# 4. Projections and factors

For this section, $H$ is a Hilbert space and $M \subseteq B(H)$ is a von Neumann algebra. We denote the set of projections of $M$ by $P(M)$ and the group of unitaries in $M$ by $U(M)$.

## 4.1. **Compressions and ideals.**

**Theorem 4.1.1.** *Suppose $p \in P(M)$. Then $pMp \subseteq B(pH)$ is a von Neumann algebra with commutant $(pMp)' = M'p$.*

*Proof.* Clearly $pMp \subseteq B(pH)$ is an SOT-closed unital $*$-subalgebra and thus a von Neumann algebra.
If $y \in M'$, then for all $x \in M$,

$$(yp)(pxp) = ypxp = pypx = (pxp)(yp)$$

so $yp \in (pMp)'$. For the converse, we use a clever trick. First, it suffices to prove every unitary in $(pMp)'$ lies in $M'p$, as every element of $(pMp)'$ is a linear combination of 4 unitaries (why?). Suppose $u \in (pMp)'$ and set $K := \overline{MpH}$. Since $K$ is both $M$ and $M'$-invariant, $p_K \in M' \cap M = Z(M)$.

**Claim.** *We may extend $u$ to $K$ by $\widetilde{u} \sum x_i p\xi := \sum x_i up\xi$.*

*Proof of claim.* To see $\widetilde{u}$ is well-defined, we prove it is isometric:

$$\begin{aligned}
\left\| \widetilde{u} \sum x_i p \xi_i \right\|^2 &= \sum_{i,j} \langle x_i up\xi_i, x_j up\xi_j \rangle \\
&\sum_{i,j} \langle px_j^* x_i pu\xi_i, u\xi_j \rangle && ([u,p]=0) \\
&= \sum_{i,j} \langle upx_j^* x_i p\xi_i, u\xi_j \rangle && (u \in (pMp)') \\
&= \sum_{i,j} \langle px_j^* x_i p\xi_i, \xi_j \rangle = \cdots = \left\| \sum \sum x_i p \xi_i \right\|^2 .
\end{aligned}$$

□

Now by construction, $\widetilde{u}$ commutes with the action of $M$ on $K = \overline{MpH}$, and thus $\widetilde{u}p_K \in M' \subseteq B(H)$; indeed, for all $x \in M$ and all $\xi \in H$,

$$\widetilde{u}p_K x\xi = \widetilde{u}x \underbrace{p_K \xi}_{\in K} = x\widetilde{u}p_K\xi.$$

Finally, we claim that $u = \widetilde{u}p_K p \in M'p$; indeed, as $u = up \in M'p$, for all $\xi \in H$,

$$\widetilde{u}p_K p\xi = \widetilde{u}(1_M p\xi) = up\xi = u\xi.$$

□

**Definition 4.1.2.** We call $pMp, M'p$ corners, compressions, or reductions of $M, M'$ respectively.

**Lemma 4.1.3.** *If $J \subseteq M$ is a $\sigma$-WOT closed left ideal, then $J = Mp$ for a unique projection $p \in M$.*

*Proof.* If $p$ is any projection such that $J = Mp$, then since $(xp)p = xp$ for all $x \in M$, $yp = y$ for all $y \in J$. It follows that if $Mq = J = Mp$, then $p \leq q$ and $q \leq p$, so $p = q$. This also tells us how to construct $p$: find the largest projection in $J$.
If $x \in J$, then so are $|x| = u^*x$ and $\chi_{[\varepsilon,\|x\|]}(|x|)$ for all $\varepsilon > 0$. Since $\chi_{[\varepsilon,\|x\|]}(|x|) \nearrow \operatorname{supp}(|x|) = \operatorname{supp}(x)$ as $\varepsilon \searrow 0$, $\operatorname{supp}(x) \in J$.
Now observe that if there is a maximal projection $p$ in $J$, then $p \geq \operatorname{supp}(x)$ for all $x \in J$, so $x = x \cdot \operatorname{supp}(x) \cdot p = xp$ for all $x \in J$. We thus have $J \subseteq Mp \subseteq J$, and thus equality holds.
Finally, to construct the maximal projection, since $J$ is $\sigma$-WOT closed, it is a norm-closed left ideal, and thus contains a right approximate identity $(e_i)$ such that $0 \leq e_i \leq 1$ for all $i$, $i \leq j$ implies $e_i \leq e_j$, and $\|x - xe_i\| \to 0$ for all $x \in J$. Since $J$ is $\sigma$-WOT closed, $p := \bigvee e_i \in J$, which is automatically self-adjoint. Since $\|p - pe_i\| \to 0$, we see that $p = p^2$, so $p$ is a projection, and since $\|x - xe_i\| \to 0$, $x = xp$ for all $x \in J$. We conclude that $p$ is the largest projection in $J$. □

**Corollary 4.1.4.** *A left ideal $J \subseteq M$ is SOT/WOT-closed if and only if it is $\sigma$-SOT/$\sigma$-WOT closed.*

