Problem 1. Suppose A is a unital Banach algebra and fix $a, b \in A$.

(1) Show that $1 \notin \text{sp}_A(ab)$ if and only if $1 \notin \text{sp}_A(ba)$ using the identity $(1-ba)^{-1} = 1 + b(1-ab)^{-1}a$. Deduce that $\text{sp}_A(ab) \cup \{0\} = \text{sp}_A(ba) \cup \{0\}$.

(2) Show that for any Banach subalgebra $B \subseteq A$ with $1_A \in B$, for every $a \in B$, the spectral radius in B of a is equal to the spectral radius in A of a, i.e., $r_B(a) = r_A(a)$.

(3) Suppose $a, b \in A$ commute. Prove that $r(ab) \le r(a)r(b)$ and $r(a+b) \le r(a) + r(b)$.
Hint: By (2), this computation can be performed in the unital commutative Banach subalgebra $B \subseteq A$ generated by a and b. In B, there is a helpful characterization of the spectrum.

(4) Deduce from part (3) that if A is commutative, the spectral radius $r : A \to [0, \infty)$ is continuous.

Problem 2. Let A be a unital Banach algebra. Suppose we have a norm convergent sequence $(a_n) \subset A$ with $a_n \to a$. Prove that for every open neighborhood U of $\text{sp}(a)$, there is an $N > 0$ such that $\text{sp}(a_n) \subset U$ for all $n > N$.

Problem 3. Let $A \in M_n(\mathbb{C})$.

(1) As best as you can, describe $f(A)$ where $f \in \mathcal{O}(\text{sp}(A))$.
Hint: First consider the case that A is a single Jordan block.

(2) Determine as best you can which matrices $A \in M_n(\mathbb{C})$ have square roots, i.e., when there is a $B \in M_n(\mathbb{C})$ such that $B^2 = A$.
Note: Such a B is not necessarily unique.

Problem 4. Suppose A is a C*-algebra and $a \in A$ is normal.

(1) Show a is self-adjoint if and only if $\text{sp}(a) \subset \mathbb{R}$.
(2) Show a is unitary if and only if $\text{sp}(a) \subset \mathbb{T}$.
(3) Show a is a projection if and only if $\text{sp}(a) \subset \{0, 1\}$.

Problem 5. Let A be a C*-algebra.

(1) Show that the following are equivalent for a self-adjoint $a \in A$:
  (a) $\text{sp}(a) \subset [0, \infty)$,
  (b) For all $\lambda \ge \|a\|$, $\|a - \lambda\| \le \lambda$, and
  (c) There is a $\lambda \ge \|a\|$ such that $\|a - \lambda\| \le \lambda$.

For now, we will call such elements *spectrally positive.*
Note: It is implicit here that a spectrally positive element is self-adjoint.

(2) Deduce that the spectrally positive elements in a C*-algebra form a closed cone, i.e., $A_+ = \{a \in A | a \ge 0\}$ is closed, and for all $\lambda \in [0, \infty)$ and $a, b \in A_+$, we have $\lambda a + b \in A_+$.

(3) Show a is positive ($a = b^*b$ for some b) if and only if a is spectrally positive ($a = a^*$ and $\text{sp}(a) \subset [0, \infty)$).
Hint: First, if $\text{sp}(a) \subset [0, \infty)$, we can define $a^{1/2}$ via the continuous functional calculus. Now suppose $a = b^*b$ for some $b \in B$. Use the continuous functions $r \mapsto \max\{0, z\}$ and $r \mapsto -\min\{0, z\}$ on $\text{sp}(a)$ to write $a = a_+ - a_-$ where $\text{sp}(a_\pm) \subset [0, \infty)$ and $a_+a_- = a_-a_+ = 0$. Now look at $c = ba_-$. Prove that $\text{sp}(c^*c) \subset (-\infty, 0]$ and $\text{sp}(cc^*) \subset [0, \infty)$ using part (1) of this problem. Use part (1) of Problem 1 to deduce that $c^*c = 0$. Finally, deduce $a_- = 0$, and thus $a = a_+$.

Problem 6. For $a, b \in A$, we say $a \le b$ if $b - a \ge 0$.

(1) Show that $\le$ is a partial order.
(2) Show that if $a \le b$, then for all $c \in A$, $c^*ac \le c^*bc$.

(3) Suppose $0 \leq a \leq b$. Prove that $\|a\| \leq \|b\|$.

Problem 7. Let A be a C*-algebra. By the hint to part (4) of Problem 4 that for $a \geq 0$, we can define an $a^{1/2} \geq 0$ such that $(a^{1/2})^2 = a$.

(1) Show that if $b \geq 0$ such that $b^2 = a$, then $b = a^{1/2}$.
(2) Prove that if $0 \leq a \leq b$, then $a^{1/2} \leq b^{1/2}$.
(3) Prove that if $0 < a$ ($0 \leq a$ and a is invertible), then $0 < a^{-1}$.
(4) Prove that if $0 < a \leq b$, then $0 < b$ and $0 < b^{-1} \leq a^{-1}$.

Problem 8 (Rieffel, "Preventative Medicine")**.** Consider $a = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $b = \begin{pmatrix} s & 0 \\ 0 & t \end{pmatrix}$ for $s, t \geq 0$.

(1) Determine for which $s, t \geq 0$ we have $b \geq a$.
(2) Determine for which $s, t \geq 0$ we have $b \geq a_+$.
Note: Since $a = a^*$, a_+ is the positive part defined as in the hint to part (4) of Problem 4.
(3) Find values of $s, t \geq 0$ for which $b \geq a$, $b \geq 0$, and yet $b \not\geq a_+$.
(4) Find values of $s, t \geq 0$ such that $b \geq a_+ \geq 0$, and yet $b^2 \not\geq a_+^2$.
(5) Can you find $s, t \geq 0$ such that $b \geq a_+$ and yet $b^{1/2} \not\geq a_+^{1/2}$?
Note: $a_+^{1/2}$ is the unique positive square root of a_+ from part (1) Problem 7.
(6) Suppose $c, p \in M_2(\mathbb{C})$ such that $c \geq 0$ and $p^2 = p^* = p$ is a projection. Is it always true that $pcp \leq c$?

Problem 9. Let $L^2(\mathbb{T})$ denote the space of complex-valued square-integrable 1-periodic functions on $\mathbb{R}$, and let $C(\mathbb{T}) \subset L^2(\mathbb{T})$ denote the subspace of continuous 1-periodic functions.

(a) Prove that $\{e_n(x) := \exp(2\pi i n x) | n \in \mathbb{Z}\}$ is an orthonormal basis for $L^2(\mathbb{T})$.
(b) Define $\mathcal{F} : L^2(\mathbb{T}) \to \ell^2(\mathbb{Z})$ by $\mathcal{F}(f)_n := \langle f, e_n \rangle_{L^2(\mathbb{T})} = \int_0^1 f(x) \exp(-2\pi i n x)\, dx$. Show that if $f \in L^2(\mathbb{T})$ and $\mathcal{F}(f) \in \ell^1(\mathbb{Z})$, then $f \in C(\mathbb{T})$, i.e., f is a.e. equal to a continuous function.

Problem 10. Recall that each $T \in B(H, K)$ induces a bounded sesquilinear form $K \times H \to \mathbb{C}$ given by $B_T(\xi, \eta) = \langle \xi, T\eta \rangle$.

(1) Prove that $T \mapsto B_T$ is an isometric bijective correspondence between operators in $B(H, K)$ and bounded sesquilinear forms $K \times H \to \mathbb{C}$.
Hint: Adapt the proof Lemma 3.2.2 in Analysis Now (see also Exercise 3.2.15 therein).
(2) For $T \in B(H, K)$ corresponding to $B_T : K \times H \to \mathbb{C}$, we define $T^* \in B(K, H)$ to be the unique operator corresponding to the adjoint sesquilinear form $B_T^* : H \times K \to \mathbb{C}$ defined by

$$B_T^*(\eta, \xi) := \overline{B_T(\xi, \eta)} \qquad \Longleftrightarrow \qquad \langle \eta, T^*\xi \rangle = \langle T\eta, \xi \rangle \qquad \eta \in H, \xi \in K.$$

Show that $T \mapsto T^*$ is a conjugate linear isometry of $B(H, K)$ onto $B(K, H)$, and that $\|T^*T\| = \|T\|^2 = \|TT^*\|$.
(3) In the case that $H = K$, deduce the following:
(a) $B(H)$ with involution $T \mapsto T^*$ is a C*-algebra.
(b) $T = T^*$ if and only if B_T is self-adjoint. That is, show $T = T^*$ if and only if $\langle T\xi, \xi \rangle \in \mathbb{R}$ for all $\xi \in H$.
(c) $T \geq 0$ if and only if B_T is positive. That is, show $T \geq 0$ if and only if $\langle T\xi, \xi \rangle \geq 0$ for all $\xi \in H$.
Hint: Use that for $T = T^*$, we have $\inf\{\langle T\xi, \xi \rangle | \xi \in H, \|\xi\| = 1\} = \min\{\lambda | \lambda \in \mathrm{sp}(T)\}$.
(d) (optional) $T \geq 0$ and T injective if and only if B_T is positive definite.
Hint: For $S \in B(H)$, $\ker(S) = \ker(S^*S)$, so $T \geq 0$ is injective if and only if $T^{1/2}$ is injective.

