Day 7: Properties of Matrix Operations

There are useful properties of matrix operations. In some ways, they act like numbers. In other ways, they're very different.

Additive Properties

Let A, B, and C be $(m \times n)$-matrices, so they're all the same size:

1) $A + B = B + A$ (commutative)
2) $(A+B) + C = A + (B+C)$ (associative)
3) Let $\mathbb{O}$ be the $(m \times n)$-matrix of all 0's

$$\mathbb{O} = \begin{array}{c c} & \overbrace{\hphantom{0 \; 0 \; \cdots \; 0}}^{n} \\ m \left\{ \vphantom{\begin{matrix} 0 \\ 0 \\ \vdots \\ 0 \end{matrix}} \right. & \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} \end{array}$$

Then $A + \mathbb{O} = A$ (additive identity)

4) If

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix},$$

set

$$-A = \begin{bmatrix} -a_{11} & -a_{12} & \cdots & -a_{1n} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ -a_{m1} & -a_{m2} & \cdots & -a_{mn} \end{bmatrix}$$

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Then $A + (-A) = \mathcal{O}$. (additive inverse)

So addition acts like addition of real numbers.
It's easy to verify these properties.

<u>Multiplicative Properties</u>

<u>Recall</u> If A is an $(m\times n)$-matrix and
B is an $(r\times s)$-matrix,
AB only makes sense if $n=r$.

So AB may make sense but BA may not make sense.

<u>Fact</u> AB and BA both make sense exactly when
A and B are $(n\times n)$-matrices for some n.
(square matrices of the same size)

<u>Warning</u> Even when AB and BA both make sense,
they may not be equal. Matrix multiplication
is not commutative.

<u>Ex</u> ~~A=~~ $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$$

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$$BA = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$$

So $AB \neq BA$.

Properties

1) Let A be $(m \times n)$-matrix
   B be $(n \times p)$-matrix
   C be $(p \times q)$-matrix

   $(AB)C = A(BC)$ (associativity)

2) If r and s are scalars
   $r(sA) = (rs)A$

3) $r(AB) = (rA)B = A(rB)$

Addition and Multiplication Working Together

1) Let A, B be $(m \times n)$-matrices
   and C an $(n \times p)$-matrix.

   Then $A(B+C) = AB + AC$.

2) Let A be an $(m \times n)$-matrix
   B, C be $(n \times p)$-matrices then

   $\cancel{A(B\ \ C) = AB + AC}$ $A(B+C) = AB + AC$

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3) Let r, s be scalars and A be an $(m \times n)$-matrix, then $(r+s)A = rA + sA$

4) Let r be a scalar and A, B be $(m \times n)$-matrices, then $r(A+B) = rA + rB$

Transpose

The transpose of a matrix interchanges rows and columns.

Ex $\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 2 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 1 & 6 \end{bmatrix}$

Def If $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{bmatrix}$$

In nerd talk: if $A = (a_{ij})$ is an $(m \times n)$-matrix, the transpose A^T is the $(n \times m)$-matrix $A^T = (b_{ij})$ where $b_{ij} = a_{ji}$ for $1 \le j \le m$, $1 \le i \le n$.

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Thm Let A and B be $(m\times n)$-matrices and C be a $(n\times p)$-matrix, then

1) $(A^T)^T = A$
2) $(A+B)^T = A^T + B^T$
3) $(AC)^T = C^T A^T$

Q What does 1) say?

A Taking the transpose twice gives you the original matrix back.

2) is easy to check.

3) is harder to verify. You have to get used to writing matrix products in Σ notation.

Ex of 3:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 3 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix}$$

$$\cancel{(AC} \quad AC = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 4 & 0 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 6 & 4 \\ 8 & 0 \\ 13 & 2 \end{bmatrix}$$

write it as sums of products

$$(AC)^T = \begin{bmatrix} 6 & 8 & 13 \\ 4 & 0 & 2 \end{bmatrix}$$

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$$C^TA^T = \begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 8 & 13 \\ 4 & 0 & 2 \end{bmatrix}$$

So $(AC)^T = C^TA^T$.

Symmetric Matrices

Def A matrix A is symmetric if $A^T = A$.

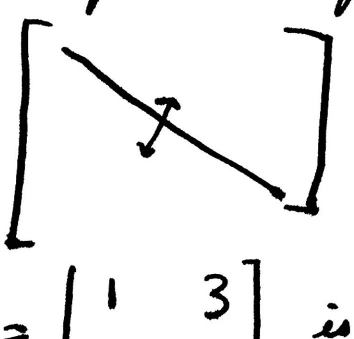
Only square matrices can be symmetric.

Ex

$$A = \begin{bmatrix} 1 & 2 & 7 \\ 2 & 3 & 0 \\ 7 & 0 & 4 \end{bmatrix} \text{ is symmetric.}$$

Its transpose $A^T = \begin{bmatrix} 1 & 2 & 7 \\ 2 & 3 & 0 \\ 7 & 0 & 4 \end{bmatrix}$ is equal to A.

It has reflection symmetry along main diagonal:



Ex $B = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$ is not symmetric since

$B^T = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ which is not B.

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Fact If Q is an $(m\times n)$-matrix
Q^T " " $(n\times m)$-matrix

Q^TQ is an $(n\times n)$-matrix

It is always symmetric

$$(Q^TQ)^T \overset{(3)}{=} Q^T(Q^T)^T = Q^TQ$$

(You take the transpose and get the same matrix back)

The Indentity Matrix

Let's think about multiplication of real numbers

For any real number x (ordinary mult not dot product)

$$1\cdot x = x\cdot 1 = x$$

We say 1 is <u>the (multiplicative) identity</u>

The $(n\times n)$ identity matrix is

$$I_n = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

1's along main diagonal, 0's elsewhere

Ex

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

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Thm Let A be an $(m \times n)$-matrix. Then

$$I_m A = A \quad \text{and} \quad A I_n = A.$$

(We pick I_m and I_n because we need matrices of this size for multiplication to make sense)

Ex Let A be the (2×3)-matrix

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 7 \end{bmatrix}$$

so $m=2$, $n=3$.

So $I_m A$ is $I_2 A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 7 \end{bmatrix} =$

Work out multiplication cross-down

This $= \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 7 \end{bmatrix}.$

And AI_n means

$$AI_3 = \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 7 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Cross-down multiplication

This equals $\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 7 \end{bmatrix}$

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Ex Let $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ be a vector.

Treat it as $(n \times 1)$-matrix

$$I_n \vec{x}$$

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 \cdots & 0 \\ \vdots & & \ddots & \vdots \\ & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$