Day 8: Matrix Operations & Linear Independence

Scalar Product

Let $\vec{x}, \vec{y} \in \mathbb{R}^3$, treated as column vectors.

What can we say about $\vec{x}^T \vec{y}$?

Ex $\vec{x} = \begin{bmatrix} 6 \\ 1 \\ 4 \end{bmatrix}$, $\vec{y} = \begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix}$ (3×1)-matrices

Then $\vec{x}^T = [6 \ 1 \ 4]$, a (1×3)-matrix.

So $\vec{x}^T \vec{y}$ is a (1×1)-matrix (that's a number!)

$$\vec{x}^T \vec{y} = [6 \ 1 \ 4]\begin{bmatrix} 5 \\ 1 \\ 3 \end{bmatrix} = [6(5) + 1(1) + 4(3)] = [43].$$

This is the same thing as dot product!

Thm Let $\vec{x}, \vec{y} \in \mathbb{R}^n$, viewed as $(n\times 1)$-matrices.

Then $\vec{x}^T \vec{y} = [x \cdot y]$.

Pf

Write $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ $\vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$

Note $\vec{x} \cdot \vec{y} = x_1 y_1 + x_2 y_2 + \dots + x_n y_n$.

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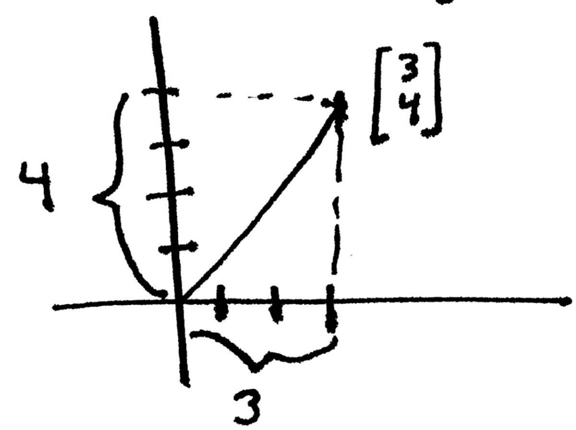
$$\vec{x}^T\vec{y} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1y_1 + x_2y_2 + \cdots + x_ny_n \end{bmatrix}$$

$$= \begin{bmatrix} \vec{x}\cdot\vec{y} \end{bmatrix}. \quad \blacksquare$$

<u>Recall</u> From HS geometry

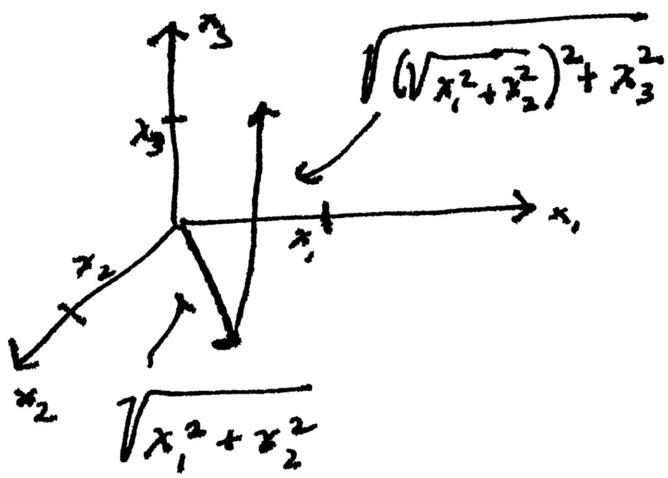


The vector has length $\sqrt{3^2+4^2} = \sqrt{25} = 5$

More generally $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ has length $\sqrt{x_1^2+x_2^2}$.

In 3d, $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ has length $\sqrt{x_1^2+x_2^2+x_3^2}$

Day 8: Matrix Operations & Linear Independance



Apply Pythagorean theorem twice.

In general, the length of $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \in \mathbb{R}^n$ is

$$\sqrt{x_1^2 + x_2^2 + \dots + x_n^2}$$

Now observe

$$\vec{x} \cdot \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1^2 + x_2^2 + \dots + x_n^2$$

$$= (\text{length of } \vec{x})^2$$

__Def__ Let $\vec{x} \in \mathbb{R}^n$. The norm (length of $\vec{x}$) is

$$\|\vec{x}\| := \sqrt{\vec{x} \cdot \vec{x}} = \sqrt{\vec{x}^T x}.$$

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Note We'll see for vectors $\vec{x}, \vec{y} \in \mathbb{R}^n$,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \vec{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix},$$

the distance between $\vec{x}$ and $\vec{y}$ is

$$\|\vec{x} - \vec{y}\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \dots + (x_n - y_n)^2}.$$

Linear ~~Independence~~ Combinations

Let $\vec{A}_1, \vec{A}_2, \dots, \vec{A}_n \in \mathbb{R}^m$ be vectors. A vector $\vec{b}$ that can be written

$$\vec{b} = x_1 \vec{A}_1 + x_2 \vec{A}_2 + \dots + x_n \vec{A}_n$$

for some real numbers $x_1, x_2, \dots, x_n$ is called a *linear combination* of $\vec{A}_1, \vec{A}_2, \dots, \vec{A}_n$.