*Proof.* If $J$ is $\sigma$-WOT closed, then $J = Mp$ for some projection $p \in J$, so $J$ is WOT closed. The converse is trivial as WOT-closed sets are $\sigma$-WOT closed. □

**Corollary 4.1.5.** *If $J \subseteq M$ is a $\sigma$-WOT closed 2-sided ideal, then $J = Mz$ for some projection $z \in Z(M)$.*

*Proof.* Since $J$ is $\sigma$-WOT closed, it is also WOT and hence SOT-closed. By Lemma 4.1.3, $J = Mz$ for some projection $z \in M$. But as $J$ is 2-sided, for every unitary $u \in M$, $J = uJu^*$. It follows that $J = uJu^* = uMzu^* = uMu^*(uzu^*) = Muzu^*$, so $z = uzu^*$ by the uniqueness statement in Lemma 4.1.3. We conclude $z \in M' \cap M = Z(M)$. □

## 4.2. **Central support of a projection.**

**Definition 4.2.1.** A factor is a von Neumann algebra with trivial center, i.e., $Z(M) = M' \cap M = \mathbb{C}1$.

**Remark 4.2.2.** By Corollary 4.1.5, factors have no non-trivial $\sigma$-WOT closed 2-sided ideals.

Just as von Neumann algebras come in pairs $M, M'$, so do factors as $Z(M) = M' \cap M = Z(M')$.

Recall that for $p, q \in P(M) \subseteq B(H)$, $p \wedge q$ is the projection onto $pH \cap qH$ and $p \vee q$ is the projection onto $\overline{pH + qH}$. Observe we have the relation

$$p \vee q = 1 - (1-p) \wedge (1-q). \tag{4.2.3}$$

For homework, you will show that $p \wedge q, p \vee q \in M$. Thus $P(M)$ is a lattice under these operations.

**Lemma 4.2.4.** *For $p, q \in P(M)$ and $u \in U(M)$, $u^*(p \vee q)u = u^*pu \vee u^*qu$ and $u^*(p \wedge q)u = u^*pu \wedge u^*qu$.*

*Proof.* Observe that $\xi \in pH \cap qH$ if and only if $u^*\xi \in u^*puH \cap u^*quH$, and $\eta \perp pH \cap qH$ if and only if $u^*\eta \perp u^*puH \cap u^*quH$. Thus $u^*(p \wedge q)u = u^*pu \wedge u^*qu$. Now apply (4.2.3) to get $u^*(p \vee q)u = u^*pu \vee u^*qu$. ☐

**Definition 4.2.5.** Given $p \in P(M)$, we define its central support

$$z(p) := \bigvee_{u \in U(M)} u^*pu := \operatorname{lub} p_F$$

where $p_F := \bigvee_{u \in F} u^*pu$ for finite subsets $F \subset U(M)$, ordered by inclusion. By Lemma 4.2.4, for all $w \in U(M)$,

$$w^*p_F w = \bigvee_{u \in F} w^*u^*puw = \bigvee_{v \in Fw} v^*pv = p_{Fw}.$$

As $z(p)$ is the SOT-limit of the $p_F$ and multiplication is separately SOT-continuous,

$$w^*z(p)w = w^*(\lim{}^{SOT} p_F)w = \lim{}^{SOT} w^*p_F w = \lim{}^{SOT} p_{Fw} = z(p).$$

This means $wz(p) = z(p)w$ for all $w \in U(M)$, so $z(p) \in M' \cap M = Z(M)$.

**Lemma 4.2.6.** *Suppose $p \in P(M)$.*

*(1) For $x \in M$, $xup = 0$ for all $u \in U(M)$ if and only if $xz(p) = 0$.*
*(2) For $y \in M'$, $yp = 0$ if and only if $yz(p) = 0$. Hence the map $M'z(p) \to M'p$ given by multiplication by $p$ is a $*$-isomorphism.*

*Proof.*

(1) If $xup = 0$ for all $u \in U(M)$, then $xupu^* = 0$ for all such $u$. Then $xp_F = 0$ where $p_F = \bigvee_{u \in F} upu^*$ for any finite $F \subset U(M)$,[a] and taking SOT limits, we have $xz(p) = x \lim^{SOT} p_F = 0$.

Conversely, if $xz(p) = 0$, then $x(upu^*)u = xz(p)(upu^*)u = 0$ for all $u \in U(M)$.

(2) Since $yp = ypz(p) = yz(p)p$, $yz(p) = 0$ implies $yp = 0$. Conversely, if $yp = 0$, then $yupu^* = 0$ for all $u \in U(M)$ as $y \in M'$. The argument from (1) shows $yp_F = 0$ for all finite $F \subset U(M)$, so taking SOT limits, $yz(p) = 0$. ☐

[a] If $u_1, \ldots, u_n \in U(M)$ and $\xi_i \in u_ipu_i^*H$ for $i = 1, \ldots, n$, then $x\sum \xi_i = \sum x\xi_i = \sum xu_ipu_i^*\xi_i = 0$, so $xp_F = 0$.

