6. Analytic and approximation properties

We discuss various analytic and approximation properties for countable discrete groups. In this section, Γ always denotes a countable discrete group.

6.1. **Positive definite functions and cp multipliers.**

This section follows a mini-course I took from Narutaka Ozawa at IMSc in February 2009. Let Γ be a discrete countable group.

Definition 6.1.1. A function $f:\Gamma\to\mathbb{C}$ is called *positive definite* if for every $g_1,\dots,g_n\in\Gamma$, $[f(g_i^{-1}g_j)]$ is positive in $M_n(\mathbb{C})$.

Lemma 6.1.2. *Suppose $a\in M_n(\mathbb{C})$ is positive and constant along the diagonal. Then $|a_{ij}|\leq a_{kk}$ for all $1\leq i,j,k\leq n$.*

Proof. Let $b\in M_n(\mathbb{C})$ such that $a=b^*b$. Then for all i,j,

$$|a_{ij}|^2=|\langle e_i|ae_j\rangle|^2=|\langle be_i|be_j\rangle|^2\underset{\text{(CS)}}{\leq}\|be_i\|^2\|be_j\|^2=\langle e_i|ae_i\rangle\cdot\langle e_j|ae_j\rangle=a_{ii}a_{jj}.$$

Since $a_{ii}=a_{jj}$, we have $|a_{ij}|\leq a_{ii}$. □

Proposition 6.1.3. *If $f:\Gamma\to\mathbb{C}$ is positive definite, then $f\in\ell^\infty\Gamma$ with $\|f\|_\infty=f(e)$.*

Proof. For $g\in\Gamma$, $|f(g)|=|a_{12}|\leq a_{11}=f(e)$ for $a=\begin{pmatrix}f(e)&f(g)\\f(g^{-1})&f(e)\end{pmatrix}\geq 0$. □

Definition 6.1.4. Given $f:\Gamma\to\mathbb{C}$, we get a *multiplier* $M_f:\mathbb{C}\Gamma\to\mathbb{C}\Gamma$ by

$$M_f\sum x_g g:=\sum f(g)x_g g.$$

Theorem 6.1.5. *For $f:\Gamma\to\mathbb{C}$, the following are equivalent:*

1. *f is positive definite.*
2. *The sesquilinear form $\langle\sum x_g g,\sum y_h h\rangle_f:=\sum f(h^{-1}g)x_g\overline{y_h}$ on $\mathbb{C}\Gamma$ is positive.*
3. *f is a coefficient of a unitary representation, i.e., there is a Hilbert space H and group homomorphism $\pi:\Gamma\to U(H)$ and $\eta\in H$ such that $f(g)=\langle\pi_g\eta,\eta\rangle$.*
4. *M_f extends to a normal cp map $L\Gamma\to L\Gamma$.*

Proof.
(1) ⇔ (2): Observe that $[f(g_i^{-1}g_j]\in M_n(\mathbb{C})$ is positive if and only if for all $x\in\mathbb{C}^n$, $x^*[f(g_i^{-1}g_j)]x\geq 0$. This condition is equivalent to $\langle\cdot,\cdot\rangle_f\geq 0$.
(2) ⇒ (3): Let $\ell^2_f\Gamma$ denote the completion of the quotient of $\mathbb{C}\Gamma$ under the length zero vectors under $\langle\cdot,\cdot\rangle_f$. We get a Γ-action $\pi:\Gamma\to U(\ell^2_f\Gamma)$ as usual by $(\pi_g\xi)(h):=\xi(g^{-1}h)$. Indeed, $\pi_g^{-1}=\pi_{g^{-1}}$, and π_g is isometric:

$$\|\pi_g\xi\|_f^2=\sum_{h,k}f(k^{-1}h)\xi(g^{-1}h)\overline{\xi(g^{-1}k)}=\sum_{h,k}f((g^{-1}k)^{-1}(g^{-1}h))\xi(g^{-1}h)\overline{\xi(g^{-1}k)}=\|\xi\|_f^2.$$

Finally, note $f(g) = \langle \pi_g \delta_e, \delta_e \rangle$ for all $g \in \Gamma$.

(3) $\Rightarrow$ (4): We will use Fell's Absorption Principle, which you proved on homework, which states that if (H, π) is any unitary representation of Γ and $\lambda : \Gamma \to U(\ell^2\Gamma)$ is the left regular representation, then $(\ell^2\Gamma \otimes H, \lambda \otimes \pi)$ is unitarily equivalent to $(\ell^2\Gamma \otimes H, \lambda \otimes 1)$.

The $L\Gamma$-representation $\widetilde{\pi} : L\Gamma \to B(\ell^2\Gamma \otimes H)$ given by

$$g \mapsto \lambda_g \otimes 1 \mapsto \lambda_g \otimes \pi_g$$

is normal as it is a composite of normal unitary $*$-homomorpisms. Define $v : \ell^2\Gamma \to \ell^2\Gamma \otimes H$ by $\xi \mapsto \xi \otimes \frac{\eta}{\|\eta\|}$, which is an isometry. Observe that for all $g, h \in \Gamma$,

$$v^* \widetilde{\pi}(\lambda_g) v \delta_h = v^* \widetilde{\pi}(\lambda_g) \delta_h \otimes \frac{\eta}{\|\eta\|} = v^* \delta_{gh} \otimes \pi_g \frac{\eta}{\|\eta\|} = \frac{1}{\|\eta\|^2} \langle \pi_g \eta, \eta \rangle \delta_{gh} = \frac{1}{\|\eta\|^2} f(g) \lambda_g \delta_h.$$

Thus by linearity, for all $x \in \mathbb{C}\Gamma$, $M_f x = \|\eta\|^2 v^* \widetilde{\pi}(x) v$, which is manifestly normal and cp.

(4) $\Rightarrow$ (1): Let $g_1, \ldots, g_n \in \Gamma$. Then

$$[\lambda_{g_i^{-1} g_j}] = \begin{bmatrix} \lambda_{g_1} \\ \vdots \\ \lambda_{g_n} \end{bmatrix}^* \begin{bmatrix} \lambda_{g_1} & \cdots & \lambda_{g_n} \end{bmatrix} \geq 0$$

in $M_n(L\Gamma)$. Now since M_f is cp, $[M_f \lambda_{g_i^{-1} g_j}] \geq 0$ in $M_n(L\Gamma)$, so

$$[f(g_i^{-1} g_j)] = \begin{bmatrix} \lambda_{g_1} & & \\ & \ddots & \\ & & \lambda_{g_n} \end{bmatrix} [M_f \lambda_{g_i^{-1} g_j}] \begin{bmatrix} \lambda_{g_1} & & \\ & \ddots & \\ & & \lambda_{g_n} \end{bmatrix}^* \geq 0$$

in $M_n(L\Gamma)$, and thus also in $M_n(\mathbb{C})$. □

Example 6.1.6. Suppose $\varphi : L\Gamma \to L\Gamma$ is cp. Define $f(g) := \operatorname{tr}(\varphi(\lambda_g) \lambda_g^*)$. We claim that M_f is cp as it is the composite of the following cp maps:

$$\begin{array}{ccccccc} L\Gamma & \xrightarrow{\Delta} & L\Gamma \otimes L\Gamma & \xrightarrow{\mathrm{id} \otimes \varphi} & L\Gamma \otimes L\Gamma & \xrightarrow{\mathrm{Ad}(v)} & L\Gamma \\ \lambda_g & \longmapsto & \lambda_g \otimes \lambda_g & & \lambda_g \otimes \lambda_h & \longmapsto & \delta_{g=h} \lambda_g \\ & & x \otimes y & \longmapsto & x \otimes \varphi(y) & & \end{array}$$

where $v \delta_g := \delta_g \otimes \delta_g$. The above composite applied to λ_g is

$$\lambda_g \mapsto \lambda_g \otimes \lambda_g \mapsto v^* (\lambda_g \otimes \varphi(\lambda_g)) v.$$

If $\varphi(\lambda_g) \delta_e = \sum y_h \delta_h$, then applying the above operator to the separating vector δ_e, we obtain

$$v^* (\lambda_g \otimes \varphi(\lambda_g)) v \delta_e = v^* (\lambda_g \otimes \varphi(\lambda_g)) (\delta_e \otimes \delta_e) = v^* \sum_h y_h \delta_g \otimes \delta_h = y_g \delta_g = y_g \lambda_g \delta_e.$$

Finally we know that $y_g = \operatorname{tr}(\lambda_g^* \varphi(\lambda_g))$, verifying the claim.

Example 6.1.7. If $\Lambda \leq \Gamma$ is a subgroup, then the characterisic function $\chi_\Lambda(g) := \langle \pi_g \delta_\Lambda, \delta_\Lambda \rangle$ is positive definite, where $\pi : \Gamma \to U(\ell^2\Gamma/\Lambda)$. In this case, $M_{\chi_\Lambda} = E_{L\Lambda}$, the canonical trace-preserving conditional expectation.

Recall that the reduced group C*-algebra $C^*_r\Gamma$ is the norm closure of $\operatorname{span} \lambda\Gamma \subset B(\ell^2\Gamma)$.

Definition 6.1.8. The *universal* group C*-algebra $C^*\Gamma$ is the closure of the group algebra $\mathbb{C}\Gamma$ under the uniform norm

$$\|x\|_u := \sup \{ \|\pi(x)\| | (H, \pi) \text{ a unitary representation of } \Gamma \}.$$

Observe $\| \cdot \|_u$ is well-defined as $\|\pi(g)\|_u = 1$ for all $g \in \Gamma$.

Remark 6.1.9. The proof of (3) $\Rightarrow$ (4) in Theorem 6.1.5 also shows that if $f : \Gamma \to \mathbb{C}$ is positive definite, we also get a cp multiplier on $C^*_r\Gamma$ and $C^*\Gamma$. Moreover, we have $\|M_f\| \leq \|f\|_\infty$ as a cp multiplier on either of $C^*_r\Gamma, C^*\Gamma$.

6.2. **Amenability for discrete groups.** The following is the main result of this section.

Theorem 6.2.1. *The following are equivalent for a countable discrete group* Γ. *If any/all are satisfied, we call* Γ amenable.

(A1) *There is a state* $m \in (\ell^\infty\Gamma)^*$ *such that* $m(g \cdot f) = m(f)$ *for all* $g \in \Gamma$, *where* $(g \cdot f)(h) := f(g^{-1}h)$.