Aside This will lead to a generalization of coords.

Ex
$$\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

So we can write $\begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}$ as a lin. comb of $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

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Ex Let $\vec{A}_1, \vec{A}_2, \vec{A}_3 \in \mathbb{R}^3$ be vectors given by

$$\vec{A}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{A}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}.$$

Then, $\begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix}$ is a linear comb. of $\vec{A}_1, \vec{A}_2, \vec{A}_3$,

because $$\begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 4\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 0\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$$

$$= 5\vec{A}_1 + 4\vec{A}_2 + 0\vec{A}_3.$$

It's a lin. comb. of the same vectors in a different way:

$$\begin{bmatrix} 5 \\ 0 \\ 4 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 1\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}.$$

But $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ is not a linear combination of $\vec{A}_1, \vec{A}_2, \vec{A}_3$.

Why not?

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Let's try to find $x_1, x_2, x_3 \in \mathbb{R}$ with

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = x_1 \vec{A}_1 + x_2 \vec{A}_2 + x_3 A_3$$

$$= x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}.$$

Row-by-row:

$$0 = x_1 + 3x_3$$
$$1 = 0$$
$$0 = x_2 + 2x_3.$$

Inconsistent! No solutions!

Fact Let $\vec{A}_1, \vec{A}_2, \ldots, \vec{A}_n, \vec{b} \in \mathbb{R}^m$. Let $x_1, x_2, \ldots, \vec{x}_n \in \mathbb{R}$.

We have

$$x_1 \vec{A}_1 + x_2 \vec{A}_2 + \ldots + x_n \vec{A}_n = \vec{b}$$

exactly when

$$\underbrace{\begin{bmatrix} | & | & & | \\ \vec{A}_1 & \vec{A}_2 & \ldots & \vec{A}_n \\ | & | & & | \end{bmatrix}}_{\text{matrix whose columns are } A_i\text{'s.}} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \vec{b}$$

$\boxed{\text{Why?}}$

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__Fact__ In $\mathbb{R}^n$, let

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \dots, \vec{e}_n = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Any vector $\vec{v} \in \mathbb{R}^n$ can be written as a linear combination of $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$.

__Pf__ Since $\vec{v} \in \mathbb{R}^n$, we can write

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$ for real numbers $v_1, v_2, \dots, v_n$.

Then,
$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = v_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + v_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}.$$

So, $\vec{v} = v_1\vec{e}_1 + v_2\vec{e}_2 + \dots + v_n\vec{e}_n$. ∎

__Fact__ $\vec{v}$ can be written as a lin. comb. of $\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n$ in exactly one way. We'll talk about this later.

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Ex (for lin. comb.)

Let $\vec{A}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, $\vec{A}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{A}_3 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $\vec{b} = \cancel{\begin{bmatrix} 5 \\ 2 \\ 1 \\ 0 \end{bmatrix}}$. $\begin{bmatrix} 5 \\ 2 \\ 1 \\ 3 \end{bmatrix}$

Can I write $\vec{b}$ as a linear combination of $\vec{A}_1, \vec{A}_2, \vec{A}_3$?

We have to solve $x_1\vec{A}_1 + x_2\vec{A}_2 + x_3\vec{A}_3 = \vec{b}$.

$$\begin{bmatrix} 3 & 0 & 2 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 1 \\ 3 \end{bmatrix}$$

means
$$\begin{bmatrix} 3x_1 + 0x_2 + 2x_3 \\ 0x_1 + 1x_2 + 0x_3 \\ 1x_1 + 0x_2 + 0x_3 \\ 0x_1 + 1x_2 + 1x_3 \end{bmatrix} = \cancel{\begin{bmatrix} 5 \\ 2 \\ 0 \\ 1 \end{bmatrix}} \cancel{\begin{bmatrix} 5 \\ 2 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 5 \\ 2 \\ 1 \\ 3 \end{bmatrix}$$

which is equivalent to

$$x_1 \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \cancel{\begin{bmatrix} 5 \\ 2 \\ 1 \\ 0 \end{bmatrix}} \begin{bmatrix} 5 \\ 2 \\ 1 \\ 3 \end{bmatrix}$$

$$x_1\vec{A}_1 + x_2\vec{A}_2 + x_3\vec{A}_3 = \vec{b}.$$

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$$\left[\begin{array}{ccc|c} 3 & 0 & 2 & 5 \\ 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 3 \end{array}\right] \xrightarrow{\text{row reduce}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 2 & 5 \\ 0 & 1 & 1 & 3 \end{array}\right]$$

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 2 & 2 \\ 0 & 1 & 1 & 3 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

skip steps

$$\to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow x_1 = 1,\ x_2 = 2,\ x_3 = 1.$$