**Proposition 4.2.7.** *For a von Neumann algebra $M$ and $p, q \in P(M) \setminus \{0\}$, the following are equivalent.*

*(1) $z(p)z(q) \neq 0$,*
*(2) there is a $u \in U(M)$ such that $puq \neq 0$, and*
*(3) there is a non-zero partial isometry $v \in M$ such that $vv^* \leq p$ and $v^*v \leq q$.*

*Proof.*
$\neg(2) \Rightarrow \neg(1)$: If $puq = 0$ for all $u \in U(M)$, then $pz(q) = 0$ by Lemma 4.2.6(1). But then $0 = qz(p)u = quz(p)$ for all $u \in U(M)$, so by (the adjoint of) Lemma 4.2.6(1) again, $z(q)z(p) = 0$.
$(2) \Rightarrow (3)$: If $puq \neq 0$, consider the polar decomposition $puq = v|puq|$. By construction, $vv^*H = vH = \overline{puqH} \subset pH$, so $vv^* \leq p$. Since $\ker(v) = \ker(puq) \supset \ker(q)$, we have $v^*v = p_{\ker(v)^\perp} \leq p_{\ker(q)^\perp} = q$.
$(3) \Rightarrow (1)$: We prove that if $z(p)z(q) = 0$ and $v \in M$ is a partial isometry such that $vv^* \leq p$ and $v^*v \leq q$, then $v = 0$. Since $vv^* \leq p \leq z(p)$, $vv^* = vv^*z(p)$. Since $v^*v \leq q \leq z(q)$, $v^*v = v^*vz(q)$. Then
$$v = vv^*v = z(p)vv^*vz(q) = vz(q)z(p) = 0. \qquad \square$$

**Corollary 4.2.8** (Ergodic property of factors)**.** *Suppose $M$ is a factor and $p, q \in P(M)\setminus\{0\}$. There is a unitary $u \in U(M)$ such that $puq \neq 0$.*

*Proof.* Since $p, q \neq 0$, $z(p) = z(q) = 1$. Now apply Proposition 4.2.7. $\square$

**Corollary 4.2.9.** *Suppose $M$ is a factor and $p, q \in P(M) \setminus \{0\}$. There is a non-zero partial isometry $u \in M$ such that $uu^* \leq p$ and $u^*u \leq q$. Moreover, we can find $u \in M$ such that $uu^* = p$ or $u^*u = q$.*

*Proof.* The first part is immediate as $z(p) = 1 = z(q)$. Consider the set of partial isometries $u \in M$ such that $uu^* \leq p$ and $u^*u \leq q$. We can order this set by $u \leq v$ if $uu^* \leq vv^*$, $u^*u \leq v^*v$, and $v|_{uu^*H} = u$.

**Claim.** *Any increasing chain has an upper bound.*

*Proof of Claim.* If $(v_i)$ is an increasing chain, then the operator $w : \bigcup v_i^*v_iH \to \bigcup v_jv_j^*H$ given by $\xi \mapsto v_k\xi$ whenever $\xi \in v_k^*v_kH$ is well-defined and unitary. It thus extends to an isometry $K := \overline{\bigcup v_i^*v_iH} \to H$, and thus to a partial isometry on $H$ by defining $w|_{K^\perp} = 0$. Clearly $ww^* \leq p$, $w^*w \leq q$, and $v_i \leq w$ for all $i$. $\square$

We claim a maximal element satisfies $uu^* = p$ or $u^*u = q$. Indeed, if $p - uu^* \neq 0 \neq q - u^*u$, then there is a non-zero partial isometry $w \in M$ such that $ww^* \leq p - uu^*$ and $w^*w \leq q - w^*w$. Observe then that $u + w$ is a partial isometry (why?) with $(u + w)(u + w)^* \leq p$ and $(u + w)^*(u + w) \leq q$ contradicting maximality. $\square$

**Exercise 4.2.10.** Show that the central support $z(p)$ is the smallest projection in $Z(M)$ such that $p \leq z(p)$.

**Corollary 4.2.11.** $Z(pMp) = Z(M)p$.

*Proof (Dixmier).* Clearly $Z(M)p = p(M' \cap M)p \subset Z(pMp)$. Suppose $x \in Z(pMp) = pMp \cap M'p$. Then there is a $y \in M'$ such that $x = yp$. Since $p = z(p)p$, replacing $y$ with $yz(p)$, we may assume $y = yz(p)$. We claim that $y \in Z(Mz(p))$ so that

$y \in M' \cap Mz(p) \subset M' \cap M = Z(M)$. Indeed, the map $M'z(p) \to M'p$ given by multiplication by $p$ is an isomorphism by Lemma 4.2.6(2), and thus maps the center onto the center. Since $yp = x \in Z(pMp)$, we conclude $y = yz(p) \in Z(Mz(p))$, as desired. □

## 4.3. **Classification of type I factors and their subfactors.**

**Definition 4.3.1.** A (nonzero) projection $p \in P(M)$ is called:

- minimal if $q \in P(M)$ with $q \leq p$ implies $q \in \{0, p\}$,
- abelian if $pMp$ is abelian, and
- diffuse if there is no minimal projection $q \leq p$.