(e) (optional) $T>0$ ($T\geq 0$ and T is invertible) if and only if B_T is positive definite, and H is complete in the norm $\|\xi\|_T := B_T(\xi,\xi)^{1/2}$.
Hint: When B_T is positive definite and H is complete for $\|\cdot\|_T$, apply part (d) and look at the isometry $(H,\|\cdot\|_T)\to(H,\|\cdot\|)$ by $\xi\mapsto T^{1/2}\xi$.

Problem 11 (Challenge!)**.** Suppose H is a Hilbert space. A *quadratic form* on H is a function $q: H\to\mathbb{C}$ such that:

(1) (quadratic) $q(\lambda\xi)=|\lambda|^2q(\xi)$ for all $\lambda\in\mathbb{C}$ and $\xi\in H$,
(2) (parallelogram identity) $q(\eta+\xi)+q(\eta-\xi)=2(q(\eta)+q(\xi))$ for all $\eta,\xi\in H$, and
(3) (continuous) There is a $C>0$ such that $|q(\xi)|\leq C\|\xi\|^2$ for all $\xi\in H$.

Prove that

$$(\eta,\xi) := \frac{1}{4}\sum_{k=0}^{3} i^k q(\eta+i^k\xi)$$

is a bounded sesquilinear form on H such that $q(\xi)=(\xi,\xi)$.

Problem 12. For a Hilbert space H, we can define the *conjugate* Hilbert space $\overline{H}=\left\{\overline{\xi}\middle|\xi\in H\right\}$ which has the conjugate vector space structure $\lambda\overline{\xi}+\overline{\eta}=\overline{\overline{\lambda}\xi+\eta}$ and the conjugate inner product $\langle\overline{\eta},\overline{\xi}\rangle_{\overline{H}}=\langle\xi,\eta\rangle_H$.

(1) Prove that $\overline{H}$ is a Hilbert space.
(2) For $T\in B(H,K)$, define $\overline{T}:\overline{H}\to\overline{K}$ by $\overline{T}\overline{\xi}=\overline{T\xi}$. Prove that $\overline{T}\in B(\overline{H},\overline{K})$, and $\|T\|=\|\overline{T}\|$.
(3) Prove that $\overline{\cdot}$ is an endofunctor on the the category Hilb of Hilbert spaces with bounded operators ($\overline{\cdot}$ is a functor $\mathsf{Hilb}\to\mathsf{Hilb}$).
(4) For each $H\in\mathsf{Hilb}$, construct a linear isometry u_H of H^* onto $\overline{H}$ satisfying $u_HT^t=\overline{T}u_H$ for all $T\in B(H,K)$ where $T^t\in B(K^*,H^*)$ is the Banach adjoint of T.

Problem 13. For $T\in B(H)$, we define its *numerical radius* as

$$R(T):=\sup_{\|\xi\|\leq 1}|\langle T\xi,\xi\rangle|.$$

Prove that $r(T)\leq R(T)\leq\|T\|\leq 2R(T)$. Deduce that if T is normal, then $\|T\|=R(T)$.

Problem 14. Let A be a C*-algebra. An element $u\in A$ is called a *partial isometry* if u^*u is a projection.

(1) Show that the following are equivalent:
   (a) u is a partial isometry.
   (b) $u=uu^*u$.
   (c) $u^*=u^*uu^*$.
   (d) u^* is a partial isometry.
   Hint: For (a) ⇒ (b), apply the C-axiom to $u-uu^*u$.*
(2) We say two projections $p,q\in A$ are *(Murray-von Neumann) equivalent*, denoted $p\approx q$, if there is a partial isometry $u\in A$ such that $uu^*=p$ and $u^*u=q$. Prove that $\approx$ is an equivalence relation on $P(A)$, the set of projections of A.
(3) Describe the set of equivalence classes $P(A)/\approx$ for $A=B(\ell^2)$.

Problem 15. Suppose $x=u|x|$ is the polar decomposition of $x\in B(H)$. Show that $x^*=u^*|x^*|$ is the polar decomposition.

Problem 16 (MO:325725)**.** Suppose A is a unital C*-algebra and $I\leq A$ is an ideal. Let $q:A\to A/I$ be the canonical surjection.

(1) Show that unital $*$-homomorphisms $C[0,1] \to A$ are in canonical bijection with positive elements of A with norm at most 1.
(2) Show that if $a + I \in A/I$ is positive with norm at most 1, there is a positive $\widetilde{a} \in A$ with norm at most 1 such that $\widetilde{a} + I = a + I$.
Hint: Since $\operatorname{sp}_{A/I}(a + I) \subseteq \operatorname{sp}_A(a)$, $f(q(a)) = q(f(a))$ *and thus* $f(a + I) = f(a) + I$ *for all* $f \in C(\operatorname{sp}_A(a))$. *Now pick* f *carefully.*
(3) Deduce that for every unital $*$-homomorphism $\phi : C[0,1] \to A/I$, there is a unital $*$-homomorphism $\widetilde{\varphi} : C[0,1] \to A$ with $\phi = q \circ \widetilde{\phi}$.
(4) Discuss the connection between the above statement and the Tietze Extension Theorem when A is commutative.

Problem 17. Let H be a Hilbert space. Compute the extreme points of the unit balls of
(1) $\mathcal{K}(H)$,
(2) $\mathcal{L}^1(H)$, and
(3) $B(H)$.

Problem 18. Let H be a Hilbert space. Prove that the trace Tr induces isometric isomorphims:
(1) $\mathcal{K}(H)^* \cong \mathcal{L}^1(H)$, and
(2) $\mathcal{L}^1(H)^* \cong B(H)$.

Problem 19. Suppose H is a Hilbert space and $K \subseteq H$ is a closed subspace. Let $p_K \in B(H)$ be associated orthogonal projection onto K.
(1) Suppose $x \in B(H)$. Prove that:
 (a) $xK \subseteq K$ if and only if $xp_K = p_K x p_K$.
 (b) $x^*K \subseteq K$ if and only if $p_K x = p_K x p_K$.
 (c) $xK \subseteq K$ and $x^*K \subseteq K$ if and only if $[x, p_K] = 0$.
(2) Prove that if $M \subseteq B(H)$ is a $*$-closed subalgebra, then $MK \subseteq K$ if and only if $p_K \in M'$.

Problem 20. Suppose H is a Hilbert space.
(1) Suppose K is another Hilbert space. Define the tensor product Hilbert space $H \overline{\otimes} K$ by completing the algebraic tensor product vector space $H \otimes K$ in the 2-norm associated to the sesquilinear form $\langle \eta \otimes \xi, \eta' \otimes \xi' \rangle := \langle \eta, \eta' \rangle \langle \xi, \xi' \rangle$. Find a unitary isomorphism $H \overline{\otimes} K \cong \bigoplus_{i=1}^{\dim K} H$.
(2) Find a unital $*$-isomorphism $B(\bigoplus_{i=1}^n H) \cong M_n(B(H))$.
Hint: use orthogonal projections.
(3) Suppose $S \subseteq B(H)$, and let $\alpha : B(H) \to M_n(B(H))$ be the amplification

$$x \longmapsto \begin{pmatrix} x & & \\ & \ddots & \\ & & x \end{pmatrix}.$$

Prove that:
 (a) $\alpha(S)' = M_n(S')$, and
 (b) If $0, 1 \in S$, then $M_n(S)' = \alpha(S')$.
 (c) Deduce that when $0, 1 \in S$, $\alpha(S)'' = \alpha(S'')$.

Problem 21. Let (X, μ) be a σ-finite measure space, and consider the map $M : L^\infty(X, \mu) \to B(L^2(X, \mu))$ given by $(M_f \xi)(x) = f(x)\xi(x)$ for $\xi \in L^2(X, \mu)$.
(1) Prove that M is an isometric unital $*$-homomorphism.
(2) Let $A \subset B(L^2(X, \mu))$ be the image of the map M. Prove that $A = A'$.
Hint: If you're stuck with (2), try the case $X = \mathbb{N}$ *with counting measure.*

Problem 22. Let H be a Hilbert space. The *weak operator topology (WOT)* on $B(H)$ is the topology induced by the separating family of seminorms $T \mapsto |\langle T\eta, \xi\rangle|$ for $\eta, \xi \in H$. The *strong operator topology (SOT)* on $B(H)$ is induced by the separating family of seminorms $x \mapsto \|T\xi\|_H$ for $\xi \in H$.

(1) Prove that every WOT open set is SOT open. Equivalently, prove that if a net $(T_\lambda)_{\lambda\in\Lambda} \subset B(H)$ converges to $T \in B(H)$ SOT, then $T_\lambda \to T$ WOT.
(2) Prove that the WOT is equal to the SOT on $B(H)$ if and only if H is finite dimensional.
(3) Show that the following are equivalent for a linear functional φ on $B(H)$:
   (a) There are $\eta_1, \ldots, \eta_n, \xi_1, \ldots, \xi_n \in H$ such that $\varphi(T) = \sum_{i=1}^n \langle T\eta_i, \xi_i\rangle$.
   (b) φ is WOT-continuous.
   (c) φ is SOT-continuous.

Problem 23. Suppose $M \subset B(H)$ is a unital $*$-subalgebra. A vector $\xi \in H$ is called:

- *cyclic* for M if $M\xi$ is dense in H.
- *separating* for M if for every $x, y \in M$, $x\xi = y\xi$ implies $x = y$.

