p-adic Integration on Curves of Bad Reduction

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Motivation:The Chabauty-Coleman method

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Theorem: (Chabauty, Coleman, Lorenzini-Tucker, McCallum-Poonen) If $\mathrm{MWR} < g$ and $p > 2g$ then

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One can replace $2g - 2$ by $2\,\mathrm{MWR}$ by results of Stoll and K-Zureick-Brown.

Idea of proof of Chabauty-Coleman:

First, work p-adically. If C has a rational point x_0 , use it as the base-point of the Abel-Jacobi map $C \to J$. If $\mathrm{MWR} < g$, by an argument involving p-adic Lie groups, we can suppose that that $\overline{J(\mathbb{Q})}$ lies in an Abelian subvariety $A_{\mathbb{Q}_p} \subset J_{\mathbb{Q}_p}$ with $\dim(A_{\mathbb{Q}_p}) \leq \mathrm{MWR} < g$.

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We might expect $C(\mathbb{Q}_p)$ to intersect $A_{\mathbb{Q}_p}$ in finitely many points. In fact, there is a 1-form ω on $J_{\mathbb{Q}_p}$ that vanishes on A, hence on the images of all points of $C(\mathbb{Q})$ under the Abel-Jacobi map. Pull back ω to $C_{\mathbb{Q}_p}$.

Idea of proof of Chabauty-Coleman (cont'd)

Define a function $\eta : C(\mathbb{Q}_p) \to \mathbb{Q}_p$ by a p-adic integral,

$$\eta(x) = \int_{x_0}^{x} \omega$$

that vanishes on points of $C(\mathbb{Q})$.
By a Newton polytope argument, for any residue class $\tilde{x} \in \mathcal{C}_0^{\mathrm{sm}}(\mathbb{F}_p)$,

$$\#(\eta^{-1}(0) \cap [\tilde{x}[) \leq 1 + \mathrm{ord}_{\tilde{x}}(\omega|_{\mathcal{C}_0}).$$

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Summing over residue classes $\tilde{x} \in \mathcal{C}_0^{\mathrm{sm}}(\mathbb{F}_p)$, we get the desired result.

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Moreover, how does the reduction type of the curve influence the reduction of rational points? If the curve has bad reduction, maybe the rational points like to reduce to particular components?

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Specifically, if the curve C has good reduction, we pick a smooth model $\mathcal{C}$ and a self-map of $\mathcal{C}$ that extends Frobenius on the central fiber. We then mandate that the integral obeys a change-of-variables formula with respect to Frobenius. This produces a primitive on the affinoid (so path independent!). It is not analytic but is more than locally analytic. Coleman-analytic!

p-adic integration on curves of bad reduction

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The preimage of closed components of $\mathcal{C}_0$ turn out to be *basic wide opens*, the complement of some discs in the analytification of a proper curve. We can extend the 1-form to the proper curve if we allow poles in the removed discs. Within any affinoid in this basic wide open we can find a primitive by the standard Coleman integration. But a new subtlety arises!

Integration on annuli

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we need to integrate $a_{-1} z^{-1}$!

p-adic logarithm

To integrate $a_{-1}z^{-1}$, we nee to pick a branch of p-adic logarithm. Logarithm is uniquely defined as a map

$$\mathrm{Log} : \mathcal{O}^* \to \mathbb{K}$$

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1. Pick a value of $\text{Log}(\pi)$ (a branch) once and for all for all annuli, or
2. Impose the condition that the integral is a pull-back of a univalent logarithm on $\text{Jac}(C)$.

A consistent choice of logarithm

If we pick a value of $\mathrm{Log}(\pi)$ for every annulus, we have resolved the ambiguity. We have to enlarge the class of Coleman functions to allow them to behave like an analytic function plus a multiple of a branch of logarithm in annuli. This leads to an integral defined for Mumford curves by Schneider (and later studied by Teitelbaum), studied in greater generality by Coleman-de Shalit, and used a basis for a very general theory of integration by Berkovich.

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And it's very strange to me, at least, that the familiar phenomena of periods only exist at primes of bad reduction.

Logarithms on Abelian Lie groups

Let's quickly review logarithms on Abelian Lie groups G over p-adic fields. Let $G(\mathbb{K})_f$ be the smallest open subgroup of $G(\mathbb{K})$ such that $G(\mathbb{K})/G(\mathbb{K})_f$ contains no non-zero torsion elements. Then there is a $\mathbb{K}$-analytic homomorphism

$$\log_{G(\mathbb{K})} : G(\mathbb{K})_f \to \mathrm{Lie}(G)$$

that induces an isomorphism on tangent spaces of the identity. Then, we must extend log to $G(\mathbb{K})$. In the case of Abelian varieties $G(\mathbb{K})_f = G(\mathbb{K})$.