**Examples 4.3.2.** Here are examples of such projections.

(1) The minimal projections in $B(H)$ are the rank 1 projections.
(2) Every projection is diffuse in $L^\infty([0,1], \lambda)$ where $\lambda$ is Lebesgue measure.

**Exercise 4.3.3.** Suppose $\mu$ is a regular finite Borel measure on a compact Hausdorff space $X$. Show that the minimal projections of $L^\infty(X, \mu)$ correspond to atoms of $X$, i.e., $x \in X$ such that $\mu(\{x\}) > 0$.

**Exercise 4.3.4.** Suppose $p \in P(M)$ is minimal and $u \in M$ is a non-zero partial isometry such that $uu^* \leq p$. Show that $uu^* = p$ and that $u^*u$ is a minimal projection.

**Definition 4.3.5.** A von Neumann algebra $M$ is called type I if for all $z \in P(Z(M)) \setminus \{0\}$, there is an abelian $p \in P(M) \setminus \{0\}$ such that $p \leq z$, i.e., every non-zero central projection majorizes an abelian projection.

**Examples 4.3.6.** Examples of type I von Neumann algebras include abelian von Neumann algebras and $B(H)$.

**Exercise 4.3.7.** Here are some exercises on minimal projections.

(1) $p \in P(M)$ is minimal if and only if $pMp = \mathbb{C}p$.
(2) If $M$ is a factor and $p$ is abelian, then $p$ is minimal.
(3) If $M$ is a factor, then $M$ is type I if and only if $M$ has a minimal projection.

**Theorem 4.3.8** (Classification of type I factors)**.** *If $M$ is a type* I *factor acting on a Hilbert space $H$, there are Hilbert spaces $K, L$ and a unitary $u \in B(K \otimes L \to H)$ such that $uMu^* = B(K) \otimes 1$.*

To prove this theorem, we will construct a system of matrix units for $M$, i.e., a family $\{e_{ij} | i, j \in I\}$ such that

- $e_{ij}^* = e_{ji}$,
- $e_{ij}e_{k\ell} = \delta_{j=k} e_{i\ell}$, and
- $\sum e_{ii} = 1$ converging in SOT.

**Lemma 4.3.9.** *If $\{e_{ij}\}_{i,j \in I}$ is a system of matrix units in $B(H)$, then setting $K := e_{11}H$ which should be viewed as a 'multiplicity space,' there is a unitary $u \colon \ell^2 I \otimes K \to H$ such that $u^* e_{ij} u = |\delta_i\rangle\langle\delta_j| \otimes 1$ for all $i, j$. Thus $u^*(\{e_{ij}\}'')u = B(\ell^2 I) \otimes 1$, and $\dim(H) = |I| \dim(K)$.*

*Proof.* Let $\{\xi_j\}_{j\in J}$ be an ONB of $K = e_{11}H$. Since $e_{1i}$ may be viewed as a unitary from $e_{ii}H$ onto $e_{11}H$, we see that $\{e_{i1}\xi_j | j \in J\}$ is an ONB for $e_{ii}H$. Since $H = \bigoplus e_{ii}H$, we see that $\{e_{i1}\xi_j | i \in I,\ j \in J\}$ is an ONB of $H$. Thus the map $u\colon \ell^2 I \otimes K \to H$ by $\delta_i \otimes \xi_j \mapsto e_{i1}\xi_j$ is a unitary isomorphism. Finally, we calculate

$$u^* e_{ij} u(\delta_k \otimes \xi_\ell) = u^* e_{ij} e_{k1} \xi_\ell = \delta_{j=k} u^* e_{i1} \xi_\ell = \delta_{j=k}(\delta_i \otimes \xi_\ell),$$

so $u^* e_{ij} u = |\delta_i\rangle\langle\delta_j| \otimes 1$ as claimed. □

**Remark 4.3.10.** Observe that if $\{p_i\}$ is a family of mutually orthogonal projections such that $\sum p_i = 1$ SOT, and $\{e_{1j}\}_{j\neq 1}$ is a family of partial isometries such that $e_{1j}e_{1j}^* = p_1$, and $e_{1j}^* e_{1j} = p_j$, then setting $e_{11} := p_1$ and $e_{ij} := e_{1i}^* e_{1j}$ for all $i, j$ with $i \neq 1$ completes $\{e_{1j}\}$ to a system of matrix units.

*Proof of Theorem 4.3.8.* Since $M$ is a type I factor, it has a minimal projection $p_1$. Let $\{p_i\}$ be a maximal family of mutually orthogonal minimal projections.

