Day 9: Linear Independence

Linear Dependence:

In $\mathbb{R}^n$, there's a special vector $\vec{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, all components 0. Zero vector!

Observe that for the vectors

$$\vec{A}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \vec{A}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \vec{A}_3 = \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix},$$

we can write $\vec{A}_3$ as a linear combination of $\vec{A}_1$ and $\vec{A}_2$:

$$\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = 3\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$(*) \quad \vec{A}_3 = 3\vec{A}_1 + 2\vec{A}_2$$

I can also express $\vec{A}_1$ as a linear combination of $\vec{A}_2$ and $\vec{A}_3$:

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} - \frac{2}{3}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{A}_1 = \frac{1}{3}A_3 - \frac{2}{3}A_2.$$

I can write (*) as

$$3\vec{A}_1 + 2\vec{A}_2 - \vec{A}_3 = \vec{0}$$

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$$3\begin{bmatrix}1\\0\\0\end{bmatrix} + 2\begin{bmatrix}0\\0\\1\end{bmatrix} - \cancel{2}\begin{bmatrix}3\\0\\2\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

These vectors are linearly dependent.

Def Let $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n \in \mathbb{R}^m$. We say that the set $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is <u>linearly dependent</u> if there are constants $c_1, c_2, \dots, c_n \in \mathbb{R}$, not all equal to zero such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = \vec{0}.$$

$\boxed{\text{SAVE}}$

Lemma If $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ are linearly dependent, then one of the $\vec{v}_i$'s can be written as a linear combination of the others.

Pf Since the vectors are linearly dependent, there are $c_1, c_2, \dots, c_n$ not all zero such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = \vec{0}.$$

Now, at least one c_i is not zero, say c_j.

Write $\quad c_j\vec{v}_j = -c_1\vec{v}_1 - c_2\vec{v}_2 - \dots - c_{j-1}\vec{v}_{j-1} - c_{j+1}\vec{v}_{j+1} - \dots - c_n\vec{v}_n.$

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~~Lemma~~

$$\cdot \frac{1}{c_j} : \vec{v}_j = -\frac{c_1}{c_j}\vec{v}_1 - \frac{c_2}{c_j}\vec{v}_2 - \dots - \frac{c_{j-1}}{c_j}\vec{v}_{j-1} - \frac{c_{j+1}}{c_j}v_{j+1} - \dots - \frac{c_n}{c_j}\vec{v}_n . \square$$

<u>Note</u> This Lemma says

Lin Dep. $\Rightarrow$ One vector is a lin. comb. of the others

The opposite implication $\Leftarrow$ is also true.

So to check lin. dep., we can find $c_1, c_2, \dots, c_n$ w/

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = \vec{0}$$

or try to write one vector as a lin. comb. of others.

<u>Def</u> $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ ~~are~~ is <u>linearly independent</u> if they are not linearly dependent. In other words, if you try solve for $c_1, c_2, \dots c_n$ such that

$$c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_n\vec{v}_n = \vec{0},$$

you'll only get the solution

$$c_1 = 0,\ c_2 = 0, \dots, c_n = 0.$$

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Ex Are $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 5 \\ -6 \\ -11 \end{bmatrix}$

linearly dependent?

Can we solve the vector system

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix} + c_3 \begin{bmatrix} 5 \\ -6 \\ -11 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

with not all c_i's equal to 0?

If we can, vectors are lin. dep.

If not, vectors are lin. ind.

We can write the vector system as a linear system

$$1c_1 + 0c_2 + 5c_3 = 0$$
$$0c_1 + 3c_2 - 6c_3 = 0$$
$$2c_1 + 5c_2 - 11c_3 = 0$$

or as an augmented matrix

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 3 & -6 & 0 \\ 2 & 5 & -11 & 0 \end{array}\right].$$

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Row reduce

$$\xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -2 & 0 \\ 2 & 5 & -11 & 0 \end{array}\right] \xrightarrow{R_3+(-2)R_1} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 5 & -21 & 0 \end{array}\right]$$

$$\xrightarrow{R_3+(-5)R_2} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & -11 & 0 \end{array}\right] \xrightarrow{-\frac{1}{11}R_3} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\xrightarrow{R_2+2R_3} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] \xrightarrow{R_1+(-5)R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\Rightarrow$ $c_1 = 0$ This is the only soln.