(A2) Γ *has a* left invariant mean, *i.e., there is a finitely additive (left)* Γ*-invariant probability measure on* 2^Γ, *the power set of* Γ.

(A3) Γ *has an* approximate invariant mean, *i.e., for every finite* $F \subset \Gamma$ *and* $\varepsilon > 0$, *there is a*

$$\mu \in \operatorname{Prob}(\Gamma) := \left\{ \mu \in \ell^1\Gamma \middle| \mu \geq 0 \text{ and } \sum_g \mu(g) = 1 \right\}$$

such that $\max_{g \in F} \|g \cdot \mu - \mu\| < \varepsilon$, *where* $(g \cdot \mu)(A) := \mu(g^{-1}A)$.

(A4) *(Følner sequence) there is a sequence of finite subsets* $\emptyset \neq F_n \subset \Gamma$ *with* $\Gamma = \bigcup F_n$ *such that*

$$\frac{|gF_n \triangle F_n|}{|F_n|} \to 0 \qquad \forall\, g \in \Gamma.$$

Here, $\triangle$ *denotes the symmetric difference of sets.*

(A5) *The left regular representation* $\lambda : \Gamma \to U(\ell^2\Gamma)$ *has* almost invariant vectors, *i.e., for every finite* $F \subset \Gamma$ *and* $\varepsilon > 0$, *there is a* $\xi \in \ell^2\Gamma$ *such that* $\|\lambda_g\xi - \xi\| < \varepsilon\|\xi\|$ *for all* $g \in F$.

(A6) *The trivial representation is* weakly contained *in the left regular representation, i.e., there is sequence of unit vectors* $(\xi_n) \subset \ell^2\Gamma$ *such that* $\|\lambda_g\xi_n - \xi_n\| \to 0$ *for all* $g \in \Gamma$.

(A7) *There is a sequence* (f_n) *of finitely supported positive definite functions on* Γ *such that* $f_n \to 1$ *pointwise.*

(A8) $C^*_r\Gamma \cong C^*\Gamma$

(A9) *There is a 1-dimensional representation of* $C^*_r\Gamma$.

(A10) *(Kesten Criterion) For all finite* $F \subset \Gamma$,

$$\left\| \frac{1}{|F|} \sum_{g \in F} \lambda_g \right\|_{B(\ell^2\Gamma)} = 1.$$

(A11) *($L\Gamma$ amenable) There is a conditional expectation $E : B(\ell^2\Gamma) \to L\Gamma$.*
(A12) *(Hypertrace) There is a state $\varphi \in B(\ell^2\Gamma)^*$ such that*
- *$\varphi(x\lambda_g) = \varphi(\lambda_g x)$ for all $g \in \Gamma$ and $x \in B(\ell^2\Gamma)$, and*
- *$\varphi|_{L\Gamma} = \mathrm{tr}_{L\Gamma}$ (recall that $\mathrm{tr}_{L\Gamma} = \omega_{\delta_e} = \langle \cdot\, \delta_e, \delta_e\rangle$).*

Non-example 6.2.2. The free group $\mathbb{F}_n$ for $n \geq 2$ is not amenable. For $n = 2$, suppose $\mathbb{F}_2 = \langle a, b\rangle$. For $x \in \{a, b, a^{-1}, b^{-1}\}$, let W_x be the set of reduced words starting with x, so that $\mathbb{F}_2$ can be written as a disjoint union

$$\mathbb{F}_2 = \{e\} \sqcup W_a \sqcup W_b \sqcup W_{a^{-1}} \sqcup W_{b^{-1}}.$$

But since $W_b \sqcup W_{a^{-1}} \sqcup W_{b^{-1}} \subset aW_{a^{-1}}$ and $W_a \sqcup W_{a^{-1}} \sqcup W_{b^{-1}} \subset bW_{b^{-1}}$, we also have

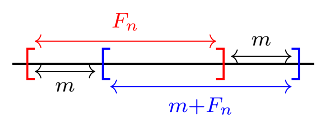
$$W_a \sqcup aW_{a^{-1}} = \mathbb{F}_2 = W_b \sqcup bW_{b^{-1}}$$

so that $\mathbb{F}_2$ has no invariant mean.

Example 6.2.3. Finite groups are amenable.

Example 6.2.4. The sets $F_n := [-n, n]$ give a Følner sequence for $\mathbb{Z}$. Indeed, for all $m \in \mathbb{Z}$, eventually $n \geq m$, for which

$$\frac{|(m + F_n)\triangle F_n|}{|F_n|} = \frac{2m}{2n+1} \xrightarrow{n\to\infty} 0.$$



Example 6.2.5. A discrete countable group Γ is called *locally finite* if $\Gamma = \varinjlim \Gamma_n$ where each Γ_n is finite, i.e., every finite subset $F \subset \Gamma$ is contained in a finite subgroup. Let m_n be the uniform measure on Γ_n and let ω be a non-principal/free ultrafilter on $\mathbb{N}$, i.e., $\omega \in \beta\mathbb{N}\setminus\mathbb{N}$. For $f \in \ell^\infty\Gamma$, we define

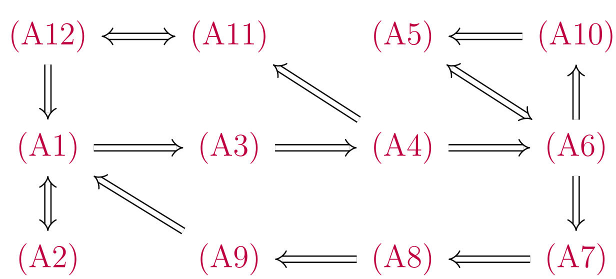
$$m(f) := \lim_{\omega} m_n(f|_{\Gamma_n}),$$

and one checks $m(g \cdot f) = m(f)$ for all $g \in \Gamma$.

Example 6.2.6. The class of amenable groups is closed under products, extensions, subgroups, quotients, and direct limits.

Example 6.2.7. Combining Examples 6.2.3, 6.2.4, and 6.2.6, all abelian groups are amenable. Indeed, every group is the direct limit of its finitely generated subgroups.

We now prove the following implications:



(A1)⇒(A2). If $m \in (\ell^\infty\Gamma)^*$ is a left Γ-invariant state, define $\mu : 2^\Gamma \to [0, 1]$ by $\mu(A) := m(\chi_A)$. $\square$

(A2)⇒(A1). If $\mu : 2^\Gamma \to [0,1]$ is a left Γ-invariant mean, define $m(f) := \int f\, d\mu$, which is a left Γ-invariant state on $\ell^\infty\Gamma$. Here, $\int f\, d\mu$ is defined in the usual way, first for positive functions as a sup over simple $0 \leq \phi \leq f$, and then extending to all bounded functions. $\square$

Exercise 6.2.8. Prove (A1)⇒(A3) (originally due to Day) and (A3)⇒(A4) (originally due to Namioka).

Exercise 6.2.9. Show (A5)⇔(A6).

(A4)⇒(A6). Suppose (F_n) is a Følner sequence for Γ. Consider the unit vectors $\xi_n := |F_n|^{-1/2}\chi_{F_n} \in \ell^2\Gamma$. For all $g \in \Gamma$,

$$\begin{aligned}\|\lambda_g \xi_n - \xi_n\|_2^2 &= \sum_h |\xi_n(g^{-1}h) - \xi_n(h)|^2 \\ &= \frac{1}{|F_n|}\sum_h |\chi_{F_n}(g^{-1}h) - \chi_{F_n}(h)|^2 \\ &= \frac{gF_n \triangle F_n}{|F_n|} \xrightarrow{n\to\infty} 0. \end{aligned}$$ $\square$

(A6)⇒(A7). Let $(\xi_n) \subset \ell^2\Gamma$ be a sequence of unit vectors such that $\|\lambda_g\xi_n - \xi_n\| \to 0$ for all $g \in \Gamma$. For $n \in \mathbb{N}$, define $\varphi_n(g) := \langle \lambda_g \xi_n, \xi_n\rangle$, which is positive definite by Theorem 6.1.5. Moreover, for all $g \in \Gamma$,

$$|\varphi_n(g) - 1| = |\langle \lambda_g\xi_n, \xi_n\rangle - \langle \xi_n, \xi_n\rangle| = |\langle \lambda_g \xi_n - \xi_n, \xi_n\rangle| \leq \|\lambda_g\xi_n - \xi_n\| \xrightarrow{n\to\infty} 0.$$

We can inductively construct finite subsets $E_n \subset \Gamma$ with $E_n \subseteq E_{n+1}$ and $\bigcup E_n = \Gamma$ such that $\|\eta_n - \xi_n\| < 2^{-n}$, where $\eta_n := \xi_n|_{E_n}$. Setting $f_n(g) := \langle \lambda_g \eta_n, \eta_n\rangle$, we have f_n is positive definite, finitely supported, and for all $g \in \Gamma$,

$$\begin{aligned}|\varphi_n(g) - f_n(g)| &= |\langle \lambda_g\xi_n, \xi_n\rangle - \langle \lambda_g \eta_n, \eta_n\rangle| = |\langle \lambda_g\xi_n, \xi_n - \eta_n\rangle - \langle \lambda_g(\eta_n - \xi_n), \eta_n\rangle| \\ &\leq 2\|\xi_n - \eta_n\| = 2^{1-n} \xrightarrow{n\to\infty} 0.\end{aligned}$$ $\square$

Definition 6.2.10 (Banach limits in $B(H)$)**.** Let Lim denote any positive extension of $\lim_{n\to\infty}$ from c to ℓ^∞ obtained from Hahn-Banach. If $(x_n) \subset B(H)$ is a norm-bounded sequence, define $\operatorname{Lim} x_n$ by $\langle \operatorname{Lim} x_n \eta, \xi\rangle := \operatorname{Lim}\langle x_n\eta, \xi\rangle$. Observe $\operatorname{Lim} x_n$ lies in the WOT-closure of $\operatorname{Conv}\{x_n\}$, so if $(x_n) \subset M \subseteq B(H)$ for some von Neumann algebra, then $\operatorname{Lim} x_n \in M$. Moreover, if $x_n \geq 0$ for all n, then $\operatorname{Lim} x_n \geq 0$ also.