**Claim.** *$\sum p_i = 1$ SOT.*

*Proof.* Otherwise, by Corollary 4.2.9, there is a non-zero partial isometry $u \in M$ such that $uu^* \leq p_1$ and $u^*u \leq 1 - \sum p_i$, so $u^*u \perp p_i$ for all $i$. By minimality, $uu^* = p_1$, so $u^*u$ is also minimal. Then $\{p_i\} \subsetneq \{p_i\} \cup \{u^*u\}$, contradicting maximality. □

Now by Corollary 4.2.8, for each $i$, there is a non-zero partial isometry $e_{1i}$ such that $e_{1i}e_{1i}^* \leq p_1$ and $e_{1i}^* e_{1i} \leq p_i$. My minimality, we must have $e_{1i}e_{1i}^* = p_1$ and $e_{1i}^* e_{1i} = p_i$ Setting $e_{ii} := p_i$ for all $i$, we can construct a system of matrix units $\{e_{ij}\}$ as in Remark 4.3.10.

**Claim.** *$M = \{e_{ij}\}''$.*

*Proof.* If $x \in M$, then $x = \left(\sum p_i\right) x \left(\sum p_j\right) = \sum_{ij} p_i x p_j$ SOT. But by minimality, each

$$p_i x p_j = e_{1i}^* e_{1i} x e_{1j}^* e_{1j} = e_{i1} \underbrace{p_1 e_{1i} x e_{j1} p_1}_{=: \lambda_{ij} p_1 \in \mathbb{C} p_1} e_{1j} = \lambda_{ij} e_{i1} p_1 e_{1j} = \lambda_{ij} e_{ij}.$$

Hence $x = \sum_{ij} \lambda_{ij} e_{ij}$, and $M = \{e_{ij}\}''$. □

The final claim follows now from Lemma 4.3.9 □

**Definition 4.3.11.** We say a type I factor $M$ is type $\mathrm{I_n}$ if $M \cong B(H)$ with $\dim(H) = n$.

**Fact 4.3.12.** If $u, v$ are two partial isometries with $uu^* \perp vv^*$ and $u^*u \perp v^*v$, then $u^*v = 0 = uv^*$ and $u + v$ is a partial isometry.

**Corollary 4.3.13.** *Suppose $M, N$ are two type I subfactors of $B(H)$. Let $p \in M$ and $q \in N$ be minimal projections. The following are equivalent.*

*(1) There is a unitary $u \in U(H)$ such that $u^*Mu = N$.*
*(2) There are minimal $p \in P(M)$ and $q \in P(N)$ and a $u \in U(H)$ such that $u^*pu = q$.*
*(3) There are minimal $p \in P(M)$ and $q \in P(N)$ and a partial isometry $v \in B(H)$ such that $vv^* = p$ and $v^*v = q$. (Note that this $v$ is a unitary isomorphism between the multiplicity spaces $pH$ and $qH$ for $M$ and $N$ respectively.)*

*Proof.*
<u>(1) ⇒ (2):</u> If $p \in P(M)$ is minimal, then so is $u^*pu \in P(N)$.
<u>(2) ⇒ (3):</u> Take $v = pu$.
<u>(3) ⇒ (1):</u> Extend $\{p\}$ and $\{q\}$ to systems of matrix units $\{e_{ij}\}_{i,j\in I}$ for $M$ with $e_{11} = p$ and $\{f_{k,\ell}\}_{k,\ell\in K}$ for $N$ with $f_{11} = q$ respectively. Observe that for each $i \in I$ and $k \in K$,

$$(e_{i1}vf_{1k})(e_{i1}vf_{1k})^* = e_{i1}\underbrace{vqv^*}_{=p}e_{i1}^* = e_{ii} \quad \text{and} \quad (e_{i1}vf_{1k})^*(e_{i1}vf_{1k}) = f_{1k}^*\underbrace{v^*pv}_{=q}f_{1k} = f_{kk}.$$

Since $\sum e_{ii} = 1 = \sum f_{kk}$, we see that $|I| = |K|$, and we may identify the two index sets. By Fact 4.3.12, $u := \sum e_{i1}vf_{1i}$ is a unitary such that $uf_{ij}u^* = e_{ij}$ for all $i, j$. □

4.4. **Comparison of projections.**

**Definition 4.4.1.** For $p, q \in P(M)$, we say $p \preccurlyeq q$ if there is a partial isometry $u \in M$ such that $uu^* = p$ and $u^*u \leq q$. We say $p \approx q$ if there is a partial isometry $u \in M$ such that $uu^* = p$ and $u^*u = q$.

**Example 4.4.2.** For $x \in M$ and $x = u|x|$ the polar decomposition, $u \in M$ with $u^*u = \operatorname{supp}(x)$ and $uu^* = \operatorname{range}(x)$. Hence $\operatorname{supp}(x) \approx \operatorname{range}(x)$.

**Example 4.4.3.** Suppose $u$ is a partial isometry such that $uu^* = p$. Then for all $q \leq p$, $qu$ is a partial isometry such that $quu^*q = qpq = q$, so $u^*qu \approx q$.

