The Hodge theory of degenerating hypersurfaces

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Hypersurfaces

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A description of Hodge-theoretic invariants was given by Danilov-Khovanskii and then refined by Batyrev-Borisov much later.

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Question: What is a combinatorial description of algebraic geometric invariants of the degenerating hypersurface?

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We can package the face data in the f-polynomial

$$f(x) = f_{d-1} + f_{d-2}x + \dots f_0 x^{d-1} + f_{-1}x^d.$$

Definition The h-polynomial is

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So $f_{k-1} = \sum_{i=0}^{k} h_i \binom{d-i}{k-i}$.

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Theorem (part of McMullen's conjecture, Stanley '80) $h_{k-1} \leq h_k$ for $1 \leq k \leq \frac{d}{2}$.

The full conjecture involves a more detailed description of h_k.

Stanley's Theorem

Stanley's theorem is proved using algebraic geometry. Perturb the polytope so that all of its vertices are rational. Since P is simplicial, this will not change the face lattice. Translate so $0 \in \mathring{P}$. Let Δ be the fan consisting of cones on the faces. Then the h-polynomial is the Poincaré polynomial of the intersection cohomology.

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It makes sense that h and f should be related in that way. There are models for the cohomology of a toric variety involving counting a certain number of "interesting" cones and then getting all the other cones by looking at faces. The binomial coefficients were counting faces.

Non-simplicial polytopes

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Consider the poset $\hat{B}$ of faces of the polytope ordered under inclusion and graded by dimension. It is Eulerian. Let 0 be the empty face and 1 be the polytope. Let $B = \hat{B} \setminus \{1\}$.

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This is very opaque, but the formula for H is doing inclusion/exclusion on toric open affine U_σ (the $t-1$-factors are just splitting off tori). The formula for G is computing the Poincaré polynomial of U_σ in terms of a quotient toric variety $U_\sigma/\mathbb{C}^*$.

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A purely combinatorial definition of intersection cohomology was given in this case by Karu. So the rationality hypothesis is no longer necessary.

Mixed Hodge structure

We now review the approach of Danilov-Khovanskii ('78) to the cohomology of $Z_f \subset (\mathbb{C}^*)^n$ for f a polynomial. Since Z is not compact, we have to work with cohomology with compact supports, $H_c^*(Z)$. This cohomology has a mixed Hodge structure.

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There is an increasing filtration W and a decreasing filtration F on H^k such that the associated gradeds with respect to W have a pure Hodge structure. We define

$$h^{p,q}(H^k(Z)) = \dim \operatorname{Gr}_F^p \operatorname{Gr}_{p+q}^W (H_c^k(Z)).$$

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Warning: Note that we may have $h^{p,q}(H^k(Z)) \neq 0$ even though $p + q \neq k$. So there's a lot more data.

Danilov-Khovanskii's approach

To throw out some of the excess data, we take the Hodge-Deligne numbers

$$e^{p,q}(Z) = \sum_k (-1)^k h^{p,q}(H_c^k(Z)).$$

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Therefore one may compactify $(\mathbb{C}^*)^n$ to the toric variety X_P given by the Newton polytope of f. Let $\overline{Z}$ be the closure of Z in X_P. One can remove the stuff that we added later. Now, we can define the genericity of f which means that f is generic among polynomials with Newton polytope P so that the strata of $\overline{Z}$ are smooth.

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Secondly, one has a Lefschetz hyperplane theorem: for $p, q > n-1$,

$$e^{p,q}(Z) = e^{p+1,q+1}((\mathbb{C}^*)^n).$$

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The specialization $E(Z; u, 1)$ can be computed by taking the Euler characteristic of an ideal sheaf sequence (twisted by differentials) together with an adjunction exact sequence.

Danilov-Khovanskii's approach (cont'd)

We end up getting

$$uE(V(P)^\circ; u, 1) = (u-1)^{\dim P} + (-1)^{\dim P+1} h_P^*(u).$$

where $h_P^*(u)$ is defined by

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where $\mathrm{Ehr}_P(u)$ is the Ehrhart series of the Newton polytope P

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Danilov-Khovanskii provide an algorithm for finding $e^{p,q}$ by using inclusion/exclusion along the faces of P.