Now suppose $\Phi_n : M \to M$ is a sequence of ucp maps. Then map $(\operatorname{Lim}\Phi_n)(x) := \operatorname{Lim}\Phi_n(x)$ is manifestly ucp. Indeed, if $(x_{ij}) \in M_n(M)_+$, then for all $\xi_1, \ldots, \xi_n \in H$,

$$\left\langle [(\operatorname{Lim}\Phi_n)(x_{ij})]\begin{bmatrix}\xi_1\\ \vdots \\ \xi_n\end{bmatrix}, \begin{bmatrix}\xi_1\\ \vdots \\ \xi_n\end{bmatrix}\right\rangle = \operatorname{Lim}\left\langle [\Phi_n(x_{ij})]\begin{bmatrix}\xi_1\\ \vdots \\ \xi_n\end{bmatrix}, \begin{bmatrix}\xi_1\\ \vdots \\ \xi_n\end{bmatrix}\right\rangle \geq 0.$$

(A4)⇒(A11). Given a Følner sequence (F_n), define $\Phi_n := \frac{1}{|F_n|}\sum_{g\in F_n} \rho_g x \rho_g^*$ where $\rho : \Gamma \to B(\ell^2\Gamma)$ is the right regular representation. Setting $E := \operatorname{Lim} \Phi_n$, we see $E(x) \in R\Gamma' = L\Gamma$ as

$$\left\| \rho_h \left(\frac{1}{|F_n|} \sum_{g\in F_n} \rho_g x \rho_g^* \right) \rho_h^* - \frac{1}{|F_n|} \sum_{g\in F_n} \rho_g x \rho_g^* \right\| \leq \frac{|hF_n|\triangle|F_n|}{|F_n|} \cdot \|x\| \xrightarrow{n\to\infty} 0.$$

Since each Φ_n is cp and preserves $L\Gamma$, E is cp and preserves $L\Gamma$. □

(A11)⇒(A12). Immediate from the more general Theorem 6.2.11 below. □

Theorem 6.2.11. *Suppose $M \subset B(H)$ is a von Neumann algebra with normal faithful tracial state* tr*. The following are equivalent:*

- *There is a conditional expectation $E : B(H) \to M$, i.e., a unital completely positive map $B(H) \to M$ which is M-bimodular.*
- *There is a hypertrace for M, i.e., there is a state $\varphi \in B(H)^*$ such that $\varphi(xm) = \varphi(mx)$ for all $x \in B(H)$ and $m \in M$ and $\varphi|_M = \mathrm{tr}_M$.*

Proof.
⇒: Set $\varphi := \mathrm{tr}_M \circ E$. Then for all $x \in B(H)$ and $m \in M$,

$$\varphi(xm) = \mathrm{tr}_M(E(xm)) = \mathrm{tr}_M(E(x)m) = \mathrm{tr}_M(mE(x)) = \mathrm{tr}_M(E(mx)) = \varphi(mx).$$

Since $E(1) = 1$, it also follows that $\varphi(m) = \mathrm{tr}_M(m)$.
⇐: For $x \in B(H)$, define ψ_x on M by $\psi_x(m) := \varphi(mx)$.

Claim. *When $x \geq 0$, ψ_x is a state on M such that $0 \leq \psi_x \leq \|x\| \cdot \mathrm{tr}_M = \|x\| \cdot \omega_{\Omega_M}$.*

Proof of claim. For $m \in M_+$, observe that

$$\begin{aligned} \psi_x(m) &= \varphi(m^{1/2} x m^{1/2}) \\ &= |\langle x m^{1/2}\Omega, m^{1/2}\Omega\rangle_\varphi| \\ &\underset{\text{(CS)}}{\leq} |\langle x m^{1/2}\Omega, x m^{1/2}\Omega\rangle_\varphi|^{1/2} \cdot |\langle m^{1/2}\Omega, m^{1/2}\Omega\rangle_\varphi|^{1/2} \\ &= \varphi(m^{1/2} x^2 m^{1/2})^{1/2} \varphi(m)^{1/2} \\ &\leq \|x\| \cdot \varphi(m). \end{aligned}$$

Since $\varphi|_M = \mathrm{tr}_M$, $\psi_x(m) \leq \|x\| \,\mathrm{tr}(m)$ for all $x \in B(H)_+$ and $m \in M_+$. □

Claim. *When $x \geq 0$, ψ_x is normal.*

Proof of claim. If $(m_i) \subset M_+$ such that $m_i \nearrow m$, then

$$\psi_x(m - m_i) \leq \|x\| \cdot \mathrm{tr}(m - m_i) \searrow 0. \qquad \square$$

Claim. *For each $x \in B(H)_+$, there is a unique $E(x) \in M_+$ such that $\psi_x(m) = \mathrm{tr}_M(mE(x))$ for all $m \in M$.*

Proof of claim.
Uniqueness: If $y, z \in M$ such that $\mathrm{tr}_M(my) = \mathrm{tr}_M(mz)$ for all $m \in M$, then

$$\langle y\Omega, m^*\Omega\rangle_{L^2M} = \langle z\Omega, m^*\Omega\rangle_{L^2M} \qquad \forall\, m \in M.$$

It follows that $y\Omega = z\Omega$, which implies $y = z$ as Ω is separating.
Existence: First, suppose $x \geq 0$. Since $0 \leq \psi_x \leq \|x\| \,\mathrm{tr}_M = \|x\|\omega_{\Omega_M}$, there is a unique $x' \in M'$ with $0 \leq x' \leq \|x\|$ such that

$$\psi_x(m) = \langle mx'\Omega_M, \Omega_M\rangle_{L^2M} \qquad \forall\, m \in M.$$

Since $M' = JMJ$, there is a unique $E(x) \in M_+$ such that $x' = JE(x)J$, and thus

$$\psi_x(m) = \langle mJE(x)J\Omega_M, \Omega_M\rangle_{L^2M} = \langle mE(x)\Omega_M, \Omega_M\rangle_{L^2M} = \mathrm{tr}_M(mE(x)) \qquad \forall\, m \in M.$$

□

Claim. *The right action of M on $L^2(B(H), \varphi)$ given by $x\Omega_\varphi \mapsto xm\Omega_\varphi$ is bounded.*

Proof. For all $x \in B(H)$,

$$\begin{aligned}\|xm\Omega\|_\varphi^2 &= \varphi(m^*x^*xm) = \varphi(mm^*x^*x) = \mathrm{tr}_M(mm^*E(x^*x)) \\ &= \mathrm{tr}_M(E(x^*x)^{1/2}mm^*E(x^*x)^{1/2}) \leq \|mm^*\| \cdot \mathrm{tr}_M(E(x^*x)) \\ &= \|mm^*\| \cdot \varphi(x^*x) = \|m\|^2 \cdot \|x\Omega\|_\varphi^2.\end{aligned}$$

□

We now mimic the proof of Stinespring's Theorem. Observe that the map $v : L^2M \to L^2(B(H), \varphi)$ given by $m\Omega_M \mapsto m\Omega_\varphi$ is an $M - M$ bilinear isometry. It follows immediately that $E(x) := v^*xv \in B(L^2M)$ commutes with the right M-action and thus lies in M, thus giving our $M - M$ bimodular ucp map. It remains to prove that our new definition of $E(x)$ agrees with our old definition, i.e., $\mathrm{tr}_M(mv^*xv) = \varphi(mx)$ for all $m \in M$:

$$\mathrm{tr}_M(mv^*xv) = \langle v^*mxv\Omega_M, \Omega_M\rangle_{L^2M} = \langle mx\Omega_\varphi, \Omega_\varphi\rangle_\varphi = \varphi(mx).$$

□

(A12)⇒(A1). Recall $\ell^\infty\Gamma \hookrightarrow B(\ell^2\Gamma)$ by $(f\xi)(g) := f(g)\xi(g)$. Observe that if $f \in \ell^\infty\Gamma$ and $g \in \Gamma$, then

$$(\lambda_g f\lambda_g^*\xi)(h) = (f\lambda_{g^{-1}}\xi)(g^{-1}h) = f(g^{-1}h)(\lambda_{g^{-1}}\xi)(g^{-1}h) = f(g^{-1}h)\xi(h) = ((g\cdot f)\xi)(h).$$

Restricting the $L\Gamma$-hypertrace φ to $\ell^\infty\Gamma \subset L\Gamma$, we have

$$\varphi(g \cdot f) = \varphi(\lambda_g f\lambda_g^*) = \varphi(f),$$

so φ yields a Γ-invariant state on $\ell^\infty\Gamma$. □

(A7)⇒(A8). First, note that $\|\lambda_x\| \leq \|x\|_u$ for all $x \in \mathrm{C}^*_r\Gamma$, and thus $\lambda : \mathbb{C}\Gamma \to B(\ell^2\Gamma)$ extends to a surjective unital $*$-homomorphism $\widetilde{\lambda} : \mathrm{C}^*\Gamma \to \mathrm{C}^*_r\Gamma \subset B(\ell^2\Gamma)$. We must show $\widetilde{\lambda}$ is injective.

Suppose (f_n) is a sequence of finitely supported positive definite functions on Γ which converges to 1 pointwise. By Remark 6.1.9, we get cp multipliers $M_n, M_{n,r}$ on $C^*\Gamma, C^*_r\Gamma$ respectively. To prove $\widetilde{\lambda}$ is injective, we will use the following two facts.

(1) $\widetilde{\lambda} \circ M_n = M_{n,r}\widetilde{\lambda}$ on $C^*\Gamma$, since both are continuous with respect to $\|\cdot\|_u$ and they agree on the dense subspace $\mathbb{C}\Gamma$.

(2) Since $f_n \to 1$ pointwise, $M_n x \to x$ for $x \in \mathbb{C}\Gamma$. Since $\|f_n\|_\infty$ are uniformly bounded by $\sup f_n(e)$ as $f_n(e) \to 1$, $M_n x \to x$ for all $x \in C^*\Gamma$ by density of $\mathbb{C}\Gamma$ in $C^*\Gamma$ by a standard $\varepsilon/3$ argument.