**Exercise 4.4.4.** Show that $\approx$ is an equivalence relation on $P(M)$ up to $\approx$.

**Theorem 4.4.5.** *$\preccurlyeq$ is a partial order on $P(M)$.*

*Proof.*
<u>reflexive:</u> $p \preccurlyeq p$ via partial isometry $p$.
<u>transitive:</u> Suppose $uu^* = p$, $u^*u \leq q = vv^*$, and $v^*v \leq r$. Then

$$uvv^*u^* = uqu^* = uu^*uqu^* = uu^*uu^* = uu^* = p \qquad \text{and}$$
$$v^*u^*uv \leq v^*qv = v^*vv^*v = v^*v \leq r.$$

<u>anti-symmetric:</u> Suppose $p \preccurlyeq q$ and $q \preccurlyeq p$. Let $u, v \in M$ br partial isometries such that $uu^* = p$, $u^*u \leq q$, $vv^* = q$, and $v^*v \leq p$. Then for each $p' \leq p$,

$$u^*p'u \leq u^*pu = u^*uu^*u = u^*u \leq q,$$

and similarly, for each $q' \leq q$, $v^*q'v \leq p$. That is, we have order preserving maps

$$\{\text{projections } \leq p\} \underset{\operatorname{Ad}(v)}{\overset{\operatorname{Ad}(u)}{\rightleftarrows}} \{\text{projections } \leq q\}\ .$$

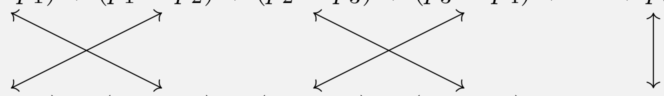
It immediately follows that inductively defining

$$p_{n+1} := v^*q_nv \qquad\qquad p_0 := p$$
$$q_{n+1} := u^*p_nu \qquad\qquad q_0 := q$$

yields two decreasing sequences of projections in $M$. Define $p_\infty := \lim^{SOT} p_n = \bigwedge p_n$ and $q_\infty := \lim^{SOT} q_n = \bigwedge q_n$, the orthogonal projections onto $\bigcap p_nH$ and $\bigcap q_nH$

respectively. The clever trick here is to write $p = p_0$ and $q = q_0$ as telescoping sums of mutually orthogonal projections, which converge SOT:

$$p = (p_0 - p_1) + (p_1 - p_2) + (p_2 - p_3) + (p_3 - p_4) + \cdots + p_\infty$$



$$q = (q_0 - q_1) + (q_1 - q_2) + (q_2 - q_3) + (q_3 - q_4) + \cdots + q_\infty$$

We then pair up projections and sum up the partial isometries with orthogonal domains and ranges.
First, since multiplication is separately SOT-continuous,

$$v^* q_\infty v = v^*(\lim{}^{SOT} q_n)v = \lim{}^{SOT} v^* q_n v = \lim{}^{SOT} p_n = p_\infty.$$

Moreover, since $q_\infty \leq q$, $q_\infty = q_\infty q q_\infty = q_\infty v v^* q_\infty$. Hence $p_\infty \approx q_\infty$ via the partial isometry $q_\infty v$. Finally, observe that

$$\begin{aligned} \operatorname{Ad}(u)(p_n - p_{n+1}) &= u^*(p_n - p_{n+1})u = u^* p_n u - u^* p_{n+1} u = q_{n+1} - q_{n+2} \\ \operatorname{Ad}(v)(q_n - q_{n+1}) &= p_{n+1} - p_{n+2}. \end{aligned}$$

Thus $(p_n - p_{n+1})u$ is a partial isometry witnessing $p_n - p_{n+1} \approx q_{n+1} - q_{n+2}$, and $(q_n - q_{n+1})v$ is a partial isometry witnessing $q_n - q_{n+1} \approx p_{n+1} - p_{n+2}$. ☐

**Corollary 4.4.6.** *If $M$ is a factor, then $\preccurlyeq$ is a total order up to $\approx$.*

*Proof.* This is a restatement of Corollary 4.2.9. ☐

**Definition 4.4.7.** A projection $p \in P(M)$ is called:

- finite if for all projections $q \leq p$, $q \approx p$ implies $q = p$.
- infinite if there is a $q \leq p$ with $q \neq p$ such that $q \approx p$ (not infinite). An infinite projection is called:
  - purely infinite if there is no non-zero finite $q \leq p$, and
  - properly infinite if for all $z \in P(Z(M))$ such that $zp \neq 0$, $zp$ is infinite.

A von Neumann algebra $M$ is called finite or (purely/properly) infinite if $1_M$ is respectively.

**Exercise 4.4.8.** Prove that abelian von Neumann algebras are finite. Deduce that $p$ abelian implies $p$ is finite.

**Definition 4.4.9.** A von Neumann algebra $M$ is called:

- type III if $M$ is purely infinite.
- type II if $M$ has no abelian projections and any non-zero central projection majorizes a non-zero finite projection. In this case, we call $M$:
  - type $\mathrm{II}_1$ if $M$ is finite, and
  - type $\mathrm{II}_\infty$ if there is no non-zero finite central projection.