Batyrev-Borisov formula

Much later, Batyrev-Borisov gave an explicit formula (inspired by intersection cohomology) in terms of the face-poset of P:

$$\begin{aligned} E(Z;u,v) &= (1/uv)[(uv-1)^{d+1} \\ &\quad +(-1)^d \sum_{Q\subseteq P} u^{\dim Q+1}\tilde{S}(Q,u^{-1}v)G([Q,P]^*,uv)]. \end{aligned}$$

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So observe that the first term in the formula for E comes from the ambient space. The second term is an inclusion/exclusion along the poset of faces and where each term has been factored into a poset-combinatorial term multiplied by an Ehrhart term.

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Naive Question: Is the machinery of $\tilde{S}$ a combinatorial abstraction of the resolution of singularities for the dual fan of P which is equivalent to finding a normal crossings compactification?

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Now, the ambient $\mathbb{P}^1 \times \mathbb{P}^1$ degenerates to two $\mathbb{P}^2$'s joined along a line. Our curve degenerates into two twice-punctured lines joined along a node.

Newton subdivision

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For $a_{\mathbf{u}} \in \mathbb{C}((t))$, let $\mathrm{val}(a_{\mathbf{u}})$ be the smallest exponent of t with non-zero coefficient. Consider the function

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Regular subdivisions can be studied as an object like Newton polytopes.

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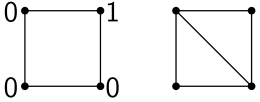

Here you can see the ambient $\mathbb{P}^1 \times \mathbb{P}^1$ degenerating into two $\mathbb{P}^2$'s joined along a $\mathbb{P}^1$.

Monodromy Filtration

In general, if we have a family Z_t, there is an additional filtration on the cohomology. View the family over the punctured disc. The cohomology $H^*(Z_t)$ gives a locally trivial fiber bundle over the punctured disc. Consequently, one can parallel transport around the puncture. This gives a monodromy operation $T : H^*(Z_t) \to H^*(Z_t)$. By possibly replacing T by T^m for some $m \in \mathbb{Z}_{\geq 1}$, we can suppose T is unipotent. Set $N = \log(T)$ which is nilpotent. There is an additional filtration coming from the Jordan decomposition of N.

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If Z_t were compact, then one could put an increasing monodromy filtration M on $H^k(Z_t)$,

$$0 \subseteq M_0 \subseteq M_1 \subseteq \cdots \subseteq M_{2k} = H^k(Z),$$

with associated graded pieces $\mathrm{Gr}_l^M := M_l/M_{l-1}$, satisfying the following properties for any non-negative integer l,

1. $N(M_l) \subseteq M_{l-2}$,
2. the induced map $N^l : \mathrm{Gr}_{k+l}^M \to \mathrm{Gr}_{k-l}^M$ is an isomorphism.

Mixed Monodromy Filtration

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By results of Steenbrink-Zucker, there is an increasing monodromy filtration M on $H^k_c(Z_t)$ and an increasing weight filtration W and a decreasing Hodge filtration F. Note that the original construction used a normal-crossings degeneration of the family and looked at log forms on components of that degeneration.

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(A twist of) the monodromy filtration has the above properties on the W-associated gradeds.

This gives us tons of structure. We can refine the Hodge numbers even further:

$$h^{p,q,r}(Z)_k = \dim(\mathrm{Gr}^p_F \mathrm{Gr}^{M(r)}_{p+q} \mathrm{Gr}^W_r H^k(Z)).$$

and form refined Hodge-Deligne numbers:

$$e^{p,q,r}(Z) = \sum (-1)^k h^{p,q,r}(Z)_k.$$

Specializations

We can play lots of different games with these refined Hodge numbers. We can forget the monodromy filtration or the weight filtration. And it's always fun to have decompositions of non-negative numbers into smaller non-negative numbers.

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Observation: $E(Z_{\text{gen}}; u, 1) = E(Z_\infty; u, 1)$ since this forgets both M and W.

Degeneration formula

There's a degeneration formula for $E(Z_\infty; u, v)$.
Observation: For generic f, $E(Z; u, 1)$ only depends on the Newton polytope of Z. So we may write Z_P

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Theorem (K-Stapledon)

$$E((Z_P)_\infty; u, v) = \sum_{\mathrm{Int}(Q) \subseteq \mathrm{Int}(P)} E(Z_Q; u, v)(1 - uv)^{\operatorname{codim} Q}.$$

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1. ϕ obeys inclusion/exclusion over faces in a regular subdivsion,
2. $\phi(\emptyset) = 0$, and
3. $\phi(P) = \phi(UP + u)$ for $P \in \mathcal{P}_{\mathbb{Z}^n}$, $U \in \mathrm{Sl}_n(\mathbb{Z})$, $u \in \mathbb{Z}^n$.