(1) Prove that ξ is cyclic for M if and only if ξ is separating for M'.
(2) Prove that H can be orthogonally decomposed into M-invariant subspaces $H = \bigoplus_{i\in I} K_i$, such that each K_i is cyclic for M (has a cyclic vector). Prove that if H is separable, this decomposition is countable.
(3) Prove that if M is abelian and H is separable, then there is a separating vector in H for M.

Problem 24. Suppose H is a Hilbert space, and (x_λ) is an increasing net of positive operators in $B(H)$ which is bounded above by the positive operator $x \in B(H)$, i.e., $\lambda \le \mu$ implies $x_\lambda \le x_\mu$, and $0 \le x_\lambda \le x$ for all λ. Prove that the following are equivalent.

(1) $x_\lambda \to x$ SOT.
(2) $x_\lambda \to x$ WOT.
(3) For every $\xi \in H$, $\omega_\xi(x_\lambda) = \langle x_\lambda\xi, \xi\rangle \nearrow \langle x\xi, \xi\rangle = \omega_\xi(x)$.
(4) There exists a dense subspace $D \subset H$ such that for every $\xi \in D$, $\omega_\xi(x_\lambda) = \langle x_\lambda\xi, \xi\rangle \nearrow \langle x\xi, \xi\rangle = \omega_\xi(x)$.

We say an increasing net of positive operators (x_λ) *increases to* $x \in B(H)_+$, denoted $x_\lambda \nearrow x$, if any of the above equivalent conditions hold.
Hint: Show it suffices to prove (3) $\Rightarrow$ (1) *and* (4) $\Rightarrow$ (3). *Try proving these implications.*

Problem 25. Let H be a Hilbert space and let $T \in B(H)$. Prove that the following are equivalent. (You may use any results from last semester that you'd like without proof.)

(1) T is compact and normal.
(2) T has an orthonormal basis of eigenvectors $(e_i)_{i\in I}$ such that the corresponding eigenvalues $\lambda_i \to 0$, with at most countably many of the $\lambda_i \neq 0$.
(3) There is a countable orthonormal subset $(\xi_n)_{n\in\mathbb{N}} \subset H$ and a sequence $(\lambda_n) \subset \mathbb{C}$ such that $\lambda_n \to 0$ and $T = \sum_{n\in\mathbb{N}} \lambda_n |\xi_n\rangle\langle\xi_n|$, which converges in operator norm.
(4) There is a sequence $(\lambda_n) \subset \mathbb{C}$ such that $\lambda_n \to 0$ and a countable family of finite rank projections $E_n \subset B(H)$ such that $T = \sum_{n\in\mathbb{N}} \lambda_n E_n$, which converges in operator norm.
(5) There is a discrete set X equipped with counting measure ν, a function $f \in c_0(X)$, and a unitary $U \in B(\ell^2 X, H)$ such that $T = UM_fU^*$ where $M_f\xi = f\xi$ for $\xi \in \ell^2 X$.
   Note: $U \in B(K, H)$ *is unitary if* $UU^* = \mathrm{id}_H$ *and* $U^*U = \mathrm{id}_K$.

Problem 26. Suppose A is a unital C*-algebra. A linear map $\Phi : A \to B(H)$ is called *completely positive* if for every $a = (a_{i,j}) \ge 0$ in $M_n(A)$, $(\Phi(a_{i,j})) \ge 0$ in $M_n(B(H)) \cong B(H^n)$. Such a map is *unital* if $\Phi(1) = 1$.

(1) Show that $\langle x\otimes\eta, y\otimes\xi\rangle := \langle\Phi(y^*x)\eta,\xi\rangle_H$ on $A\otimes H$ linearly extends to a well-defined positive sesquilinear form.
(2) Show that for V a vector space with positive sesquilinear form $B(\,\cdot\,,\,\cdot\,)$, $N_B = \{v \in V | B(v,v) = 0\}$ is a subspace of V, and B descends to an inner product on V/N_B.
(3) Define K to be completion of $(A\otimes H)/N_{\langle\,\cdot\,,\,\cdot\,\rangle}$ in $\|\cdot\|_2$. Find a unital $*$-homormophism $\Psi : A \to B(K)$, and an isometry $v \in B(H,K)$ such that $\Phi(m) = v^*\Psi(m)v$.

Problem 27. Suppose $y \in B(H)$ is positive.

(1) Show that if $y \notin K(H)$, then there is a $\lambda > 0$ and a projection p with infinite dimensional range such that $y \geq \lambda p$.
(2) Deduce that if $x \mapsto \mathrm{Tr}(xy)$ is bounded on $\mathcal{L}^p(H)$ where $1 \leq p < \infty$, then $y \in K(H)$.

Problem 28. Suppose $A \subseteq B(H)$ is a unital C*-subalgebra and $\xi \in H$ is a cyclic vector for A. Consider the vector state $\omega_\xi = \langle\,\cdot\,\xi,\xi\rangle$. Prove there is a bijective correspondence between:

(1) positive linear functionals φ on A such that $0 \leq \varphi \leq \omega_\xi$ ($\omega_\xi - \varphi \geq 0$), and
(2) operators $0 \leq x \leq 1$ in A'.

Hint: For $0 \leq x \leq 1$ in A', define $\varphi_x(a) := \langle ax\xi,\xi\rangle$ for $a \in A$. (Why is $0 \leq \varphi_x \leq \omega_\xi$?) For the reverse direction, use the bijective correspondence between sesquilinear forms and operators.

Problem 29.

(1) Prove that a unital $*$-subalgebra $M \subseteq B(H)$ is a von Neumann algebra if and only if its unit ball is σ-WOT compact.
(2) Let $M \subset B(H)$ be a von Neumann algebra and $\Phi : M \to B(K)$ a unital $*$-homomorphism. Deduce that if Φ is σ-WOT continuous and injective, then $\Phi(M)$ is a von Neumann subalgebra of $B(K)$.

Problem 30. Suppose X is a compact Hausdorff topological space and $E : (X,\mathcal{M}) \to B(H)$ is a Borel spectral measure. Prove that the following conditions are equivalent.

(1) E is regular, i.e., for all $\xi \in H$, $\mu_{\xi,\xi}(S) = \langle E(S)\xi,\xi\rangle$ is a finite regular Borel measure.
(2) For all $S \in \mathcal{M}$, $E(S) = \sup\{E(K) | K \text{ is compact and } K \subseteq S\}$.
(3) For all $S \in \mathcal{M}$, $E(S) = \inf\{E(U) | U \text{ is open and } S \subseteq U\}$

Problem 31. Suppose $x \in B(H)$ is normal. Show that $\chi_{\{0\}}(x) = p_{\ker(x)}$ and $\chi_{\mathrm{sp}(x)\setminus\{0\}} = p_{\overline{xH}}$.

Problem 32. Let H be a separable Hilbert space and $A \subseteq B(H)$ an abelian von Neumann algebra. Prove that the following are equivalent.

(1) A is maximal abelian, i.e., $A = A'$.
(2) A has a cyclic vector $\xi \in H$.
(3) For every norm separable SOT-dense C*-subalgebra $A_0 \subset A$, A_0 has a cyclic vector.
(4) There is a norm separable SOT-dense C*-subalgebra $A_0 \subset A$ such that A_0 has a cyclic vector.
(5) There is a finite regular Borel measure μ on a compact Hausdorff second countable space X and a unitary $u \in B(L^2(X,\mu),H)$ such that $f \mapsto uM_f u^*$ is an isometric $*$-isomorphism $L^\infty(X,\mu) \to A$.

Hints:
For (1) $\Rightarrow$ (2), use Problem 23.
For (3) $\Rightarrow$ (4) it suffices to construct a norm separable SOT-dense C-algebra. First show that $A_* = \mathcal{L}^1(H)/A_\perp$ is a separable Banach space. Then show that A is σ-WOT separable, which implies SOT-separable. Take A_0 to be the unital C*-algebra generated by an SOT-dense sequence.*
For (4) $\Rightarrow$ (5) show that A_0 separable implies $X = \widehat{A}_0$ is second countable. Define $\mu = \mu_{\xi,\xi}$ on X, and show that the map $C(X) \to H$ by $f \mapsto \Gamma^{-1}(f)\xi$ is a $\|\cdot\|_2 - \|\cdot\|_H$ isometry with dense range.

Problem 33. Suppose $E : (X, \mathcal{M}) \to P(H)$ is a spectral measure with H separable, and let $A \subset B(H)$ be the unital C*-algebra which is the image of $L^\infty(E)$ under $\int \cdot \, dE$. Suppose there is a cyclic unit vector $\xi \in H$ for A.

1. Show that $\omega_\xi(f) = \langle (\int f dE)\xi, \xi \rangle$ is a faithful state on $L^\infty(E)$ ($\omega_\xi(|f|^2) = 0 \Longrightarrow f = 0$).
2. Consider the finite non-negative measure $\mu = \mu_{\xi,\xi}$ on $(X, \mathcal{M})$. Show that a measurable function f on $(X, \mathcal{M})$ is essentially bounded with respect to E if and only if f is essentially bounded with respect to μ.
3. Deduce that for essentially bounded measurable f on $(X, \mathcal{M})$, $\|f\|_E = \|f\|_{L^\infty(X,\mathcal{M},\mu)}$.
4. Construct a unitary $u \in B(L^2(X, \mathcal{M}, \mu), H)$ such that for all $f \in L^\infty(E) = L^\infty(X, \mathcal{M}, \mu)$, $(\int f dE)u = uM_f$.
5. Deduce that $A \subset B(H)$ is a maximal abelian von Neumann algebra.