Suppose $x \in C^*\Gamma$ such that $\widetilde{\Lambda}(x) = 0$. Then by (1) above,

$$\widetilde{\lambda}(M_n x) = M_{n,r}\widetilde{\lambda}(x) = 0 \qquad \forall n \in \mathbb{N}.$$

But since f_n is finitely supported, $M_n x \in \mathbb{C}\Gamma$ for all n, and thus $\widetilde{\lambda}(M_n x) = 0$ implies $M_n x = 0$. Thus $x = \lim M_n x = 0$ by (2). □

(A8)⇒(A9). Note that $C^*\Gamma$ has a 1-dimensional representation as the trivial representation $\mathbb{C}\Gamma \to \mathbb{C}$ by $\sum x_g g \mapsto \sum x_g$ on $\mathbb{C}$ is subordinate to $\|\cdot\|_u$. □

Lemma 6.2.12. *Let A be a unital C^*-algebra. Suppose $\varphi \in A^*$ is a state and $a \in A$ such that $\varphi(a^*a) = |\varphi(a)|^2$. Then for all $b \in A$, $\varphi(a)\varphi(b) = \varphi(ba)$.*

Proof. Let $(H_\varphi, \pi_\varphi, \Omega_\varphi)$ be the cyclic GNS representation of A with respect to φ. Note that

$$\|\pi_\varphi(a)\Omega_\varphi\|^2 = \varphi(a^*a) = |\varphi(a)|^2 = |\langle \pi_\varphi(a)\Omega_\varphi, \Omega_\varphi\rangle|^2 \underset{\text{(CS)}}{\leq} \|\pi_\varphi(a)\Omega_\varphi\|^2,$$

and thus the Cauchy-Schwarz inequality above is an equality. Thus there is an $\alpha \in \mathbb{C}$ such that

$$\pi_\varphi(a)\Omega_\varphi = \alpha\Omega_\varphi.$$

It follows immediately that

$$\varphi(ba) = \langle \pi_\varphi(b)\pi_\varphi(a)\Omega_\varphi, \Omega_\varphi\rangle = \alpha\langle \pi_\varphi(b)\Omega_\varphi, \Omega_\varphi\rangle = \varphi(a)\varphi(b).$$ □

(A9)⇒(A1). Let $\phi : C^*_r\Gamma \to \mathbb{C}$ be a 1-dimensional representation. Then ϕ is a state, and we can extend ϕ to a state $\varphi \in B(\ell^2\Gamma)^*$ by Hahn-Banach. Note that for every $g \in \Gamma$,

$$\varphi(\lambda_g\lambda_g^*) = \varphi(\lambda_g^*\lambda_g) = \varphi(1) = 1 = |\varphi(\lambda_g)|^2.$$

Then for all $f \in \ell^\infty\Gamma$, $g \cdot f = \lambda_g f \lambda_g^*$, and thus by Lemma 6.2.12,

$$\varphi(g \cdot f) = \varphi(\lambda_g f \lambda_g^*) = \varphi(\lambda_g)\varphi(f)\varphi(\lambda_g^*) = \varphi(f)$$

and thus φ restricts to a Γ-invariant state on $\ell^\infty\Gamma$. □

(A6)⇒(A10). Let $(\xi_n) \subset \ell^2\Gamma$ be a sequence of almost invariant vectors. Then for every finite $F \subset \Gamma$,

$$1 = \lim_n \left\| \frac{1}{|F|} \sum_{g\in F} \xi_n \right\|_2 = \lim_n \left\| \frac{1}{|F|} \sum_{g\in F} \lambda_g \xi_n \right\|_2 \leq \left\| \frac{1}{|F|} \sum_{g\in F} \lambda_g \right\| \leq 1. \qquad \square$$

(A10)⇒(A5). Let $F \subset \Gamma$ be finite such that $F = F^{-1}$. Then $x = \frac{1}{|F|}\sum_{g\in F} \lambda_g$ is self-adjoint and has operator norm equal to 1. Let $\varepsilon > 0$. There is a $\xi \in \ell^2\Gamma$ such that $|\langle x\xi, \xi\rangle| > 1 - \varepsilon'$, where $\varepsilon' > 0$ is to be determined in terms of ε and $|F|$. Let $|\xi| \in \ell^2\Gamma$ be the pointwise absolute value of ξ: $|\xi|(g) := |\xi(g)|$. We calculate

$$\begin{aligned} 1 - \varepsilon' < |\langle x\xi, \xi\rangle| &= \left| \sum_{g\in\Gamma} (x\xi)(g)\overline{\xi(g)} \right| \\ &\leq \sum_{g\in\Gamma} |(x\xi)(g)| \cdot |\xi(g)| \leq \sum_{g\in\Gamma} (x|\xi|)(g) \cdot |\xi|(g) \\ &= \langle x|\xi|, |\xi|\rangle \frac{1}{F} \sum_{g\in F} \underbrace{\langle \lambda_g|\xi|, |\xi|\rangle}_{\leq 1\, \forall g\in F}. \end{aligned}$$

Thus for all $g \in F$, $\langle \lambda_g|\xi|, |\xi|\rangle > 1 - |F|\varepsilon'$, and we have

$$\begin{aligned} \|\lambda_g|\xi| - |\xi|\|^2 &= \|\lambda_g|\xi|\|^2 + \||\xi|\|^2 - \langle \lambda_g|\xi|, |\xi|\rangle - \langle \lambda_{g^{-1}}|\xi|, |\xi|\rangle \\ &= 1 - \langle \lambda_g|\xi|, |\xi|\rangle + 1 - \langle \lambda_{g^{-1}}|\xi|, |\xi|\rangle \\ &< 2|F|\varepsilon' < \varepsilon^2 \end{aligned}$$

whenever $\varepsilon' < \min\{\frac{\varepsilon^2}{2|F|}, \frac{1}{|F|}\}$. □

6.3. **Amenability for von Neumann algebras.** TODO:

6.4. **The Haagerup property for discrete groups and tracial von Neumann algebras.** For this section, Γ is a discrete countable group.

Definition 6.4.1. We say Γ has the *Haagerup property* if

[HP] there is a sequence (φ_n) of positive definite c_0 functions on Γ such that $\varphi_n \to 1$ pointwise.

Example 6.4.2.

(1) All amenable gropus have [HP], as finitely supported implies c_0.
(2) Free groups $\mathbb{F}_n$ with $n \geq 2$ have [HP]. We will prove this once we have a second equivalent characterization of [HP].
(3) $SL(2,\mathbb{Z}) = \mathbb{Z}/4 *_{\mathbb{Z}/2} \mathbb{Z}/6 \supset \mathbb{F}_2$ as an index 12 subgroup.
(4) $PSL(2,\mathbb{Z}) = \mathbb{Z}/2 * \mathbb{Z}/3$
(5) Groups which act on trees (e.g. $\mathbb{F}_n$ acting on its Cayley graph)
(6) Coxeter groups $\langle g_1, \ldots, g_n | (g_ig_j)^m_{ij}$ where $m_{ii} = 1$, $m_{ij} \geq 2\, i \neq j\rangle$. Here, $m_{ij} = \infty$ is ok, which means there is no relation of this form.

(7) The class of groups with [HP] is closed under taking subgroups, direct products, free products.
(8) If H has [HP] and $H \leq G$ with finite index, then G has [HP].

Definition 6.4.3. A *cocycle* of Γ is a triple (H, π, β) where (H, π) is a unitary representation and $\beta : \Gamma \to H$ such that

$$\beta(hg) = \beta(h) + \pi_h \beta(g) \qquad\qquad \forall\, g, h \in \Gamma.$$

A cocycle is called *inner* if there is a $\xi \in H$ such that $\beta(g) = \xi - \pi_g \xi$ for all $g \in \Gamma$.

Facts 6.4.4. We have the following facts about cocycles.

(β1) $\beta(e) = \beta(e^2) = \beta(e) + \pi_e \beta(e) = 2\beta(e)$, so $\beta(e) = 0$.
(β2) $0 = \beta(e) = \beta(g^{-1}g) = \beta(g^{-1}) + \pi_{g^{-1}}\beta(g)$, so $\beta(g^{-1}) = -\pi_{g^{-1}}\beta(g)$.
(β3) $\|\beta(g^{-1}h)\| = \|\beta(g^{-1}) + \pi_{g^{-1}}\beta(h)\| = \| - \pi_{g^{-1}}\beta(g) + \pi_{g^{-1}}\beta(h)\| = \|\beta(g) - \beta(h)\|$.

The motivation for these cocycles is as follows. Let

$$\begin{aligned} \operatorname{Aff}(H) &:= \{\text{affine invertible transformations of } H\} \\ &= \{\xi \mapsto u\xi + \eta | \eta \in H,\ u \in U(H)\} \end{aligned}$$

Observe that $\operatorname{Aff}(H)$ is a group under composition:

$$\xi \mapsto u_2\xi + \eta_2 \mapsto u_1(u_2\xi + \eta_2) + \eta_1 = u_1u_2\xi + (u_2\eta_2 + \eta_1).$$

Thus we may identify $\operatorname{Aff}(H) = H \rtimes U(H)$ with multiplication $(u_1, \eta_1) \cdot (\eta_2, u_2) := (\eta_1 + u_1\eta_2, u_1u_2)$.

Definition 6.4.5. An *affine isometric action* of Γ on H is a group homomorphism $\alpha : \Gamma \to \operatorname{Aff}(H)$.

Example 6.4.6. Given a cocycle (H, π, β), we get an affine isometric action by

$$\alpha_g \xi := \pi_g \xi + \beta(g).$$

The cocycle condition implies $\alpha_g \alpha_h = \alpha_{gh}$:

$$\alpha_g\alpha_h\xi = \alpha_g(\pi_h\xi + \beta(h)) = \pi_g(\pi_h\xi + \beta(h)) + \beta(g) = \pi_{gh}\xi + \underbrace{\pi_g\beta(h) + \beta(g)}_{\beta(hg)} = \alpha_{gh}\xi.$$

Conversely, observe that an affine isometric action $\alpha : \Gamma \to \operatorname{Aff}(H)$ gives a unitary representation $\pi : \Gamma \to U(H)$ by the quotient map:

$$\pi : \Gamma \xrightarrow{\alpha} \operatorname{Aff}(H) = H \rtimes U(H) \twoheadrightarrow U(H)$$

Observe that there is a unique $\beta(g) \in H$ such that $\alpha_g = (\beta(g), \pi_g) \in \operatorname{Aff}(H)$, i.e., $\alpha_g\xi = \pi_g\xi + \beta(g)$ for all $\xi \in H$ and $g \in \Gamma$. This β is a cocycle:

$$\beta(h) + \pi_h\beta(g) = \pi_h(\alpha_g\xi) - \alpha_h(\alpha_g\xi) + \pi_h(\pi_g\xi - \alpha_g\xi) = \pi_{hg}\xi - \alpha_{hg}\xi = \beta(hg).$$

Exercise 6.4.7. Let X be a uniformly convex Banach space and $B \subset X$ a bounded set. Then

$$\inf_{x \in X} \sup_{b \in B} \|x - b\|$$

is attained at a unique $x \in X$.