**Remark 4.4.10.** The above definition is rather hard to parse, so here is another way to say it. We will informally say that a von Neumann algebra $M$ has *sufficiently many* projections with property (P) if every non-zero central projection of $M$ majorizes a non-zero projection with property (P). Then $M$ is:

- type I if $M$ has sufficiently many abelian projections,
- type II if $M$ has no abelian projections, but has sufficiently many finite projections. In this case, $M$ is:
  (1) type $\mathrm{II}_1$ if $M$ is finite and
  (2) type $\mathrm{II}_\infty$ if has no non-zero finite central projections.
- type III if $M$ has no abelian projections and no non-zero finite projections.

4.5. **$L\Gamma$ is a $\mathrm{II}_1$ factor when $\Gamma$ is icc.** Let $\Gamma$ be a countable discrete group. Recall

$$L\Gamma := \{\lambda_g | g \in \Gamma\}'' \subset B(\ell^2\Gamma) \qquad \text{where} \qquad (\lambda_g\xi)(h) := \xi(g^{-1}h).$$

The functions $\delta_g(h) := \delta_{g=h}$ give a distinguished orthonormal basis of $\ell^2\Gamma$. Observe $\lambda_g\delta_h = \delta_{gh}$. We also have a right $\Gamma$ action on $\ell^2\Gamma$ by $(\rho_g\xi)(h) := \xi(hg)$. Notice that $\rho_g \in U(\ell^2\Gamma)\cap L\Gamma'$.

**Facts 4.5.1.** We compute some basic properties about $L\Gamma$.

($L\Gamma$1) For all $x \in L\Gamma$, there a sequence $(x_g) \in \ell^2\Gamma$ such that $x\delta_e = \sum x_g\delta_g$.

($L\Gamma$2) For all $x \in L\Gamma$ and $h \in \Gamma$,

$$x\delta_h = x\rho_h\delta_e = \rho_h x\delta_e = \rho_h \sum x_g\delta_g = \sum_g x_g\delta_{gh} = \sum_g x_{gh^{-1}}\delta_g.$$

($L\Gamma$3) $x^*\delta_e = \sum \overline{x_{g^{-1}}}\delta_g$ since for all $h \in \Gamma$,

$$\langle x^*\delta_e, \delta_h\rangle = \langle \delta_e, x\delta_h\rangle \underset{(L\Gamma2)}{=} \sum \overline{x_{gh^{-1}}}\langle\delta_e, \delta_g\rangle = \overline{x_{h^{-1}}}.$$

($L\Gamma$4) If $x\delta_e = \sum x_g\delta_g$ and $y\delta_e = \sum y_g\delta_g$, then $xy\delta_e = \sum_g \left(\sum_h x_h y_{h^{-1}g}\right)\delta_g$. Thus the convolution product $(x_g) * (y_h) \in \ell^2\Gamma$.

*Proof.* For all $g \in \Gamma$,

$$\langle xy\delta_e, \delta_g\rangle = \langle y\delta_e, x^*\delta_g\rangle \underset{(L\Gamma3)}{=} \sum_{h,k} x_{h^{-1}}y_k\langle\delta_k, \rho_{g^{-1}}\delta_h\rangle = \sum_{h,k} x_{h^{-1}}y_k\delta_{k=hg},$$

which simplifies to $\sum_{h^{-1}} x_{h^{-1}}y_{hg}$. This is the claimed formula swapping $h$ with $h^{-1}$ as the index of summation. □

($L\Gamma$5) $\delta_e$ is a cylic and separating vector for $L\Gamma$.

*Proof.* Clearly $\mathbb{C}[\Gamma]\delta_e \subset L\Gamma\delta_e$ is dense in $\ell^2\Gamma$, so $\delta_e$ is cyclic. If $x \in L\Gamma$ such that $x\delta_e = 0$, then $x\delta_g = \rho_{g^{-1}}x\delta_e = 0$ for all $g$, and $x = 0$. Thus $\delta_e$ is separating. □

($L\Gamma$6) $\operatorname{tr} := \langle\,\cdot\,\delta_e, \delta_e\rangle$ is a faithful $\sigma$-WOT continuous tracial state on $L\Gamma$ with $\operatorname{tr}(x) = x_e$.

*Proof.* First, we have the tracial property as

$$\langle xy\delta_e, \delta_e\rangle \underset{(L\Gamma4)}{=} \sum_h x_h y_{h^{-1}} = \sum_h y_h x_{h^{-1}} \underset{(L\Gamma4)}{=} \langle yx\delta_e, \delta_e\rangle.$$

Next, $\operatorname{tr}(x^*x) = \sum_g |x_g|^2 = 0$ if and only if $x_g = 0$ for all $g$ if and only if $x = 0$, so tr is faithful. □

($L\Gamma$7) All projections in $L\Gamma$ are finite.