Problem 34. Suppose H is a separable infinite dimensional Hilbert space. Prove that $K(H) \subset B(H)$ is the unique norm closed 2-sided proper ideal.

Problem 35. Classify all abelian von Neumann algebras $A \subset B(H)$ when H is separable.
Hint: Use a maximality argument to show you can write $1 = p + q$ with $p, q \in P(A)$ such that q is diffuse and $p = \sum p_i$ (SOT) with all p_i minimal. Then analyze Aq and Ap.

Problem 36. Suppose $M \subseteq B(H)$ is a von Neumann algebra and $p, q \in P(M)$. Define $p \wedge q \in B(H)$ to be the orthogonal projection onto $pH \cap qH$. Prove that $p \wedge q \in M$ two separate ways:

1. Show that $pH \cap qH$ is M'-invariant, and deduce $p \wedge q \in M$.
2. Show that $p \wedge q$ is the SOT-limit of $(pq)^n$ as $n \to \infty$.
   Hint: You could proceed as follows, but a quicker proof would be much appreciated!
   (a) *Use (2) of Problem 6 to show $(pq)^n p$ is a decreasing sequence of positive operators.*
   (b) *Show $(pq)^n p$ converges SOT to a positive operator $x \in M$.*
   (c) *Show that $x^2 = x$, and deduce $x \leq p$ is an orthogonal projection.*
   (d) *Show that $xqp = x$, and deduce $xqx = x$.*
   (e) *Show that $x \leq q$, and deduce $x \leq p \wedge q$.*
   (f) *Show that $(p \wedge q)(pq)^n$ converges SOT to both $p \wedge q$ and x, and deduce $x = p \wedge q$.*
   (g) *Finally, show $(pq)^n$ converges SOT to $xq = p \wedge q$.*

Define $p \vee q$ as the projection onto $\overline{pH + qH}$. Show that $p \vee q \in M$ in two separate ways:

1. Prove that $\overline{pH + qH}$ is M'-invariant, and deduce $p \vee q \in M$.
2. Show that $p \vee q = 1 - (1 - p) \wedge (1 - q)$ and use that $p \wedge q \in M$.

Problem 37. Suppose $N \subseteq M \subset B(H)$ is a unital inclusion of von Neumann algebra and $p \in P(N)$.

1. Prove that $(N'p) \cap pMp = (N' \cap M)p$.
2. Deduce that if $p \in P(M)$, $Z(pMp) = Z(M)p$.
3. Deduce that if $p \in P(M)$ and M is a factor, then pMp is a factor.
4. Prove that when M is a factor and $p \in P(M)$, the map $M' \to M'p$ by $x \mapsto xp$ is a unital $*$-algebra isomorphism.

Problem 38. Prove that the following conditions are equivalent for a von Neumann algebra $M \subseteq B(H)$:

1. Every non-zero $q \in P(M)$ majorizes an abelian projection $p \in P(M)$.
2. M is type I (every non-zero $z \in P(Z(M))$ majorizes an abelian $p \in P(M)$).
3. There is an abelian projection $p \in P(M)$ whose central support $z(p) = \bigvee_{u \in U(M)} u^* p u \in Z(M)$ is 1_M.

Hints:
For (2) ⇒ (3), if $p \in P(M)$ is abelian with $z(p) \neq 1$, then there is an abelian projection $q \in P(M)$ such that $z(q) \leq 1 - z(p)$. Show that $pMq = 0$ and $p + q$ is an abelian projection. Now use Zorn's Lemma.
For (3) ⇒ (1), suppose $p \in P(M)$ is abelian with $z(p) = 1$ and $q \in P(M)$ is non-zero. Show there is a non-zero partial isometry $u \in M$ such that $uu^* \leq p$ and $u^*u \leq q$. Deduce that uu^* is abelian, and then prove u^*u is abelian.

Problem 39. Show that for every von Neumann algebra M, there are unique central projections z_{I}, z_{II_1}, z_{II_∞}, and z_{III} (some of which may be zero) such that

- Mz_{I} is type I, Mz_{II_1} is type II_1, Mz_{II_∞} is type II_∞, and Mz_{III} is type III, and
- $z_{\mathrm{I}} + z_{\mathrm{II}_1} + z_{\mathrm{II}_\infty} + z_{\mathrm{III}} = 1$

Hint: You could proceed as follows:

(1) First, show that if M has an abelian projection p, then $z(p)$ is type I. Then use a maximality argument to construct z_{I}. For this, you could adapt the hint for (2) ⇒ (3) in Problem 38.
(2) Replacing M, H with $M(1 - z_{\mathrm{I}}), (1 - z_{\mathrm{I}})H$, we may assume M has no abelian projections. Show that if M has a finite central projection z, then Mz is type II_1. Now use a maximality argument to construct z_{II_1}. This hinges on proving the sum of two orthogonal finite central projections is finite. (Proving this is much easier than proving the sup of two finite projections is finite!)
(3) By compression, we may now assume that M has no abelian projections and no finite central projections. Show that if M has a nonzero finite projection p, then its central support $z(p)$ satisfies $Mz(p)$ is type II_∞. Use a maximality argument to construct z_{II_∞}.
(4) Compressing one more time, we may assume M has no finite projections, and thus M is purely infinite and type III.

Problem 40. Let $M \subseteq B(H)$ be a finite dimensional von Neumann algebra.

(1) Prove M has a minimal projection.
(2) Deduce that $Z(M)$ has a minimal projection.
(3) Prove that for any minimal projection $p \in Z(M)$, Mp is a type I factor.
(4) Prove that M is a direct sum of matrix algebras.

Problem 41. Suppose H is infinite dimensional. Prove that $B(H)$ does not admit a σ-WOT continuous tracial state.
Optional: Instead, prove that $B(H)$ does not admit a non-zero tracial linear functional.

Problem 42. Suppose $M \subseteq B(H)$ and $N \subseteq B(K)$ are von Neumann algebras, and let $H \overline{\otimes} K$ be the tensor product of Hilbert spaces as in Problem 20.

(1) Show that for every $m \in M$ and $n \in N$, the formula $(m \otimes n)(\eta \otimes \xi) := m\eta \otimes n\xi$ gives a unique well-defined operator $m \otimes n \in B(H \overline{\otimes} K)$.
(2) Let $M \overline{\otimes} N = \{m \otimes n | m \in M, n \in N\}'' \subset B(H \overline{\otimes} K)$. Show that the linear extension of the map from the algebraic tensor product $M \otimes N$ to $M \overline{\otimes} N$ given by $m \otimes n \mapsto m \otimes n$ is a well-defined injective unital $*$-algebra map onto an SOT-dense unital $*$-subalgebra.
Hint for injectivity: Suppose $x = \sum_{i=1}^{k} m_i \otimes n_i$ is not zero in $M \otimes N$. Reduce to the case $\{n_1, \ldots, n_k\}$ is linearly independent and all $m_i \neq 0$. Show that for each $i = 1, \ldots, k$, there exists a $k_i > 0$ and $\{\eta_j^i, \xi_j^i\}_{j=1}^{k_i}$ such that $\sum_{j=1}^{k_i} \langle n_{i'} \eta_j^i, \xi_j^i \rangle = \delta_{i=i'}$. (Sub-hint: Consider $F = \operatorname{span}_{\mathbb{C}}\{n_1, \ldots, n_k\} \subset N$, a closed normed space, and look at $\Phi : H \times \overline{H} \to F^*$ by $(\eta, \xi) \mapsto \langle \cdot\, \eta, \xi \rangle$. Show that $\operatorname{span}_{\mathbb{C}}(\Phi(H)) = F^*$.) Now pick $\kappa, \zeta \in H$ such that $\langle m_1 \kappa, \zeta \rangle \neq 0$, and deduce $\sum_{j=1}^{k_1} \langle x(\kappa \otimes \eta_j^1), \zeta \otimes \xi_j^1 \rangle_{H \overline{\otimes} K} \neq 0$.

(3) We denote by $B(H) \otimes 1$ the image of $B(H)$ under the map $x \mapsto x \otimes 1 \in B(H \overline{\otimes} K)$. Prove that $B(H) \otimes 1$ is a von Neumann algebra.
Hint: Show that $(B(H) \otimes 1)' = 1 \otimes B(K)$. Then by symmetry, $(1 \otimes B(K))' = B(H) \otimes 1$ is a von Neumann algebra.

(4) Prove that $B(H \overline{\otimes} K) = B(H) \overline{\otimes} B(K)$.
Hint: Calculate the commutant of the image of the algebraic tensor product $(B(H) \otimes B(K))' = \mathbb{C}1$ and use (2).

Problem 43. Let S_∞ be the group of finite permutations of $\mathbb{N}$.

(1) Show that S_∞ is ICC. Deduce that LS_∞ is a II_1 factor.
(2) Give an explicit description of a projection with trace k^{-n} for arbitrary $n, k \in \mathbb{N}$.
Hint: Find such a projection in $\mathbb{C}S_\infty \subset LS_\infty$.
(3) Find an increasing sequence $F_n \subset LS_\infty$ of finite dimensional von Neumann subalgebras such that $LS_\infty = (\bigcup_{n=1}^\infty F_n)''$.

