4 are free.

$$x_1 = -3x_2 - 3x_4$$
$$x_3 = -x_4$$

$n = 4, \quad r = 2$

$4 - 2 = 2$ degrees of freedom

Consistency Again

Q For which values of b_1, b_2, b_3 is the following system consistent?

$$x_1 + 3x_2 + 2x_3 = b_1$$
$$2x_1 + 7x_2 + 5x_3 = b_2$$
$$x_2 + x_3 = b_3$$

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What do I mean by this?

$=1$

$=0$ has a soln.

$=2$

$=3$

$=1$ does not.

$=2$

Idea Write system as an augmented matrix. Row reduce. Find conditions on the b_i's so there are no pivots to the right of the bar.

$$\left[\begin{array}{ccc|c} 1 & 3 & 2 & b_1 \\ 2 & 7 & 5 & b_2 \\ 0 & 1 & 1 & b_3 \end{array}\right] \xrightarrow{R_2+(-2)R_1} \left[\begin{array}{ccc|c} 1 & 3 & 2 & b_1 \\ 0 & 1 & 1 & b_2-2b_1 \\ 0 & 1 & 1 & b_3 \end{array}\right]$$

$$\xrightarrow{R_3+(-1)R_2} \left[\begin{array}{ccc|c} 1 & 3 & 2 & b_1 \\ 0 & 1 & 1 & b_2-2b_1 \\ 0 & 0 & 0 & b_3-b_2+2b_1 \end{array}\right]$$

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Two cases: I) $b_3 - b_2 + 2b_1 \neq 0$

II) $b_3 - b_2 + 2b_1 = 0$

Case I) $\xrightarrow[b_3-b_2+2b_1]{1} R_3$: $\left[\begin{array}{ccc|c} 1 & 3 & 2 & b_1 \\ 0 & 1 & 1 & b_2 - 2b_1 \\ 0 & 0 & 0 & 1 \end{array}\right]$ Inconsistent!

Case II) $\left[\begin{array}{ccc|c} 1 & 3 & 2 & b_1 \\ 0 & 1 & 1 & b_2 - 2b \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1 + (-3)R_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 7b_1 + 3b_2 \\ 0 & 1 & 1 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Solve $x_1 - x_3 = (7b_1 + 3b_2)$
$x_2 + x_3 = (b_2 - 2b_1)$, so $x_1 = (7b_1 + 3b_2) + x_3$
$x_2 = (b_2 - 2b_1) - x_3$.

In general, if someone gives you a system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad\qquad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

tells you the a's but not the b's. And they ask you which b's make the system consistent...

You row reduce, you might get rows of 0's to the left of the bar like:

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$$\left[\begin{array}{ccc|c} & & & * \\ & * & & * \\ & & & \vdots \\ & & & * \\ 0 & \cdots & 0 & \boxed{b_1 - 3b_2 + 4b_3} \\ 0 & \cdots & 0 & \boxed{b_2 - b_1} \end{array}\right]$$

For the system to be consistent, these entries must be 0.

Matrices

We're going to do algebra with matrices

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

This is an $(m \times n)$-matrix

m rows
n columns

We'll say a_{ij} = entry in the i^{th} row and j^{th} column

Note $1 \leq i \leq m$ (there are m rows)
$1 \leq j \leq n$ (there are n columns)

We call $m \times n$ the size of the matrix.

Write $A = (a_{ij})$

Or $[A]_{ij} = a_{ij}$ (the entry of A in the i^{th} row and j^{th} col) a_{ij}.

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We can add matrices of the same size.

Ex (2×3)-matrices

$$\begin{bmatrix} 1 & 0 & 2 \\ -3 & 1 & 7 \end{bmatrix} + \begin{bmatrix} 4 & 3 & -5 \\ 2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 3 & -3 \\ -1 & 1 & 9 \end{bmatrix}$$

We add corresponding entries.

In general

$$\underset{A}{\begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & & \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{bmatrix}} + \underset{B}{\begin{bmatrix} b_{11} & b_{12} \cdots & b_{1n} \\ b_{21} & b_{22} \cdots & b_{2n} \\ \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} \cdots & b_{mn} \end{bmatrix}}$$

$$= \underset{A+B}{\begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} \cdots & a_{2n}+b_{2n} \\ \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} \cdots & a_{mn}+b_{mn} \end{bmatrix}}$$

$$[A+B]_{ij} = [A]_{ij} + [B]_{ij}$$

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We can't add matrices of different sizes.

We can also multiply matrices by numbers:

Ex $3\begin{bmatrix} 1 & 2 & 1 \\ 3 & 0 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 3 \\ 9 & 0 & 12 \end{bmatrix}.$

"scalar multiplication"

If r is a scalar

$$\underbrace{r\begin{bmatrix} a_{11} & a_{12} \cdots & a_{1n} \\ a_{21} & a_{22} \cdots & a_{2n} \\ \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} \cdots & a_{mn} \end{bmatrix}}_{\substack{m \times n \\ r \cdot A}} = \underbrace{\begin{bmatrix} ra_{11} & ra_{12} \cdots & ra_{1n} \\ ra_{21} & ra_{22} \cdots & ra_{2n} \\ \vdots & \vdots & \vdots \\ ra_{m1} & ra_{m2} \cdots & ra_{mn} \end{bmatrix}}_{\substack{m \times n \\ rA}}.$$

We can say

$$[rA]_{ij} = r[A]_{ij}.$$

Vectors

$\mathbb{R}$ = set of real numbers.

$\mathbb{R}^2$ = the plane = the set of ordered pairs (x_1, x_2).

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Ex $(2,3)$ can also be written $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

$\mathbb{R}^3$ = 3d space = set of ordered triples (x_1, x_2, x_3)

Ex $(3,1,4)$ can be written $\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix}$.

$\mathbb{R}^n$ = n-dim space, ordered n-tuples

$(x_1, x_2, \dots, x_n)$

$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ We write vectors with an arrow over them

We can add vectors: $\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 11 \end{bmatrix}$.

We can multiply vectors by scalars

$5\begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 15 \\ 5 \\ 20 \end{bmatrix}$.