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Lemma The following function is a regular unimodular valuation

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We obtain Danilov-Khovanskii's formula by checking that the right-hand side is a unimodular valuation and showing that the formula is true for unimodular simplices by (easy) explicit computation.

Application 2: Refining $\tilde{S}(P)$

We may also use this machinery to refine $\tilde{S}(P,t)$. By Batyrev-Borisov's formula, we have the following formula for coefficients of $\tilde{S}(P,t)$:

$$\tilde{S}(P)_{p+1} = h^{p,n-1-p}(H^{n-1}_{c,\mathrm{na}}((Z_P)_{\mathrm{gen}}))$$

where na refers to the non-ambient cohomology, the kernel of the map

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We have

$$\tilde{S}(P)_{p+1} = \sum_q h^{p,q,n-1}(H^{n-1}_{c,\mathrm{na}}(Z_f)).$$

Note that the right-hand side depends on the Newton subdivision of P.

Structure of $\tilde{S}(P)$

Now, by the structure of the monodromy filtration, the sequence $\{h^{l+i,i,k}(H^{n-1}_{c,\mathrm{na}}(Z_P)) | 0 \leq i \leq k-l\}$ is symmetric and unimodal. This decomposes the coefficients of $\tilde{S}(P)$ into the sum of symmetric and unimodal sequences. If we can show that some of them vanish, then we can get inequalities for $\tilde{S}$.

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For example, if P admits a regular, unimodular lattice triangulation, then the refined limit mixed Hodge numbers are concentrated in (p, p). This is because the hypersurface is degenerating into a union of hyperplanes. In this case $\tilde{S}(P)$ is symmetric and unimodal.

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Theorem: The coefficients $\{\tilde{S}(P)_i\}_{\{i=1,\ldots,n\}}$ are bounded below by the number of interior lattice points of P.

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Theorem: The coefficients $\{\tilde{S}(P)_i\}_{\{i=1,\ldots,n\}}$ are bounded below by the number of interior lattice points of P.

This is proven by picking a regular, lattice triangulation of P, showing that some refined mixed Hodge numbers vanish and using the unimodality properties of the rest.

Application 3: Computing the refined limit mixed Hodge polynomial

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2. We complete our hypersurface to a hypersurface in a complete toric variety. A similar degeneration formula holds. The Hodge structure is now pure with respect to the weight filtration. So we know the degrees in which the filtration is concentrated.

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2. We complete our hypersurface to a hypersurface in a complete toric variety. A similar degeneration formula holds. The Hodge structure is now pure with respect to the weight filtration. So we know the degrees in which the filtration is concentrated.
3. We break the terms into ambient and non-ambient contributions and apply inclusion/exclusion along the faces of the polytope.

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1. Some situations are best understood when they are combinatorially simple: simplicial polytopes, normal crossing compactifications and degenerations. However, algebraic geometry can handle the bad cases and tells us what the right answer is. This should involve subdividing/resolving singularities and picking out what is invariant of choices.

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1. Some situations are best understood when they are combinatorially simple: simplicial polytopes, normal crossing compactifications and degenerations. However, algebraic geometry can handle the bad cases and tells us what the right answer is. This should involve subdividing/resolving singularities and picking out what is invariant of choices.
2. A lot of invariants are motivic and are given by summing over strata or components of the central fiber. This involves inclusion/exclusion. So we should look for ways to decouple the combinatorics of the inclusion/exclusion from the algebraic geometry of the pieces.

Questions

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Stanley has introduced local h-vectors as a way of decomposing h-vectors along faces or along a subdivision. The Ehrhart analogues are local h^*-vectors, investigated by Karu, Nill, and Schepers.

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Question: Is the decomposition coming from Hodge theory the same?

Question: Are local h-vectors a combinatorial abstraction of semistable reduction?

Thanks!

Vladimir Danilov and Askold Khovanskii, *Newton polyhedra and an algorithm for calculating Hodge-Deligne numbers*.

K. and Alan Stapledon, *The tropical motivic nearby fiber and the Hodge theory of hypersurfaces.* in preparation.