Note: A II_1 factor which is generated by an increasing sequence of finite dimensional von Neumann subalgebras as in (3) above is called hyperfinite.

Problem 44. Let M be a von Neuman algebra. Suppose $a, b \in M$ with $0 \leq a \leq b$. Prove there is a $c \in M$ such that $a = c^*bc$. Deduce that a 2-sided ideal in a von Neumann algebra is *hereditary*: $0 \leq a \leq b \in M$ implies $a \in M$.

Problem 45. Let M be a factor. Prove that if M is finite or purely infinite, then M is algebraically simple, i.e., M has no 2-sided ideals.
Note: You may use that a II_1 factor has a (faithful σ-WOT continuous) tracial state.

Problem 46. A positive linear functional $\varphi \in M^*$ is called *completely additive* if for any family of pairwise orthogonal projections (p_i), $\varphi(\sum p_i) = \sum \varphi(p_i)$. (Here, $\sum p_i$ converges SOT.)

Suppose $\varphi, \psi \in M^*$ are completely additive and $p \in P(M)$ such that $\varphi(p) < \psi(p)$. Then there is a non-zero projection $q \leq p$ such that $\varphi(qxq) < \psi(qxq)$ for all $x \in M_+$ such that $qxq \neq 0$.
Hint: Choose a maximal family of mutually orthogonal projections $e_i \leq p$ for which $\psi(e_i) \leq \varphi(e_i)$. Consider $e = \bigvee e_i$, and show that $\psi(e) \leq \varphi(e)$. Set $q = p - e$, and show that for all projections $r \leq q$, $\varphi(r) < \psi(r)$. Then show $\varphi(qxq) < \psi(qxq)$ for all $x \in M_+$ such that $qxq \neq 0$.

Problem 47. Show that the following conditions are equivalent for a positive linear functional $\varphi \in M^*$ for a von Neumann algebra M:

(1) φ is σ-WOT continuous,
(2) φ is *normal*: $x_\lambda \nearrow x$ implies $\varphi(x_\lambda) \nearrow \varphi(x)$, and
(3) φ is *completely additive*: for any family of pairwise orthogonal projections (p_i), $\varphi(\sum p_i) = \sum \varphi(p_i)$. (Here, $\sum p_i$ converges SOT.)

Hint: For (3) $\Rightarrow$ (1), show if $p \in P(M)$ is non-zero, then pick $\xi \in H$ such that $\varphi(p) < \langle p\xi, \xi\rangle$. Use Problem 46 to find a non-zero $q \leq p$ such that $\varphi(qxq) < \langle xq\xi, q\xi\rangle$ for all $x \in M$. Use the Cauchy-Schwarz inequality to show $x \mapsto \varphi(xq)$ is SOT-continuous, and thus σ-WOT continuous. Now use Zorn's Lemma to consider a maximal family of mutually orthogonal projections $(q_i)_{i \in I}$ for which $x \mapsto \varphi(xq_i)$ is σ-WOT continuous. Show $\sum q_i = 1$. For finite $F \subseteq I$, define $\varphi_F(x) = \sum_{i \in F} \varphi(xq_i)$. Ordering finite subsets by inclusion, we get a net $(\varphi_F) \subset M_*$. Show that $\varphi_F \to \varphi$ in norm in M^*. Deduce that $\varphi \in M_*$ since $M_* \subset M^*$ is norm-closed.

Problem 48. Let $\Phi : M \to N$ be a unital $*$-homomorphism between von Neumann algebras.

(1) Prove that the following two conditions are equivalent:
  (a) Φ is *normal*: $x_\lambda \nearrow x$ implies $\Phi(x_\lambda) \nearrow \Phi(x)$.
  (b) Φ is σ-WOT continuous.

(2) Prove that if Φ is normal, then $\Phi(M) \subset N$ is a von Neumann subalgebra.
Hint: $\ker(\Phi) \subset M$ *is a* σ*-WOT closed 2-sided ideal.*
(3) Let φ be a normal state on a a von Neumann algebra M, and let $(H_\varphi, \Omega_\varphi, \pi_\varphi)$ be the cyclic GNS representation of M associated to φ, i.e., $H_\varphi = L^2(M, \varphi)$, $\Omega_\varphi \in H_\varphi$ is the image of $1 \in M$ in H_φ, and $\pi_\varphi(x)m\Omega_\varphi = xm\Omega_\varphi$ for all $x, m \in M$.
  (a) Show that π_φ is normal.
  (b) Deduce that if φ is faithful, then $M \cong \pi_\varphi(M) \subset B(H_\varphi)$ is a von Neumann algebra acting on H_φ.

Problem 49. Suppose $\Phi : M \to N$ is a unital $*$-algebra homomorphism between von Neumann algebras.

(1) Prove that the following conditions imply Φ is normal:
  (a) Φ is SOT-continuous on the unit ball of M.
  (b) Φ is WOT-continuous on the unit ball of M.
  (c) Suppose $N = N'' \subseteq B(H)$. For a dense subspace $D \subseteq H$, $m \mapsto \langle \Phi(m)\eta, \xi\rangle$ is WOT-continuous on M for any $\eta, \xi \in D$.
(2) (optional) Which of the conditions above are equivalent to normality of Φ?

Problem 50. Let M be a finite von Neumann algebra with a faithful σ-WOT continuous tracial state. Let $L^2M = L^2(M, \mathrm{tr})$ where Ω is the image of 1_M in L^2M. Identify M with its image in $B(L^2M)$ by part (3) of Problem 48.

(1) Show that $J : M\Omega \to M\Omega$ by $a\Omega \mapsto a^*\Omega$ is a conjugate-linear isometry with dense range.
(2) Deduce J has a unique extension to L^2M, still denoted J, which is a conjugate-linear unitary, i.e, $J^2 = 1$ and $\langle J\eta, J\xi\rangle = \langle \xi, \eta\rangle$ for all $\eta, \xi \in L^2M$.
Hint: Look at η, ξ *in* $M\Omega$.
(3) Calculate $Ja^*Jb\Omega$ for $a, b \in M$. Deduce that $JMJ \subseteq M'$.
(4) Show $\langle Ja^*Jb\Omega, c\Omega\rangle = \langle b\Omega, JaJc\Omega\rangle$ for all $a, b, c \in M$. Deduce $(JaJ)^* = Ja^*J$.
(5) Show $\langle Jy\Omega, a\Omega\rangle = \langle y^*\Omega, a\Omega\rangle$ for all $a \in M$ and $y \in M'$. Deduce $Jy\Omega = y^*\Omega$.
(6) Prove that for $y \in M'$, $(JyJ)^* = Jy^*J$.
Hint: Try the same technique as in (4).
(7) Show for all $a, b \in M$ and $x, y \in M'$, $\langle xJyJa\Omega, b\Omega\rangle = \langle JyJxa\Omega, b\Omega\rangle$.
(8) Deduce that $M' \subseteq (JM'J)' = JMJ$, and thus $M' = JMJ$.

Problem 51. Let Γ be a discrete group, and let $L\Gamma = \{\lambda_g\}'' \subset B(\ell^2\Gamma)$. Consider the faithful σ-WOT continuous tracial state $\mathrm{tr}(x) = \langle x\delta_e, \delta_e\rangle$ on $L\Gamma$.

(1) Show that $u\delta_g = \lambda_g$ uniquely extends to a unitary $u \in B(\ell^2\Gamma, L^2L\Gamma)$ such that for all $x \in L\Gamma$ and $\xi \in \ell^2\Gamma$, $L_x u\xi = ux\xi$ where $L_x \in B(L^2L\Gamma)$ is left multiplication by x, i.e., $L_x(y\Omega) = xy\Omega$.
(2) Deduce from Problem 50 that $L\Gamma' = R\Gamma$.

Problem 52. Use Problem 51 above to give the following alternative characterization of $L\Gamma$. Let

$$\ell\Gamma = \left\{x = (x_g) \in \ell^2\Gamma \middle| x * y \in \ell^2\Gamma \text{ for all } y \in \ell^2\Gamma\right\}$$

where $(x * y)_g = \sum_h x_h y_{h^{-1}g}$. Define a unital $*$-algebra structure on $\ell\Gamma$ by multiplication is convolution, the unit is δ_e, the the indicator function at $e \in \Gamma$ ($\delta_e(g) = \delta_{g=e}$), and the involution $*$ on $\ell\Gamma$ is given on $x \in \ell\Gamma$ by $(x^*)_g := \overline{x_{g^{-1}}}$.

(1) Show that $\ell\Gamma$ is a well-defined unital $*$-algebra under the above operations.
(2) For $x \in \ell\Gamma$ define $T_x : \ell^2\Gamma \to \ell^2\Gamma$ by $T_x y = x * y$. Prove $T_x \in B(\ell^2\Gamma)$.
Hint: Show that for all $x \in \ell\Gamma$ *and* $y, z \in \ell^2\Gamma$, $\langle T_x y, z\rangle = \langle y, T_{x^*} z\rangle$. *Then use the Closed Graph Theorem.*

(3) Prove that for all $x \in \ell\Gamma$, $T_x \in L\Gamma$.
Hint: Prove $T_x \in R\Gamma'$ and apply Problem 51.
(4) Deduce that $x \mapsto T_x$ is a unital $*$-algebra isomorphism $\ell\Gamma \to L\Gamma$.