Lemma 6.4.8. *A cocycle (H, π, β) is inner if and only if it is bounded.*

Proof.
$\Rightarrow$: If (H, π, β) is inner with $\beta(g) = \xi - \pi_g \xi$, then

$$\|\beta(g)\| = \|\xi - \pi_g \xi\| \leq 2\|\xi\| \qquad \forall g \in \Gamma.$$

$\Leftarrow$: Consider the affine action of Γ on H associated to (π, β). If β is bounded, then the orbit $\Gamma \cdot 0_H$ is bounded as

$$\alpha_g 0_H = \pi_g 0_H + \beta(g) = \beta(g).$$

By Exercise 6.4.7, there is a unique $\xi \in H$ minimizing $\sup_{g\in\Gamma} \|\beta(g) - \xi\|$. We claim that $\beta(g) = \xi - \pi_g \xi$ for all $g \in \Gamma$. Indeed, for every $\eta \in \Gamma \cdot 0_H$ and $g \in \Gamma$,

$$\|\alpha_g \xi - \underbrace{\alpha_g \eta}_{\in \Gamma\cdot 0_H}\| = \|\pi_g(\xi - \eta)\| = \|\xi - \eta\|,$$

so by uniqueness in Exercise 6.4.7, $\alpha_g \xi = \xi$ for all $g \in \Gamma$. Hence

$$\xi = \alpha_g \xi = \pi_g \xi + \beta(g) \qquad \Longleftrightarrow \qquad \beta(g) = \xi - \pi_g \xi$$

for all $g \in \Gamma$. □

Definition 6.4.9. A function $f : X \to Y$ between topological spaces is called *proper* if whenever $K \subset Y$ is compact, $f^{-1}K \subset X$ is compact. An affine action $\alpha : \Gamma \to \operatorname{Aff}(H)$ is called *proper* if the map $\Gamma \times H \to H \times H$ given by $(g, \xi) \mapsto (g\xi, \xi)$ is proper.

A cocycle $\beta : \Gamma \to H$ is called *proper* if $g \mapsto \|\beta(g)\|$ is proper, i.e., for all $N \in \mathbb{N}$, $\{g \in \Gamma | \|\beta(g)\| < N\}$ is finite.

Exercise 6.4.10. Show that an affine action $\alpha = (H, \pi, \beta)$ is proper if and only if β is proper.

Exercise 6.4.11. Suppose $a, b \in M_n(\mathbb{C}) \geq 0$. Prove that the *Schur product* $a * b \in M_n(\mathbb{C})$ is also positive, where $(a * b)_{ij} := a_{ij} b_{ij}$.

Deduce that if $a \geq 0$, then the pointwise exponential $[\exp(a_{ij})] \geq 0$.

Proposition 6.4.12 (Schoenberg). *If $\beta : \Gamma \to H$ is a cocycle, then for all $r > 0$, $f_r(g) := \exp(-r\|\beta(g)\|^2)$ is positive definite, and $f_r \to 1$ pointwise as $r \searrow 0$. Moreover,*

- *$f_r \in c_0\Gamma$ if and only if β is proper, and*
- *$f_r \to 1$ uniformly as $r \searrow 0$ if and only if β is bounded.*

Proof. By scaling β linearly, we may assume $r = 1$, and we wrte $f = f_1$. Note that

$$\begin{aligned} f(g^{-1}h) &\underset{(\beta 3)}{=} \exp(-\|\beta(g) - \beta(h)\|^2) \\ &= \exp(-\|\beta(g)\|^2) \cdot \exp(-\|\beta(h)\|^2) \cdot \exp(2\operatorname{Re}\langle \beta(g), \beta(h)\rangle). \end{aligned}$$

Fix $g_1, \ldots, g_n \in \Gamma$. First, note that $[\exp(-\|\beta(g_i)\|^2) \cdot \exp(-\|\beta(g_j)\|^2)] \geq 0$ as it equals

$$\begin{bmatrix} \exp(-\|\beta(g_1)\|^2) \\ \vdots \\ \exp(-\|\beta(g_n)\|^2) \end{bmatrix} \cdot \begin{bmatrix} \exp(-\|\beta(g_1)\|^2) & \cdots & \exp(-\|\beta(g_n)\|^2) \end{bmatrix}.$$

Second, we show that $[\exp(2\operatorname{Re}\langle\beta(g_i),\beta(g_j)\rangle)] \geq 0$ by the following steps.

Step 1: $[\langle\beta(g_i),\beta(g_j)\rangle] \geq 0$. Indeed, each $\xi \in H$ can be viewed as a bounded linear map $|\xi\rangle : \mathbb{C} \to H$ by $1 \mapsto \xi$, and for all $x = (x_i)_{i=1}^n \in \mathbb{C}^n$,

$$\sum_{i,j} x_i \langle\beta(g_i),\beta(g_j)\rangle \overline{x_j} = \left(\sum_{j=1}^n x_j|\beta(g_j)\rangle\right)^* \left(\sum_{i=1}^n x_i|\beta(g_i)\rangle\right) \geq 0.$$

Step 2: If $a \in M_n(\mathbb{C})_+$, then

$$\langle \overline{a}\xi, \xi\rangle_{\mathbb{C}^n}\rangle = \sum_{i,j=1}^n \overline{a}_{ij}\xi_j\overline{\xi_i} = \overline{\sum_{i,j=1}^n a_{ij}\overline{\xi_j}\xi_i} = \overline{\langle a\overline{\xi}, \overline{\xi}\rangle_{\mathbb{C}^n}\rangle} \geq 0,$$

and thus $\operatorname{Re}(a) = \dfrac{a+\overline{a}}{2} \geq 0$.

Step 3: Since $\exp(2\operatorname{Re}\langle\beta(g),\beta(h)\rangle) = \displaystyle\sum_{n\geq 0} \frac{(2\operatorname{Re}\langle\beta(g),\beta(h)\rangle)^n}{n!}$, by Exercise 6.4.11,

$$[2\operatorname{Re}\langle\beta(g_i),\beta(g_j)\rangle] \geq 0 \qquad \Longrightarrow \qquad [\exp(2\operatorname{Re}\langle\beta(g_i),\beta(g_j)\rangle)] \geq 0.$$

Finally, we see that the matrix in question is exactly the Schur product of two positive matrices, which is again positive by Exercise 6.4.11.

The final claims about the f_r are immediate. □

Theorem 6.4.13. *For a countable discrete group Γ, the following are equivalent:*

(1) Γ has [HP].
(2) Γ admits a proper cocycle.
(3) Γ admits a proper affine isometric action on a Hilbert space.

Proof.
(1) ⇒ (2): Omitted.
(2) ⇔ (3): Immediate from Exercise 6.4.10 above.
(2) ⇒ (1): Suppose $\beta : \Gamma \to H$ is a proper cocycle. Schoenberg's result 6.4.12 gives c_0 positive definite functions $f_{1/n}(g) := \exp(-\|\beta(g)\|^2/n)$ such that $f_r \to 1$ pointwise as $n \to \infty$. □

Theorem 6.4.14. *If Γ acts faithfully on a tree T preserving the distance of vertices, then Γ has [HP].*

Proof. Let H denote ℓ^2(*oriented* edges of T), so that each edge appears twice with opposite orientations. For vertices $u, v \in T$, define:

- $d(u,v) :=$ the length of the geodescic $[u,v]$ from u to v in T, and

- the *signed* characteristic function $\chi_{[u,v]} \in H$ by

$$\chi_{[u,v]}(\varepsilon) := \begin{cases} 0 & \text{if } \varepsilon \notin [u,v] \\ 1 & \text{if } \varepsilon \in [u,v] \\ -1 & \text{if } \varepsilon \in [v,u] \end{cases}$$

We observe the following two important relations:

$$\chi_{[u,v]} + \chi_{[v,w]} = \chi_{[u,w]} \qquad \forall \text{ vertices } u,v,w \in T \tag{6.4.15}$$

$$\|\chi_{[u,v]}\|^2 = 2d(u,v) \qquad \forall \text{ vertices } u,v \in T. \tag{6.4.16}$$

The Γ action on T gives a unitary representation $\pi : \Gamma \to B(H)$ by left translation such that

$$\pi_g \chi_{[u,v]} = \chi_{[gu,gv]} \qquad \forall \text{ vertices } u,v \in T. \tag{6.4.17}$$

Now fix a vertex $t_0 \in T$, and define $\beta : \Gamma \to H$ by $\beta(g) = \chi_{[gt_0,t_0]}$. For all $g,h \in \Gamma$,

$$\begin{aligned} \beta(hg) &= \chi_{[hgt_0,t_0]} \\ &= \chi_{[hgt_0,gt_0]} + \chi_{[ht_0,t_0]} && (6.4.15) \\ &= \pi_h \chi_{[gt_0,t_0]} + \chi_{[ht_0,t_0]} && (6.4.17) \\ &= \beta_h \beta(g) + \beta(h), \end{aligned}$$

so β is a cocycle. By (6.4.16), $\|\beta(g)\|^2 = 2d(gt_0,t_0) \to \infty$ as $g \to \infty$, so β is proper. Hence Γ has [HP] by Theorem 6.4.13. □

Example 6.4.18. The free group $\mathbb{F}_n$ acts on its Cayley graph, which is a tree.

Definition 6.4.19. Let (M, tr) be a tracial von Neumann algebra. We say (M, tr) has the *Haagerup property* if there is a sequence $(\varphi_n : M \to M)$ of normal trace-preserving cp maps such that:

- $\varphi_n \to \operatorname{id}$ pointwise-$\|\cdot\|_2$, and
- on L^2M, $\widehat{\varphi_n}(m\Omega) := \varphi_n(m)\Omega$ is compact as an operator in $B(L^2M)$.