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We can identify the dual to the Lie algebra with the global, invariant 1-forms. This allows us to rewrite the logarithm as a bilinear pairing

$$A(\mathbb{K}) \times H^0(A_{\mathbb{K}}, \Omega^1) \to \mathbb{K}.$$

Logarithms on Abelian Lie groups (cont'd)

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This integral can be pulled back by the Abel-Jacobi map

$$C \to \mathrm{Jac}(C).$$

This gives (a special case of) the Colmez integral. This is the integral that you use in bad reduction Chabauty because it will vanish on the sub-Abelian variety containing rational points of C.

Comparison Theorem

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To set up the comparison result, we will pull back integrals from the universal cover of the Jacobian.

Raynaud Uniformization

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If A is an Abelian variety, then one can form a uniformization cross

$$\begin{array}{ccccc} & & T & & \\ & & \downarrow & & \\ \Lambda & \longrightarrow & G & \xrightarrow{p} & A \\ & & \downarrow & & \\ & & B & & \end{array}$$

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where T is a torus, Λ is a discrete group, and B is an Abelian scheme with good reduction.

We should think of this (imprecisely) as writing an Abelian variety as an extension of an Abelian variety of good reduction by one of maximally degenerate reduction. We think of G as the universal cover of A.

Integrals on the Universal cover

The two integrals pull back to integrals on $G(\mathbb{K})$

$$G(\mathbb{K}) \times \Omega^1(A) \to \mathbb{K}$$

given by

$$(P, \omega) \mapsto \int_0^P \omega.$$

and so induce logarithms

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These logarithms are characterized by their extension to $T(\mathbb{K})$ in the diagram:

$$\begin{array}{ccccc} T(\mathbb{K}) & \longrightarrow & G(\mathbb{K}) & \longrightarrow & B(\mathbb{K}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathrm{Lie}(T) & \longrightarrow & \mathrm{Lie}(G) & \longrightarrow & \mathrm{Lie}(B). \end{array}$$

since the logarithm on $B(\mathbb{K})$ is already determined.

Characterization of Integrals

In the diagram

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the BCdS integral is determined by (after extending $\mathbb{K}$ to ensure that T splits) the fact that the logarithm is given by a Cartesian product of Log. Specifically if z is a unit on T, then the primitive of the invariant 1-form $\frac{dz}{z}$ is $\mathrm{Log}(z)$.

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Denote the two logarithms by $\log_{\mathrm{BCdS}}$ and $\log_{\mathrm{Colmez}}$.

Comparison of Logarithms

The two logarithms agree on $G(\mathbb{K})_f$. So we can view their difference as

$$\log_{\mathrm{BCdS}} - \log_{\mathrm{Colmez}} : (G(\mathbb{K})/G(\mathbb{K})_f) \times \Omega^1 \to \mathbb{K}$$

where Ω^1 denotes the invariant differential on $G(\mathbb{K})$.

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But $G(\mathbb{K})/G(\mathbb{K})_f = T(\mathbb{K})/T(\mathbb{K})_f = T(\mathbb{K})/T(\mathcal{O})$. Now, $T(\mathbb{K})/T(\mathcal{O})$ is an intrinsic tropicalization of an algebraic torus that should be thought of as $(\mathbb{K}^*/\mathcal{O}^*)^n = v(\mathbb{K}^*)^n$. Write the quotient as

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This completely describes the Colmez integral.

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There is a tropical Abel-Jacobi map

$$\Gamma \to ((T(\mathbb{K})/T(\mathcal{O})) \otimes \mathbb{R})/\operatorname{Trop}(\Lambda)$$

whose universal cover is the map of the central fibers of the above:

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whose universal cover is the map of the central fibers of the above:

$$\widetilde{\Gamma} \to (T(\mathbb{K})/T(\mathcal{O})) \otimes \mathbb{R}.$$

The map $\log_{\mathrm{BCdS}} - \log_{\mathrm{Colmez}}$ can be pulled back to $\widetilde{\Gamma}$ and can be used to correct Berkovich-Coleman-de Shalit integrals to Colmez integrals.

Thanks!

V. Berkovich. *Integration of one-forms on p-adic analytic spaces.*

R. Coleman and E. de Shalit. *p-adic regulators on curves and special values of p-adic L-functions.*

E. Katz and D. Zureick-Brown (and others?). *p-adic integration on curves of bad reduction.*

M. Stoll. *Uniform bounds for the number of rational points on hyperelliptic curves of small Mordell-Weil rank.*