Problem 53 (V. Jones)**.** Suppose $M = M_2(\mathbb{C})$ and φ is a state. Then $\varphi(x) = \operatorname{tr}(x\rho)$ for a unique density matrix $\rho \geq 0$ with $\operatorname{tr}(\rho) = 1$. Choosing a basis of eigenvectors for ρ, we may identify

$$\rho = \begin{pmatrix} \frac{1}{1+\lambda} & \\ & \frac{\lambda}{1+\lambda} \end{pmatrix}$$

for some $0 \leq \lambda \leq 1$. Observe that φ is faithful if and only if $0 < \lambda < 1$ if and only if ρ is invertible.

(1) Describe as best you can $L^2(M, \phi)$ in terms of λ.
(2) Show that the action of M on $L^2(M, \phi)$ is faithful.
(3) From this point on, assume $0 < \lambda < 1$. Consider $S : L^2(M, \varphi) \to L^2(M, \varphi)$ by $x\Omega \mapsto x^*\Omega$. Compute the polar decomposition $S = J\Delta^{1/2}$ where $\Delta = S^*S$.
(4) Show that $M' = JMJ = SMS$ on $L^2(M, \varphi)$.
(5) Show that for all $z \in \mathbb{C}$, $\Delta^z M \Delta^{-z} = M$.
(6) Deduce that we have a 1-parameter group of unitaries $t \mapsto \sigma_t := \Delta^{it}$ for $t \in \mathbb{R}$ which preserve M.

Problem 54. Repeat Problem 52 for the crossed product von Neumann algebra $M \rtimes_\alpha \Gamma$ acting on $L^2M \otimes \ell^2\Gamma \cong L^2(\Gamma, L^2M)$ where M is a finite von Neumann algebra with faithful normal tracial state tr, Γ is a discrete group, and $\alpha : \Gamma \to \operatorname{Aut}(M)$ is an action. Here, we define

$$\ell^2(\Gamma, M) = \left\{ x : \Gamma \to M \,\middle|\, \sum_g \|x_g\Omega\|^2_{L^2M} < \infty \right\}$$
$$\ell^2(\Gamma, L^2M) = \left\{ \xi : \Gamma \to L^2M \,\middle|\, \sum_g \|\xi_g\|^2 < \infty \right\} \text{ and}$$
$$M \propto_\alpha \Gamma = \left\{ x = (x_g) \in \ell^2(\Gamma, M) \middle| x * \xi \in \ell^2(\Gamma, L^2M) \text{ for all } \xi \in \ell^2(\Gamma, L^2M) \right\}.$$

Here, the convolution action is given by $(x * \xi)_g = \sum_h x_h v_h \xi_{h^{-1}g}$ where $v_h \in U(L^2M)$ is the unitary implementing $\alpha_u \in \operatorname{Aut}(M)$. Define an analogous unital $*$-algebra structure on $M\Gamma$ and find a unital $*$-algebra isomorphism $M \propto_\alpha \Gamma \to M \rtimes_\alpha \Gamma$.
Hint: Similar to $L\Gamma$, some people write elements of $M \rtimes_\alpha \Gamma$ as formal sums $\sum_g x_g u_g$ which does not converge in any operator topology. Rather, $\sum_g x_g u_g(\Omega \otimes \delta_e)$ converges in $L^2M \otimes \ell^2\Gamma$. These formal sums can be algebraically manipulated to obtain a unital $*$-algebra structure using the covariance condition $u_g m u_g^* = \alpha_g(m)$ for all $g \in \Gamma$ and $m \in M$. Thus

$$\left(\sum_g x_g u_g\right)^* = \sum_g u_g x_g^* = \sum_g u_g x_g^* u_g^* u_g = \sum_g \alpha_g(x_g^*) u_g.$$

Thus for $x = (x_g) \in M \propto_\alpha \Gamma$, we define $(x^*)_g = \alpha_g(x_g^*)$. A similar algebraic manipulation gives the formula for multiplication, which is similar to convolution, but involves the action.

Problem 55. Prove that a $*$-isomorphism between von Neumann algebras is automatically normal.

Problem 56. Suppose (X, μ) is a measure space and $T : X \to X$ is a measurable bijection preserving the measure class of μ. Let $\alpha_T \in \operatorname{Aut}(L^\infty(X, \mu))$ by $(\alpha_T f)(x) = f(T^{-1}x)$. Is it always the case that the condition $\mu(\{x \in X | Tx = x\}) = 0$ is equivalent to the automorphism α_T being free? If yes, give a proof, and if not, find a counterexample together with a mild condition under which it is true.

Problem 57. Let $\mathbb{F}_2 = \langle a, b\rangle$ be the free group on 2 generators.

(1) Show that $\mathbb{F}_2$ is ICC. Deduce $L\mathbb{F}_2$ is a II_1 factor.
(2) Show that the swap $a \leftrightarrow b$ extends to an automorphism σ of $L\mathbb{F}_2$.
(3) Show that σ is outer.

Problem 58.

(1) (Fell's Absorption Principle) Suppose Γ is a countable group and (H, π) is a unitary representation on a separable Hilbert space. Find a unitary $u \in B(\ell^2\Gamma\overline{\otimes}H)$ intertwining $\lambda\otimes\pi$ and $\lambda\otimes 1$, i.e., $u(\lambda_g\otimes\pi_g) = (\lambda_g\otimes 1)u$ for all $g \in \Gamma$.
(2) Consider the two definitions of $M \rtimes_\alpha \Gamma$ when (M, tr) is a tracial von Neumann algebra and $\mathrm{tr}\circ\alpha_g = \mathrm{tr}$ for all $g \in \Gamma$. The first is the von Neumann algebra generated by the π_m and u_g on $\ell^2(\Gamma, L^2M)$ where
$$(u_g\xi)(h) := \xi(g^{-1}h) \qquad\qquad (\pi_m\xi)(h) = \alpha_{h^{-1}}(m)\xi(h).$$
The second is the von Neumann algebra generated by the π_m and u_g on $L^2M\otimes\ell^2\Gamma$ given by
$$\pi_m(x\Omega\otimes\delta_h) = mx\Omega\otimes\delta_h \qquad\qquad u_g(x\Omega\otimes\delta_h) = \alpha_g(x)\Omega\otimes\delta_{gh}.$$
Find a unitary isomorphism $\ell^2(\Gamma, L^2M) \to L^2M\otimes\ell^2\Gamma$ intertwining the two M-actions and Γ-actions. Deduce the two definitions of $M\rtimes_\alpha\Gamma$ are equivalent.

Problem 59. Prove that irrational rotation on the circle (with Lebesgue/Haar measure) is free and ergodic.

Problem 60. Let M be a finite von Neumann algebra with a faithful normal tracial state.

(1) Show for all $x, y \in M$, $|\operatorname{tr}(xy)| \leq \|y\| \operatorname{tr}(|x|)$.
(2) Show for all $x \in M$, $\operatorname{tr}(|x|) = \sup\{|\operatorname{tr}(xy)| | y \in M \text{ with } \|y\| = 1\}$.
(3) Define $\|x\|_1 = \operatorname{tr}(|x|)$ on M. Show that $\|\cdot\|_1$ is a norm on M.
(4) Define a map $\varphi : M \to M_*$ by $x \mapsto \varphi_x$ where $\varphi_x(y) = \operatorname{tr}(xy)$. Show that φ is a well-defined isometry from $(M, \|\cdot\|_1) \to M_*$ with dense range.
(5) Deduce that $L^1(M, \mathrm{tr}) := \overline{M}^{\|\cdot\|_1}$ is isometrically isomorphic to the predual M_*.

Problem 61. Continue the notation of Problem 60. Let $N \subseteq M$ be a (unital) von Neumann subalgebra.

(1) Prove that the inclusion $N \to M$ extends to an isometric inclusion $i : L^1(N, \mathrm{tr}) \to L^1(M, \mathrm{tr})$.
(2) Let $E : M \to N$ be the Banach adjoint of i under the identification $M_* = L^1(M, \mathrm{tr})$ and $N_* = L^1(N, \mathrm{tr})$. Show that E is uniquely characterized by the equation
$$\operatorname{tr}_M(xy) = \operatorname{tr}_N(E(x)y) \qquad x \in M, y \in N.$$

Note: E is called the canonical trace-preserving conditional expectation $M \to N$.

Problem 62. Suppose M is a finite von Neumann algebra with normal faithful tracial state tr and $N \subseteq M$ is a (unital) von Neumann subalgebra.

(1) Prove that the inclusion $N \to M$ extends to an isometric inclusion $L^2(N, \mathrm{tr}) \to L^2(M, \mathrm{tr})$.
(2) Define $e_N \in B(L^2M, L^2N)$ be the orthogonal projection with range $L^2(N, \mathrm{tr}) = \overline{N\Omega}^{\|\cdot\|_2} \subset L^2(M, \mathrm{tr})$. Show that for all $x \in M$, $e_N x e_N^* \subset B(L^2N)$ commutes with the right action of N, and thus defines an element in N by Problem 50.
Hint: Show the inclusion $e_N^* : L^2N \to L^2M$ commutes with the right N action, and deduce e_N commutes with the right N action.
(3) For $x \in M$, define $E(x) = e_N x e_N^*$. Show that $E(x)$ is uniquely characterized by the equation
$$\operatorname{tr}_M(xy) = \operatorname{tr}_N(E(x)y) \qquad x \in M, y \in N.$$

Note: E is called the canonical trace-preserving conditional expectation $M \to N$. *Part (3) implies this definition agrees with that from Problem 61.*

Problem 63. Continue the notation of Problem 62.