$c_2 = 0$ The vectors are lin. ind.

$c_3 = 0$.

<u>Ex</u> Are $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 5 \\ -6 \\ 0 \end{bmatrix}$

linearly dependent?

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Can we solve

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix} + c_3 \begin{bmatrix} 5 \\ -6 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

with not all c_i's equal to 0?

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 3 & -6 & 0 \\ 2 & 5 & 0 & 0 \end{array}\right] \xrightarrow{R_3 - 2R_1} \cancel{\left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 3 & -6 & 0 \\ 2 & 5 & 0 & 0 \end{array}\right]}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 3 & -6 & 0 \\ 0 & 5 & -10 & 0 \end{array}\right] \xrightarrow{\frac{1}{3}R_2} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 5 & -10 & 0 \end{array}\right] \xrightarrow{R_3 - 5R_2} \left[\begin{array}{ccc|c} 1 & 0 & 5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This is in REF.

So
$$c_1 + 5c_3 = 0$$
$$c_2 - 2c_3 = 0.$$

So c_3 is a free variable

We need only one non-trivial soln (c_i's not all zero).

Choose a nonzero value for c_3, say $c_3 = 1$.

Then
$$c_1 = -5c_3 = -5$$
$$c_2 = 2c_3 = 2.$$

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So
$$-5\begin{bmatrix}1\\0\\2\end{bmatrix}+2\begin{bmatrix}0\\3\\5\end{bmatrix}+1\begin{bmatrix}5\\-6\\0\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

We can also write
$$\begin{bmatrix}5\\-6\\0\end{bmatrix}=5\begin{bmatrix}1\\0\\2\end{bmatrix}-2\begin{bmatrix}0\\3\\5\end{bmatrix}.$$

In $\mathbb{R}^n$, there are the <u>standard unit vectors</u>
$$\vec{e}_1=\begin{bmatrix}1\\0\\\vdots\\0\end{bmatrix},\quad \vec{e}_2=\begin{bmatrix}0\\1\\0\\\vdots\\0\end{bmatrix},\ \dots,\ \vec{e}_n=\begin{bmatrix}0\\\vdots\\0\\1\end{bmatrix}.$$

They are linearly independent.

For $n=3$, they are
$$\vec{e}_1=\begin{bmatrix}1\\0\\0\end{bmatrix},\quad \vec{e}_2=\begin{bmatrix}0\\1\\0\end{bmatrix},\quad \vec{e}_3=\begin{bmatrix}0\\0\\1\end{bmatrix}.$$

Let's check that they are lin. ind.

Try to solve
$$c_1\vec{e}_1+c_2\vec{e}_2+c_3\vec{e}_3=\vec{0}.$$

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$$c_1\begin{bmatrix}1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}0\\1\\0\end{bmatrix}+c_3\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

So,
$$\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}.$$

The only solution is $c_1 = 0$, $c_2 = 0$, $c_3 = 0$.

Therefore $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is linear independent.

Time-permitting

Q Are $\begin{bmatrix}1\\2\end{bmatrix}, \begin{bmatrix}3\\5\end{bmatrix}, \begin{bmatrix}7\\-9\end{bmatrix}$ lin ind?

Try to solve

$$c_1\begin{bmatrix}1\\2\end{bmatrix}+c_2\begin{bmatrix}3\\5\end{bmatrix}+c_3\begin{bmatrix}7\\9\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

with ~~some~~ not all c_i's equal to 0.

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$$\left[\begin{array}{ccc|c} 1 & 3 & 7 & 0 \\ 2 & 5 & 9 & 0 \end{array}\right]$$

Can anyone see why there's a nontrivial solution?

There are at most 2 pivots, so there's a free variable.