This second condition is analogous to the c_0 condition for Γ.

Remark 6.4.20. Suppose (φ_n) is a sequence of trace-preserving ucp maps on $L\Gamma$. If $\varphi_n \to \operatorname{id}_{L\Gamma}$ pointwise-$\|\cdot\|_2$, then the positive definite functions $f_n(g) := \operatorname{tr}(\varphi_n(\lambda_g)\lambda_g^*)$ from Example 6.1.6 converge to 1 pointwise. Indeed,

$$\begin{aligned} |f_n(g) - 1| &= |\langle \varphi_n(\lambda_g), \lambda_g \rangle_{L^2(L\Gamma)} - \langle \lambda_g, \lambda_g \rangle_{L^2(L\Gamma)}| \\ &= |\langle (\widehat{\varphi}_n - 1)\delta_g, \delta_g \rangle_{\ell^2\Gamma} \underset{\text{(CS)}}{\leq} \|(\widehat{\varphi}_n - 1)\delta_g\|_{\ell^2\Gamma} \xrightarrow{n\to\infty} 0. \end{aligned}$$

Lemma 6.4.21. *If $x \in K(H)$ and (e_i) is an ONB for H, then $|\omega_{e_i}(x)| = |\langle xe_i, e_i \rangle| \to 0$ as $i \to \infty$.*

Proof. Since every $x \in K(H)$ is a linear combination of 4 positive compact operators, we may assume $x \geq 0$. Let $x = \sum s_n |f_n\rangle\langle f_n|$ be a Schmidt decomposition of x with $s_n \searrow 0$ as $n \to \infty$. Let $\varepsilon > 0$ and pick $N > 0$ such that $n \geq N$ implies $s_n < \varepsilon/2$.

Since $|\langle e_i, f_n\rangle|^2 \to 0$ as $i \to \infty$, there is an i_0 such that $i > i_0$ implies

$$\sum_{n=0}^{N-1} s_n |\langle e_i, f_n\rangle|^2 < \frac{\varepsilon}{2}.$$

We now calculate that whenever $i > i_0$,

$$\langle xe_i, e_i\rangle = \sum_{n=0}^{\infty} s_n|\langle e_i, f_n\rangle|^2 < \sum_{n=0}^{N-1} s_n|\langle e_i, f_n\rangle|^2 + s_N \sum_{n\geq N} |\langle e_i, f_n\rangle|^2 < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

Theorem 6.4.22. *A countable discrete group* Γ *has* [HP] *if and only if* $L\Gamma$ *has* [HP].

Proof.
$\Rightarrow$: Let (f_n) be a sequence of c_0 positive definite functions $\Gamma \to \mathbb{C}$ which converges to 1 pointwise. Without loss of generality, we may assume $f_n(e) = 1$ for all n; otherwise replace f_n with $f_n/f_n(e)$. The cp multipliers $M_{f_n} : L\Gamma \to L\Gamma$ witness that $L\Gamma$ has [HP]. Indeed, $\widehat{M}_{f_n} \in B(\ell^2\Gamma)$ is clearly compact as it is diagonal with eigenvalues going to 0, and

$$\|(M_{f_n}(x) - x)\delta_e\|_2^2 = \|(\widehat{M}_{f_n} - 1)x\delta_e\|_2^2 = \sum_g |f_n(g) - 1|^2|x_g|^2 \xrightarrow{n\to\infty} 0$$

as each $f_n \in c_0\Gamma$ with $\|f_n\| = f_n(e) = 1$ for all n. Explicitly, $|f_n(g) - 1|^2 \leq 4$ for all n, so we may choose $h \in \Gamma$ large in some ordering so that $\sum_{g>h} |x_g|^2 < \varepsilon^2/8$, and we may then choose N so that $n > N$ implies

$$\sum_{g\leq h} |f_n(g) - 1|^2|x_g|^2 < \varepsilon^2/2.$$

$\Leftarrow$: Suppose (φ_n) witness that $L\Gamma$ has [HP]. Then $f_n(g) := \operatorname{tr}_{L\Gamma}(\varphi_n(\lambda_g)\lambda_g^*)$ is positive definite by Example 6.1.6. To see that $f_n \in c_0\Gamma$, we have that

$$|f_n(g)| = |\operatorname{tr}_{L\Gamma}(\varphi_n(\lambda_g)\lambda_g^*)| = |\langle \varphi_n(\lambda_g), \lambda_g\rangle_{L^2(L\Gamma)}| = |\omega_{\lambda_g}(\widehat{\varphi}_n)| \xrightarrow{g\to\infty} 0$$

by Lemma 6.4.21. Since $\varphi_n \to \mathrm{id}_{L\Gamma}$ pointwise-$\|\cdot\|_2$, $f_n \to 1$ pointwise by Remark 6.4.20 $\square$

6.5. **Kazhdan's Property (T) for discrete groups.** For this section, Γ is a countable discrete group, and $\Lambda \leq \Gamma$ is a subgroup.

Definition 6.5.1. We say Γ has *property* (T) *relative to* Λ whenever (f_n) is a sequence of positive definite functions $\Gamma \to \mathbb{C}$ such that $f_n \to 1$ pointwise, then $f_n|_\Lambda \to 1$ uniformly on Λ. We say Γ has *property* (T) if Γ has property (T) relative to Γ. In other words:

(T) whenever (f_n) is a sequence of positive definite functions $\Gamma \to \mathbb{C}$ such that $f_n \to 1$ pointwise, then $f_n \to 1$ uniformly.

Example 6.5.2.
(1) All finite groups have (T).
(2) $SL(2,\mathbb{Z})$ has [HP] as $\mathbb{F}_2 \leq SL(2,\mathbb{Z})$ with index 12, but $SL(n,\mathbb{Z})$ has (T) for $n \geq 3$.

(3) $\mathbb{Z}^2 \leq \mathbb{Z}^2 \rtimes SL(2,\mathbb{Z})$ has relative (T).

$$\left\{ \begin{bmatrix} 1 & & * \\ & 1 & * \\ & & 1 \end{bmatrix} \right\} \leq \left\{ \begin{bmatrix} a & b & * \\ c & d & * \\ & & 1 \end{bmatrix} \middle| \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in SL(2,\mathbb{Z}) \right\}$$

Observe that neither of these groups has (T).

Facts 6.5.3.

(1) If Γ has (T) relative to Λ and Γ has [HP], then Λ is finite. In particular, Γ has [HP] and (T) if and only if Γ is finite.

(2) If Γ has relative (T) with respect to an infinite subgroup Λ, then Γ does not have [HP]. Thus [HP] is not preserved under extensions.

Theorem 6.5.4. *The following are equivalent.*

(T1) *Γ has (T), i.e., for all sequences (f_n) of positive deifnite functions with $f_n \to 1$ pointwise, $f_n \to 1$ uniformly.*

(T2) *Every cocycle $\beta : \Gamma \to H$ is inner (equivalently bounded).*

(T3) *Every affine Γ-action has a fixed point.*

(T4) *If (H, π) is a unitary representation of Γ with a sequence of unit vectors (ξ_n) such that $\|\pi_g \xi_n - \xi_n\| \to 0$ for all $g \in \Gamma$, then there is a non-zero $\xi \in H$ such that $\pi_g \xi = \xi$ for all $g \in \Gamma$.*

(T5) *There is a $\delta > 0$ and a finite $F \subset \Gamma$ such that for every unitary representation (H, π) and $\xi \in (H)_1$ with $\|\pi_g \xi - \xi\| < \delta$ for all $g \in F$, there is an $\xi_0 \in (H)_1$ with $\pi_g \xi_0 = \xi_0$ for all $g \in \Gamma$.*

(T6) *For all $\varepsilon > 0$, there is a $\delta > 0$ and a finite $F \subset \Gamma$ such that for every unitary representation (H, π) and $\xi \in (H)_1$ with $\|\pi_g \xi - \xi\| < \delta$ for all $g \in F$, there is an $\xi_0 \in (H)_1$ with $\|\xi - \xi_0\| < \varepsilon$ and $\pi_g \xi_0 = \xi_0$ for all $g \in \Gamma$.*

(T7) *For all $\varepsilon > 0$, there is a $\delta > 0$ and a finite $F \subset \Gamma$ such that for all positive definite $f : \Gamma \to \mathbb{C}$ with $|f(g) - 1| < \delta$ on F, we have $|f(g) - 1| < \varepsilon$ for all $g \in \Gamma$.*

We prove the following implications:

$$\begin{array}{ccccccc} & & \text{(T1)} & \Longleftarrow & \text{(T7)} & \Longleftarrow & \text{(T6)} \\ & & \Downarrow & & & & \Updownarrow \\ \text{(T3)} & \Longleftrightarrow & \text{(T2)} & \Longrightarrow & \text{(T4)} & \Longrightarrow & \text{(T5)} \end{array}$$

(T1)⇒(T2). Let $\beta : \Gamma \to H$ be a cocycle. By Schoenberg's result 6.4.12, for all $r > 0$, $f_r(g) := \exp(-r\|\beta(g)\|^2)$ is positive definite, and $f_r(g) \to 1$ pointwise as $r \to 0^+$. By (T1), $f_{1/n} \to 0$ uniformly, which implies β is bounded. Thus β is inner by Lemma 6.4.8. ☐

(T2)⇔(T3). Observe that for all $\alpha \in \text{Aff}(H)$,

$$\xi = \alpha_g \xi = \pi_g \xi + \beta(g) \qquad \Longleftrightarrow \qquad \beta(g) = \xi - \pi_g \xi \qquad \forall g \in \Gamma.$$

Hence α has a fixed point if and only if β is inner. ☐

$\neg$(T5)$\Rightarrow \neg$(T4). Let $\Gamma = \{g_1, g_2, \dots\}$ be an enumeration and set $F_n = \{g_1, \dots, g_n\} \subset \Gamma$ and $\delta_n = 1/n$. Then for each n, there is a unitary representation (H_n, π_n, ξ_n) such that $\|\xi_n\| = 1$, $\|\pi_n(g)\xi_n - \xi_n\| < 1/n$ for all $g \in F_n$, but the Γ-invariant subspace $H_n^{\pi_n} = 0$. Set $(H, \pi) := \bigoplus(H_n, \pi_n)$. Then (ξ_n) where ξ_n lives in only the n-th component is a sequence of almost invariant vectors, but there is no Γ-invariant vector in (H, π) as every projection map $(H, \pi) \to (H_n, \pi_n)$ is Γ-equivariant. □