(1) Deduce that E is normal.
(2) Deduce $E(1) = 1$ and E is $N-N$ bilinear, i.e., for all $x \in M$ and $y, z \in N$, $E(yxz) = yE(x)z$.
(3) Deduce that $E(x^*) = E(x)^*$.
(4) Show that E is completely positive, which was defined in Problem 26.
Hint: Use the characterization $E(x) = e_N x e_N^*$ from (5) of Problem 62.
(5) Show that $E(x)^*E(x) \leq E(x^*x)$ for all $x \in M$.
Hint: Use the characterization $E(x) = e_N x e_N^*$ from (5) of Problem 62. Show that $e_N^* e_N$ is an orthogonal projection.
(6) Show that E is faithful: $E(x^*x) = 0$ implies $x^*x = 0$.
Hint: Prove this by looking at the vector states $\omega_{n\Omega}$ for $n \in N$.

Problem 64. Suppose M is a finite von Neumann algebra with faithful normal tracial state tr. Suppose further that there is an increasing sequence of von Neumann subalgebras $M_1 \subset M_2 \subset \cdots M$ such that $(\bigcup M_n)'' = M$ (considered as acting on L^2M). Let $E_n : M \to M_n$ be the canonical trace-preserving conditional expectation from Problem 62.

(1) Prove that the $\|\cdot\|_2$-topology agrees with the SOT on the unit ball of M. That is, prove that $x_n \to x$ SOT if and only if $\|x_n\Omega - x\Omega\|_2 \to 0$.
(2) Prove that for all $x \in M$, $\|E_n(x)\Omega - x\Omega\|_2 \to 0$ as $n \to \infty$.
(3) Deduce that $E_n(x) \to x$ SOT as $n \to \infty$.

Problem 65. Suppose Γ is a countable group, and let $\text{Prob}(\Gamma) = \left\{\mu \in \ell^1\Gamma \middle| \mu \geq 0 \text{ and } \sum_g \mu(g) = 1\right\}$.

(1) Prove that $\text{Prob}(\Gamma)$ is weak* dense in the state space of $\ell^\infty\Gamma$.
(2) Let $F \subset \Gamma$ be finite, and consider $\bigoplus_{g\in F} \ell^1\Gamma$ with the (product) weak topology. Let K be the weak closure of $\left\{\bigoplus_{g\in F} g\cdot\mu - \mu \middle| \mu \in \text{Prob}(\Gamma)\right\} \subset \bigoplus_{g\in F} \ell^1\Gamma$. Prove K is convex and norm closed in $\bigoplus_{g\in F} \ell^1\Gamma$.
(3) Now assume Γ is amenable, i.e., there is a left Γ-invariant state on $\ell^\infty\Gamma$. Prove that $0 \in K$. Deduce that Γ has an approximately invariant mean.

Problem 66. Suppose Γ is a countable group, and let $\text{Prob}(\Gamma)$ be as in Problem 65.

(1) Prove that if $a, b \in [0, 1]$, then
$$|a - b| = \int_0^1 |\chi_{(r,1]}(a) - \chi_{(r,1]}(b)|\, dr.$$
(2) Deduce that for $\mu \in \text{Prob}(\Gamma)$ and $h \in \Gamma$,
$$\|h\cdot\mu - \mu\|_{\ell^1\Gamma} = \int_0^1 \sum_{g\in\Gamma} |\chi_{(r,1]}(\mu(h^{-1}g)) - \chi_{(r,1]}(\mu(g))|\, dr.$$
(3) For $r \in [0, 1]$ and $\mu \in \text{Prob}(\Gamma)$, let $E(\mu, r) = \{g \in \Gamma | \mu(g) > r\}$. Show that for all $h \in \Gamma$, $hE(\mu, r) = \{g \in \Gamma | (h\cdot\mu)(g) > r\}$.
(4) Calculate $\int_0^1 |E(\mu, r)|\, dr$.
(5) Show that for $r \in [0, 1]$, $\mu \in \text{Prob}(\Gamma)$, and $h \in \Gamma$,
$$|hE(\mu, r) \triangle E(\mu, r)| = \sum_{g\in\Gamma} |\chi_{(r,1]}(\mu(h^{-1}g)) - \chi_{(r,1]}(\mu(g))|.$$
Deduce that $\|h\cdot\mu - \mu\|_1 = \int_0^1 |hE(\mu, r) \triangle E(\mu, r)|\, dr$.

(6) Suppose now that Γ has an approximate invariant mean, so that for every finite subset $F \subset \Gamma$ and $\varepsilon > 0$, there is a $\mu \in \text{Prob}(\Gamma)$ such that

$$\sum_{h \in F} \|h \cdot \mu - \mu\|_1 < \varepsilon.$$

Show that for the μ corresponding to this F and ε,

$$\int_0^1 \sum_{h \in F} |hE(\mu, r) \triangle E(\mu, r)| \, dr < \varepsilon \int_0^1 |E(\mu, r)| \, dr.$$

Deduce there is an $r \in [0, 1]$ such that $|hE(\mu, r) \triangle E(\mu, r)| < \varepsilon |E(\mu, r)|$ for all $h \in F$.

(7) Use (6) above to construct a Følner sequence for Γ.

Problem 67. Recall that an *ultrafilter* ω on a set X is a nonempty collection of subsets of X such that:

- $\emptyset \notin \omega$,
- If $A \subseteq B \subseteq X$ and $A \in \omega$, then $B \in \omega$,
- If $A, B \in \omega$, then $A \cap B \in \omega$, and
- For all $A \subset X$, either $A \in \omega$ or $X \setminus A \in \omega$ (but not both!).

(1) Find a bijection from the set of ultrafilters on $\mathbb{N}$ to $\beta\mathbb{N}$, the Stone-Cech compactification of $\mathbb{N}$.

(2) Let ω be an ultrafilter on $\mathbb{N}$. Let X be a compact Hausdorff space and $f : \mathbb{N} \to X$. We say
   - $x = \lim_{n \to \omega} f(n)$ if for every open neighborhood U of x, $f^{-1}(U) \in \omega$.

   Prove that $\lim_{n \to \omega} f(n)$ always exists for any function $f : \mathbb{N} \to X$.

(3) An ultrafilter on $\mathbb{N}$ is called *principal* if it contains a finite set. Show that every principal ultrafilter on $\mathbb{N}$ contains a unique singleton set, and that any two principal ultrafilters containing the same singleton set are necessarily equal. Thus we may identify the set of principal ultrafilters on $\mathbb{N}$ with $\mathbb{N} \subset \beta\mathbb{N}$.

(4) Determine $\lim_{n \to \omega} f(n)$ for $f : \mathbb{N} \to X$ as in (2) when ω is principal.

(5) An ultrafilter on $\mathbb{N}$ is called *free* or *non-principal* if it does not contain a finite set. Let ω be a free ultrafilter on $\mathbb{N}$. Suppose $\Gamma = \bigcup \Gamma_n$ is a locally finite group and m_n is the uniform probability (Haar) measure on Γ_n. Define $m : 2^\Gamma \to [0, 1]$ by $m(A) = \lim_{n \to \omega} m_n(A \cap \Gamma_n)$. Prove that m is a left Γ-invariant finitely additive probability measure on Γ, i.e., Γ is amenable.

Problem 68. Let X be a uniformly convex Banach space and $B \subset X$ a bounded set. Prove that the function $f : X \to [0, \infty)$ given by $f(x) = \sup_{b \in B} \|b - x\|_X$ achieves its minimum at a unique point of X.

Problem 69. Let Γ be a countable discrete group. Show that an affine action $\alpha = (\pi, \beta) : \Gamma \to \text{Aff}(H)$ ($\alpha_g \xi := \pi_g \xi + \beta(g)$ for $\pi_g \in U(H)$ and $\beta(g) \in H$ such that $\alpha_g \circ \alpha_h = \alpha_{gh}$ for all $g, h \in \Gamma$) is proper if and only if the cocycle part $\beta : \Gamma \to H$ is proper ($g \mapsto \|\beta(g)\|$ is a proper map).

Problem 70. Recall that the *Schur product* of two matrices $a, b \in M_n(\mathbb{C})$ is given by the entry-wise product: $(a * b)_{i,j} := a_{i,j} b_{i,j}$.

(1) Prove that if $a, b \geq 0$, then $a * b \geq 0$.

(2) Suppose that $p \in \mathbb{R}[z]$ is a polynomial whose coefficients are all non-negative. Prove that if $a \geq 0$, then $p[a] \geq 0$, where $p[a]_{i,j} := p(a_{i,j})$ for $a \in M_n(\mathbb{C})$.

   Note: Here we use the notation $p[a]$ to not overload the functional calculus notation.

(3) Suppose that f is an entire function whose Taylor expansion at 0 has only non-negative real coefficients. Prove that is $a \geq 0$, then $f[a] \geq 0$, where again $f[a]_{i,j} := f(a_{i,j})$ for $a \in M_n(\mathbb{C})$.

Problem 71. Let A be a unital C*-algebra.

(1) Prove that a map $\Phi : A \to M_n(\mathbb{C})$ is completely positive if and only if the map $\varphi : M_n(A) \to \mathbb{C}$ given by $(a_{i,j}) \mapsto \sum_{i,j}^n \Phi(a_{i,j})_{i,j}$ is positive.
Hint: for one direction, note that $\varphi(a) = \vec{e}^*\Phi(a)\vec{e}$ where $\vec{e} \in \mathbb{C}^{n^2}$ is the vector $(e_1, e_2, \dots, e_n)$ where $e_i \in \mathbb{C}^n$ is the i-th standard basis vector. For the other direction, use GNS with respect to φ, and consider $V : \mathbb{C}^n \to L^2(M_n(A), \varphi)$ given by $Ve_i = \pi_\varphi(E_{ij})\Omega_\varphi$ where (E_{ij}) is a system of matrix units in $M_n(\mathbb{C}) \subseteq M_n(A)$. Then use Stinespring.

(2) Let $S \subset A$ be an operator subsystem, and let $\psi : S \to \mathbb{C}$ be a positive linear functional. Prove $\|\psi\| = \psi(1)$. Deduce that any norm-preserving (Hahn-Banach) extension of ψ to A is also positive.

(3) Let $S \subset A$ be an operator subsystem, and let $\Phi : S \to M_n(\mathbb{C})$ be a (unital) completely positive map. Show that Φ extends to a (unital) completely positive map $A \to M_n(\mathbb{C})$.

Problem 72. Suppose Γ is a countable discrete group, and suppose $\varphi : L\Gamma \to L\Gamma$ is a normal completely positive map. Prove that $f : \Gamma \to \mathbb{C}$ given by $f(g) := \mathrm{tr}_{L\Gamma}(\varphi(\lambda_g)\lambda_g^*)$ is a positive definite function.

Problem 73. Prove that the following are equivalent for a finite von Neumann algebra $(M, \mathrm{tr}) \subset B(H)$ with faithful normalized tracial state.

(1) M is amenable, i.e., there is a conditional expectation $E : B(H) \to M$.

(2) There is a sequence $(\varphi_n : M \to M)$ of (normal) trace-preserving completely positive maps such that $\varphi_n \to \mathrm{id}$ pointwise in $\|\cdot\|_M$, and for all $n \in \mathbb{N}$, the induced map $\widehat{\varphi}_n \in B(L^2M)$ given by $m\Omega \mapsto \varphi_n(m)\Omega$ is finite rank.

Problem 74. Suppose that Γ is a countable discrete group such that every cocycle is inner. Suppose (H, π) is a unitary representation and $(\xi_n) \subset H$ is a sequence of unit vectors such that $\|\pi_g\xi_n - \xi_n\| \to 0$ as $n \to \infty$ for all $g \in \Gamma$. Follow the steps below to find a non-zero Γ-invariant vector in H. (We may assume that no ξ_n is fixed by Γ.)

(1) Enumerate $\Gamma = \{g_1, g_2, \dots\}$. Explain why you can pass to a subsequence of (ξ_n) to assume that for all $n \in \mathbb{N}$, $\|\pi_{g_i}\xi_n - \xi_n\| < 4^{-n}$ for all $1 \leq i \leq n$.

(2) For $n \in \mathbb{N}$, consider the inner cocycles $\beta_n(g) := \xi_n - \pi_g\xi_n$. Let $(K, \sigma) = \bigoplus_{n\in\mathbb{N}}(H, \pi)$. Define $\beta : \Gamma \to K$ by $\beta(g)_n := 2^n\beta_n(g)$. Prove that $\beta(g) \in H$ is well-defined for every $g \in \Gamma$. Then show that β is a cocycle for (K, σ).

(3) Deduce β is inner and thus bounded. Thus there is a $\kappa \in K \setminus \{0\}$ such that $\beta(g) = \kappa - \sigma_g\kappa$ for all $g \in \Gamma$.

(4) Prove that $\|\beta_n(g)\| \to 0$ uniformly for $g \in \Gamma$. That is, show that for all $\varepsilon > 0$, there is an $N \in \mathbb{N}$ such that $n > N$ implies $\|\beta_n(g)\| < \varepsilon$ for all $g \in \Gamma$.

(5) Fix $N \in \mathbb{N}$ such that $\|\beta_N(g)\| = \|\xi_N - \pi_g\xi_N\| < 1$ for all $g \in \Gamma$. Show there is a $\xi_0 \in H \setminus \{0\}$ such that $\pi_g\xi_0 = \xi_0$ for all $g \in \Gamma$.
Hint: Look at $\{\pi_g\xi_N | g \in \Gamma\} \subset (H)_1$ and apply Problem 68.

(6) (optional) Use a similar trick to finish the proof of (1) $\Rightarrow$ (2) from the same theorem from class.

Problem 75 (optional)**.** As best as you can, edit the equivalent definitions I gave in class for property (T) for a countable discrete group Γ to be relative to a subgroup $\Lambda \leq \Gamma$. Then prove all the equivalences.

Problem 76. Suppose $\Gamma \curvearrowright (X, \mu)$ is a free p.m.p. action and $\mathcal{R} = \{(x, gx) | x \in X, g \in \Gamma\}$ is the corresponding countable p.m.p. equivalence relation. Follow the steps below to show $L^\infty(X, \mu) \rtimes \Gamma \cong L\mathcal{R}$.

(1) Prove that $\theta : (x, g) \mapsto (x, g^{-1}x)$ induces a unitary operator $v \in B(L^2\mathcal{R}, L^2(X \times \Gamma, \mu \times \gamma))$ where γ is counting measure on Γ.
(2) Deduce that θ is a p.m.p. isomorphism $(X \times \Gamma, \mu \times \gamma) \to (\mathcal{R}, \nu)$.
(3) Show that $v^* M_f v = \lambda(f)$ for all $f \in L^\infty(X, \mu)$. Here, $(M_f \xi)(x, g) = f(x)\xi(x, g)$ for $\xi \in L^2(X \times \Gamma, \mu \times \gamma)$.
(4) Show that $v^* u_g v = L_{\varphi_g}$ where $\varphi_g \in [\mathcal{R}]$ is the isomorphism $x \mapsto g \cdot x$. Here, $(u_g\xi)(x, h) = \xi(g^{-1}x, g^{-1}h)$ for all $\xi \in L^2(X \times \Gamma, \mu \times \gamma) \cong L^2(X, \mu) \otimes \ell^2\Gamma$.
(5) Deduce that $v^*(L^\infty(X, \mu) \rtimes \Gamma)v \subset L\mathcal{R}$.
(6) Show that conjugation by v takes the commutant of $L^\infty(X, \mu) \rtimes \Gamma$ into $R\mathcal{R}$.
Hint: Show that right multiplication by $L^\infty(X, \mu)$ and the right action of u_g are both taken into $R\mathcal{R}$.
(7) Deduce that $v^*(L^\infty(X, \mu) \rtimes \Gamma)v = L\mathcal{R}$.

Problem 77. Let $\mathcal{R}$ be a countable p.m.p. equivalence relation on (X, μ). Let $A = L^\infty(X, \mu) \subset L\mathcal{R}$. Prove that the von Neumann subalgebra of $B(L^2(\mathcal{R}, \nu))$ generated by $A \cup JAJ$ is the von Neumann algebra of multiplication operators by elements of $L^\infty(\mathcal{R}, \nu)$.

Problem 78. Let M be a von Neumann algebra. A *weight* on M is a function $\varphi : M_+ \to [0, \infty]$ such that for all $r \in [0, \infty)$ and $x, y \in B(H)_+$, $\varphi(rx + y) = r\varphi(x) + \varphi(y)$, with the convention that for $s \in [0, \infty)$,

$$\infty \cdot s = \begin{cases} \infty & \text{if } s > 0 \\ 0 & \text{if } s = 0. \end{cases}$$

Define

$$\mathfrak{p}_\varphi = \{x \in M | \varphi(x) < \infty\}$$
$$\mathfrak{n}_\varphi = \{x \in M | x^*x \in \mathfrak{p}_\varphi\}$$
$$\mathfrak{m}_\varphi = \mathfrak{n}_\varphi^* \mathfrak{n}_\varphi = \left\{ \sum_{i=1}^n x_i^* y_i \middle| x_i, y_i \in \mathfrak{n}_\varphi \text{ for all } i = 1, \ldots, n \right\}.$$

(1) Prove that
  (a) $\mathfrak{p}_\varphi$ is a hereditary subcone of M_+, i.e.,
    - (subcone) $r \geq 0$ and $x, y \in \mathfrak{p}_\varphi$ implies $rx + y \in \mathfrak{p}_\varphi$
    - (hereditary) $0 \leq x \leq y$ and $y \in \mathfrak{p}_\varphi$ implies $x \in \mathfrak{p}_\varphi$.
  (b) $\mathfrak{n}_\varphi$ is a left ideal of M.
  Hint: Prove that for all $x, y \in M$, $(x \pm y)^*(x \pm y) \leq 2(x^*x + y^*y)$.
  (c) $\mathfrak{m}_\varphi$ is algebraically spanned by $\mathfrak{p}_\varphi$.
  Hint: Use polarization.
  (d) $\mathfrak{m}_\varphi \cap M_+ = \mathfrak{p}_\varphi$.
  (e) $\mathfrak{m}_\varphi$ is a hereditary $*$-subalgebra of M (hereditary is defined the same way as above).
(2) When $M = B(H)$ and $\varphi = \mathrm{Tr}$, show $\mathfrak{m}_{\mathrm{Tr}} = \mathcal{L}^1(H)$ and $\mathfrak{n}_{\mathrm{Tr}} = \mathcal{L}^2(H)$.