(T6)$\Rightarrow$(T5). Trivial - just take an arbitrary $\varepsilon > 0$. □

(T5)$\Rightarrow$(T6). Let $\varepsilon > 0$. Pick $\delta' > 0$ and a finite set $F' \subset \Gamma$ satisfying (T5). We set $\delta = \varepsilon'\delta'$ for a to-be-determined $\varepsilon' > 0$ in terms of ε and set $F = F'$. Suppose (H, π) is a unitary Γ-representation with $\xi \in H$ a (δ, F)-almost invariant vector as in (T6). Consider the Γ-fixed points

$$H^\pi := \{\eta \in H | \pi_g \eta = \eta \quad \forall\, g \in \Gamma\}.$$

If $\xi \in H^\pi$, then we are finished. If not, our strategy will be to project ξ to H^π and show that this vector is non-zero and close to ξ after renormalizing.
To this end, let p be the orthogonal projection onto H^π so that

$$\|\pi_g\eta - \eta\| = \|\pi_g(1-p)\eta + \pi_g p\eta - \eta\| = \|\pi_g(1-p)\eta + (1-p)\eta\| \qquad \forall\, \eta \in H.$$

Note that $(H^\pi)^\perp = (1-p)H$ does not contain any non-zero invariant vectors. Since $\pi|_{(1-p)H}$ is a unitary Γ-representation, by (T5), for all unit vectors $\eta \in (1-p)H$, $\|\pi_g\eta - \eta\| \geq \delta'$ for some $g \in F$. This means

$$\|\pi_g(1-p)\xi - (1-p)\xi\| \geq \delta'\|(1-p)\xi\|.$$

We now calculate that

$$\varepsilon'\delta' = \delta \geq \|\pi_g\xi - \xi\| = \|\pi_g(1-p)\xi - (1-p)\xi\| \geq \delta'\|(1-p)\xi\| \quad \Longrightarrow \quad \|(1-p)\xi\| \leq \varepsilon'.$$

As ξ is a unit vector, this means that if $\varepsilon' < 1$, then $p\xi \neq 0$, and we may set $\xi_0 := p\xi/\|p\xi\| \in H^\pi$. It remains to show ξ_0 is close to ξ when ε' is small enough. Indeed,

$$\xi_0 - p\xi = \frac{p\xi}{\|p\xi\|} - p\xi = \frac{1 - \|p\xi\|}{\|p\xi\|} p\xi$$

which implies

$$\|\xi_0 - p\xi\| \leq 1 - \|p\xi\| = \|\xi\| - \|p\xi\| \leq \|(1-p)\xi\| \leq \varepsilon'$$

by the reverse triangle inequality. Finally, we calculate

$$\|\xi_0 - \xi\| \leq \|\xi_0 - p\xi\| + \|p\xi - \xi\| = \|\xi_0 - p\xi\| + \|(1-p)\xi\| \leq 2\varepsilon' < \varepsilon$$

as long as $\varepsilon' < \min\{\varepsilon/2, 1\}$. □

(T6)$\Rightarrow$(T7). Let $\varepsilon > 0$, and choose (F', δ') as in (T6) for $\varepsilon' > 0$ a function of ε to be determined. Set $F = F' \cup (F')^{-1} \cup \{e\}$ and let δ be a function of ε and δ' to be determined. Suppose $f : \Gamma \to \mathbb{C}$ is positive definite such that $|f(g) - 1| < \delta$ for all $g \in F$. By Theorem 6.1.5, there is a unitary Γ-representation (H, π, η) such that

$f(g) = \langle \pi_g \eta, \eta \rangle$ for all $g \in \Gamma$. Since $e \in F$,

$$|\|\eta\|^2 - 1\| = |f(e) - 1| < \delta.$$

Set $\xi := \eta/\|\eta\|$, and we record the estimate

$$|1 - \langle \pi_g \xi, \xi \rangle| \leq \underbrace{|1 - f(g)|}_{<\delta} + \underbrace{|\langle \pi_g \eta, \eta \rangle - \langle \pi_g \xi, \xi \rangle|}_{\leq |\|\eta\|^2 - 1| \cdot \langle \pi_g \xi, \xi \rangle < \delta \cdot 1} < 2\delta \qquad \forall\, g \in F.$$

Then for all $g \in F$,

$$\|\pi_g \xi - \xi\|^2 = 1 - \langle \pi_g \xi, \xi \rangle + 1 - \langle \pi_{g^{-1}} \xi, \xi \rangle \leq |1 - \langle \pi_g \xi, \xi \rangle| + |1 - \langle \pi_{g^{-1}} \xi, \xi \rangle| \leq 4\delta < \delta'^2$$

if $\delta < \delta'^2/4$. By (T6), there is a unit vector $\xi_0 \in H$ such that $\pi_g \xi_0 = \xi_0$ for all $g \in \Gamma$ and $\|\xi - \xi_0\| < \varepsilon'$. Then for all $g \in \Gamma$,

$$\begin{aligned} |1 - f(g)| &= |\langle \pi_g \xi_0, \xi_0 \rangle - \langle \pi_g \eta, \eta \rangle| \\ &= |\langle \pi_g(\xi_0 - \xi), \xi_0 \rangle + \langle \pi_g \xi, (\xi_0 - \xi) \rangle + \langle \pi_g \xi, \xi \rangle - \langle \pi_g \eta, \eta \rangle| \\ &\leq |\langle \pi_g(\xi_0 - \xi), \xi \rangle| + |\langle \pi_g \xi, (\xi_0 - \xi) \rangle| + |\langle \pi_g \xi, \xi \rangle - \langle \pi_g \eta, \eta \rangle| \\ &< 2\varepsilon' + \delta < \varepsilon \end{aligned}$$

provided we chose $\varepsilon' < \varepsilon/3$ and $\delta < \min\{\varepsilon/3, \delta'^2/4\}$. □

(T7)⇒(T1). Suppose (f_n) is a sequence of positive definite functions such that $f_n \to 1$ pointwise on Γ. Let $\varepsilon > 0$, and choose (F, δ) as in (T7). Since F is finite and $f_n \to 1$ pointwise, eventually $|f_n(g) - 1| < \delta$ for all $g \in F$. Then $|f_n(g) - 1| < \varepsilon$ for all $g \in \Gamma$ by (T7). □

Exercise 6.5.5. Prove (T2)⇒(T4).

Exercise 6.5.6. Modify all the statements in Theorem 6.5.4 for a countable discrete group Γ to be relative to a subgroup $\Lambda \leq \Gamma$. Then prove all the equivalences.

6.6. **Property (T) for tracial von Neumann algebras.** For this section, (M, tr) is a tracial von Neumann algebra with separable predual.

Definition 6.6.1. We say (M, tr) has *property* (T) if for every sequence $(\varphi_n : M \to M)$ of normal trace-preserving ucp maps with $\varphi_n \to \mathrm{id}_M$ pointwise-$\|\cdot\|_2$, $\varphi_n \to \mathrm{id}_M$ uniformly in $\|\cdot\|_2$ on $(M)_1$, the unit ball of M.

The main goal of this section is to prove that a countable discrete group Γ has (T) if and only if $L\Gamma$ with its canonical trace has (T).

Definition 6.6.2. Suppose $(A, \mathrm{tr}_A), (B, \mathrm{tr}_B)$ are tracial von Neumann algebras. An $A-B$ *bimodule* ${}_AH_B$ is a Hilbert space H equipped with commuting normal unital $*$-homomorphisms $\lambda : A \to B(H)$ and $\rho : B^{\mathrm{op}} \to B(H)$ (with $[\lambda_a, \rho_b] = 0$ for all $a \in A$, $b \in B^{\mathrm{op}}$). We typically suppress λ, ρ and simply write $a\eta b = \lambda_a \rho_b \eta$.

A *pointing* on a bimodule ${}_AH_B$ is a distinguished vector $\xi \in H$ such that $A\xi B$ is dense in H. A pointing is called *tracial* if in addition

$$\langle a\xi, \xi \rangle = \mathrm{tr}_A(a) \qquad \forall\, a \in A \qquad \text{and} \qquad \langle \xi b, \xi \rangle = \mathrm{tr}_B(b) \qquad \forall\, b \in B.$$

Construction 6.6.3. Suppose $(A, \mathrm{tr}_A), (B, \mathrm{tr}_B)$ are tracial von Neumann algebras and $({}_AH_B, \eta)$ is a tracially pointed bimodule. We can construct a trace-preserving normal ucp map $\phi : A \to B$ as follows.

First, since η is a tracial pointing, the map $L_\eta : L^2B \to H$ given by $b\Omega \mapsto \eta b$ extends to a unique isometry. Define $\phi : A \to B$ by $\phi(a) := L_\eta^* \lambda_a L_\eta \in B(L^2B)$. Since L_η and λ_a are right B-linear, so is $\phi(a)$, i.e., $\phi(a) \in JBJ' = B$. Finally, we verify

$$\mathrm{tr}_B(\phi(a)) = \langle \phi(a)\Omega, \Omega\rangle = \langle L_\eta^* \lambda_a L_\eta \Omega, \Omega\rangle = \langle b\eta, \eta\rangle = \mathrm{tr}_B(a).$$

Remark 6.6.4. Given two tracially pointed bimodules $({}_AH_B, \eta)$ and $({}_AK_B, \xi)$, there is at most one $A-B$ bimodular map $T : H \to K$ mapping η to ξ. This map will be unitary if and only if $T^* : K \to H$ also preserves the pointing. Indeed, $T^*\xi = \eta$ if and only if $T^* = T^{-1}$. This shows that the 2-category of tracial von Neumann algebras, tracially pointed bimodules, and $A - B$ bimodular unitaries preserving the pointing is 1-truncated, i.e., equivalent to a 1-category.

Construction 6.6.5. Suppose $(A, \mathrm{tr}_A), (B, \mathrm{tr}_B)$ are tracial von Neumann algebras and $\phi : A \to B$ is a trace-preserving normal ucp map. We can build a tracially pointed bimodule as follows.

Let H_ϕ be the Hilbert space obtained from taking the algebraic tensor product $A \otimes B$ with sesquilinear form $\langle a_1 \otimes b_1, a_2 \otimes b_2\rangle_\phi := \mathrm{tr}_B(b_2^* \phi(a_2^* a_1) b_1)$, quotienting out the length zero vectors, and completing in $\|\cdot\|_2$; this is the Hilbert space from the proof of the Stinespring Dilation Theorem. We calculate the left A-action descends to a bounded action:

$$\left\| a \cdot \sum_i x_i \otimes y_i \right\|_\phi^2 = \sum_{i,j} \langle \phi(x_j^* a^* a x_i) y_i \Omega, y_j \Omega\rangle_{L^2B} \leq \|a^* a\| \cdot \left\| \sum_i x_i \otimes y_i \right\|_\phi^2 .$$

where the inequality comes from the fact $[\phi(x_j^* a^* a x_i)] \leq \|a^* a\| \cdot [\phi(x_j^* x_i)]$ in $M_n(B)$. Boundedness of the right B-action is easier and omitted. These actions are normal since ϕ is normal (exercise).

Remark 6.6.6. Consider the case of $N \subset M$ an inclusion of finite von Neumann algebras where M is equipped with a faithful normal tracial state tr. Let $E : M \to N$ be the unique trace-preserving conditional expectation. We claim that the map $m \otimes n \mapsto mn$ descends to an $M - N$ bimodular unitary isomorphism $H_E \cong {}_ML^2M_N$; this is the unique map from Remark 6.6.4. Indeed, $M - N$ bimodularity is obvious, and we calculate

$$\begin{aligned} \langle m_1 \otimes n_1, m_2 \otimes n_2\rangle_E &= \mathrm{tr}(n_2^* E(m_2^* m_1) n_1) = (\mathrm{tr} \circ E)(n_2^* m_2^* m_1 n_1) \\ &= \mathrm{tr}(n_2^* m_2^* m_1 n_1) = \langle m_1 n_1 \Omega, m_2 n_2 \Omega\rangle_{L^2M}. \end{aligned}$$

Hence this map descends to a well-defined isometry with dense range, and thus uniquely extends to a unitary.

Exercise 6.6.7. Prove that Constructions 6.6.5 and 6.6.3 are mutually inverse. In more detail:

(1) Starting with a trace-preserving normal ucp map $\phi : A \to B$, show that applying Construction 6.6.5 and then Construction 6.6.3 produces exactly ϕ again.
(2) Starting with a tracially pointed bimodule $({}_AH_B, \eta)$, show that applying Construction 6.6.3 and then Construction 6.6.5 gives another tracially pointed bimodule $({}_AK_B, \xi)$ which is canonically unitarily equivalent to $({}_AH_B, \eta)$ via Remark 6.6.4.

Remark 6.6.8. Exercise 6.6.7 above shows that the 1-truncated 2-category from Remark 6.6.4 is equivalent to the 1-category of tracial von Neumann algebras with trace-preserving normal ucp maps.

Lemma 6.6.9. *Suppose $\varphi : M \to M$ is trace-preserving ucp map, and let (H, ξ) be the associated tracially pointed $M - M$ bimodule. Then for all $x \in M$, $\langle x\xi, \xi x\rangle = \operatorname{tr}_M(\varphi(x)x^*)$ and*

$$\|\varphi(x)\Omega - x\Omega\|_{L^2M} \leq \|x\xi - \xi x\|_H \leq \|\varphi(x)\Omega - x\Omega\|_{L^2M} \cdot \|x\|_2.$$

Proof. First, note that

$$\langle x\xi, \xi x\rangle = \langle x \otimes 1, 1 \otimes x\rangle_\varphi = \langle \varphi(x)\Omega, x\Omega\rangle_{L^2M} = \operatorname{tr}_M(\varphi(x)x^*).$$

We then calculate

$$\begin{aligned}\|\varphi_n(x)\Omega - x\Omega\|_{L^2M}^2 &= \|\varphi_n(x)\Omega\|_{L^2M}^2 + \|x\Omega\|_{L^2M}^2 - 2\operatorname{Re}\operatorname{tr}_M(\varphi_n(x)x^*)\\ &= \operatorname{tr}_M(\varphi_n(x)^*\varphi_n(x)) + \operatorname{tr}_M(x^*x) - 2\operatorname{Re}\operatorname{tr}_M(\varphi_n(x)x^*)\\ &\leq \operatorname{tr}_M(\varphi_n(x^*x)) + \operatorname{tr}_M(x^*x) - 2\operatorname{Re}\operatorname{tr}_M(\varphi_n(x)x^*)\\ &= 2\operatorname{tr}_M(x^*x) - 2\operatorname{Re}\langle x\xi_n, \xi_n x\rangle.\end{aligned}$$

We now see that

$$2\operatorname{tr}_M(x^*x) - 2\operatorname{Re}\langle x\xi_n, \xi_n x\rangle = \|x\xi_n\|_2^2 + \|\xi_n x\|_2^2 - 2\operatorname{Re}\langle x\xi_n, \xi_n x\rangle = \|x\xi_n - \xi_n x\|_{L^2M}^2$$

and

$$\begin{aligned}2\operatorname{tr}_M(x^*x) - 2\operatorname{Re}\langle x\xi_n, \xi_n x\rangle &= 2\operatorname{Re}\operatorname{tr}_M((\varphi_m(x) - x)x^*)\\ &\leq 2|\langle(\varphi_m(x) - x)\Omega, x\Omega\rangle|\\ &\underset{\text{(CS)}}{\leq} 2\|\varphi_m(x)\Omega - x\Omega\|_2 \cdot \|x\Omega\|_2.\end{aligned}$$

□

Theorem 6.6.10. *For a tracial von Neumann algebra (M, tr), the following are equivalent.*

(1) (M, tr) *has* (T).

(2) For all $\varepsilon > 0$, there is a $\delta > 0$ and a finite $F \subset M$ such that for every tracially pointed $M - M$ bimodule $({}_MH_M, \xi)$ satisfying

$$\max_{x \in F} \|x\xi - \xi x\| < \delta,$$

there is an M-central vector $\xi_0 \in H$ such that $\|\xi - \xi_0\| < \varepsilon$.

Proof.
(1) ⇒ (2): Omitted. **TODO: Check this!**
(2) ⇒ (1): Suppose (φ_n) is sequence of normal trace-preserving ucp maps such that $\varphi_n \to \operatorname{id}_M$ pointwise $\|\cdot\|_2$. Let $\varepsilon > 0$ and pick (F', δ') for a to-be-determined $\varepsilon' > 0$ as a function of ε. Let (H_n, ξ_n) be the tracially pointed $M - M$ bimodule associated to φ_n. Since $\varphi_n \to \operatorname{id}_M$ pointwise $\|\cdot\|_2$, there is an $N > 0$ such that $n > N$ implies

$$\|\varphi_n(x)\Omega - x\Omega\|_2 < \delta \qquad \forall x \in F',$$

where $\delta > 0$ is to be determined in terms of $\varepsilon', \delta', F'$. Then by Lemma 6.6.9, for all $n > N$ and $x \in F$,

$$\|x\xi_n - \xi_n x\|_2^2 \leq 2\|\varphi_m(x)\Omega - x\Omega\|_2 \cdot \|x\Omega\|_2 < 2\delta K$$

where $K := \max_{x\in F} \|x\Omega\|_2$. Now if $\delta < \delta'^2/2K$, then for every $n > N$, there is an M-central vector $\xi_{n,0} \in H_n$ such that $\|\xi_{n,0} - \xi_n\| < \varepsilon'$. Then again by Lemma 6.6.9, for all $n > N$ and $x \in (M)_1$,

$$\|\varphi_n(x)\Omega - x\Omega\|_2 \leq \|x\xi_n - \xi_n x\| \leq \|x\xi_n - x\xi_{n,0}\| + \|\xi_{n,0}x - \xi_n x\| \leq 2\varepsilon' < \varepsilon$$

whenever $\varepsilon' < \varepsilon/2$. □

Corollary 6.6.11. *A countable discrete group* Γ *has* (T) *if and only if* $L\Gamma$ *with its canonical trace has* (T)*.*

Proof. Suppose Γ has (T). Let $\varepsilon > 0$, and choose (F, δ) as in (T6). Let (H, ξ) be a tracially pointed $L\Gamma - L\Gamma$ bimodule such that $\max_{g\in F} \|\lambda_g\xi - \xi\lambda_g\|_2 < \delta$. We have a unitary representation $\pi : \Gamma \to B(H)$ by $\pi_g\eta := \lambda_g\eta\lambda_g^*$. Since $\xi \in (H)_1$ and

$$\|\pi_g\xi - \xi\| = \|\lambda_g\xi\lambda_g^* - \xi\| = \|\lambda_g\xi - \xi\lambda_g\| < \delta \qquad \forall\, g \in \Gamma,$$

by (T6) there is a Γ-invariant vector $\xi_0 \in (H)_1$ with $\|\xi - \xi_0\| < \varepsilon$ such that $\pi_g\xi_0 = \xi_0$ for all $g \in \Gamma$. But then $\lambda_g\xi_0 = \xi_0\lambda_g$ for all $g \in \Gamma$, and thus ξ_0 is $L\Gamma$-central as desired. We conclude $L\Gamma$ has (T).

Conversely, suppose $L\Gamma$ has (T). Let (f_n) be a sequence of positive definite functions on Γ which converge to 1 pointwise. Without loss of generality, we may assume $f_n(e) = 1$ for all n. Then (M_{f_n}) is a sequence of trace-preserving ucp maps such that $M_{f_n} \to \mathrm{id}_M$ pointwise $\|\cdot\|_2$. Since $L\Gamma$ has (T), $M_{f_n} \to \mathrm{id}_M$ uniformly in $\|\cdot\|_2$ on $(L\Gamma)_1$. In particular, for every $\varepsilon > 0$, there is an $N > 0$ such that for all $n > N$ and $g \in \Gamma$,

$$|f_n(g) - 1| = \|f_n(g)\delta_g - \delta_g\|_{\ell^2\Gamma} = \|M_{f_n}(\lambda_g)\Omega - \lambda_g\Omega\|_{L^2L\Gamma} < \varepsilon.$$

Hence $f_n \to 1$ uniformly, and Γ has (T). □