*Proof.* Suppose $uu^* = p$ and $u^*u = q \leq p$. Then $\operatorname{tr}(p-q) = \operatorname{tr}(uu^*) - \operatorname{tr}(u^*u) = 0$ which implies $p - q = 0$ as tr is faithful by ($L\Gamma$6). □

**Example 4.5.2.** If $H$ is infinite dimensional, then $B(H)$ does not admit a trace.

**Proposition 4.5.3.** *If $\Gamma$ is icc (infinite and all nontrivial conjugacy classes infinite), then $L\Gamma$ is a $\mathrm{II}_1$ factor.*

*Proof.* If $z \in Z(L\Gamma)$, then

$$\sum z_g \delta_g = z\delta_e = \lambda_{h^{-1}} z \lambda_h \delta_e = \sum z_{h^{-1}gh} \delta_g,$$

so $(z_g) \in \ell^2\Gamma$ is constant on conjugacy classes. Since $\Gamma$ is icc, $z_g = 0$ for $g \neq e$, so $z \in \mathbb{C}1$ by ($L\Gamma$6), and $L\Gamma$ is a factor.
Since $L\Gamma$ is infinite dimensional and admits a trace, it cannot be type I by Exercise 4.5.2. Since $L\Gamma$ is finite by ($L\Gamma$7) $L\Gamma$ is type $\mathrm{II}_1$. □

4.6. $\mathrm{II}_1$ **factor basics.** This subsection follows Jones' von Neumann algebra notes quite closely.

Above, we exploited the trace on $L\Gamma$ to prove Proposition 4.5.3. For this subsection, we assume a $\mathrm{II}_1$ factor comes equipped with a $\sigma$-WOT continuous tracial state. We will construct such a trace in Corollary 4.8.5 below.

**Facts 4.6.1.** Here are some elementary facts about a factor $M$ equipped with a tracial state tr, which is sometimes assumed to be faithful or $\sigma$-WOT continuous.

(tr1) A $\sigma$-WOT continuous tracial state on a factor $M$ is faithful.

*Proof.* Let $J = \{x \in M | \operatorname{tr}(x^*x) = 0\}$. Since $x^*y^*yx \leq \|y^*y\| x^*x$, $J$ is a left ideal. But since tr is a trace, $J$ is a 2-sided ideal. By Cauchy-Schwarz, $\operatorname{tr}(x^*x) = 0$ if and only if $\operatorname{tr}(xy) = 0$ for all $y$, so

$$J = \bigcap_{y \in M} \ker(\underbrace{\operatorname{tr}(\,\cdot\, y)}_{\sigma\text{-WOT cts}})$$

is $\sigma$-WOT closed. By Corollary 4.1.5, $M$ has no non-trivial $\sigma$-WOT closed 2-sided ideals, so $\ker(\operatorname{tr}) = 0$. □

(tr2) If $M$ is a factor with a faithful tracial state, then $M$ is finite.

*Proof.* The proof of ($L\Gamma$7) applies verbatim. □

(tr3) An infinite dimensional factor $M$ with a $\sigma$-WOT continuous tracial state is type $\mathrm{II}_1$.

*Proof.* The second part of the proof of Proposition 4.5.3 applies verbatim. □

(tr4) Suppose $M$ is a factor and tr is faithful.

(a) $p \precsim q$ if and only if $\operatorname{tr}(p) \leq \operatorname{tr}(q)$.
(b) $p \approx q$ if and only if $\operatorname{tr}(p) = \operatorname{tr}(q)$.

*Proof.* For the forward direction, suppose $p = uu^*$ and $u^*u \leq q$. Then

$$\operatorname{tr}(p) = \operatorname{tr}(uu^*) = \operatorname{tr}(u^*u) \leq \operatorname{tr}(q)$$

with equality if and only if $q = u^*u$ as tr is faithful.
Conversely, suppose $\operatorname{tr}(p) \leq \operatorname{tr}(q)$. Since $M$ is a factor, then $p \precsim q$ or $q \precsim p$. If $q \precsim p$, then by the forward step, $\operatorname{tr}(q) \leq \operatorname{tr}(p)$, in which case $\operatorname{tr}(p) = \operatorname{tr}(q)$ and $p = uu^*$ by faithfulness of tr. Thus $p \approx q$. □

**Lemma 4.6.2.** *Suppose $M$ is a $\mathrm{II}_1$ factor with a faithful trace. For every non-zero $p \in P(M)$ and $0 < \varepsilon < \operatorname{tr}(p)$, there is a $q \in P(M)$ with $0 \leq q \leq p$ and $0 < \operatorname{tr}(q) < \varepsilon$.*

*Proof.* Let

$$\delta := \inf \{\operatorname{tr}(q) | q \in P(M) \setminus \{0\} \text{ such that } q \leq p\}.$$

If $0 < \delta \leq \operatorname{tr}(p)$, there is a non-zero $q \in P(M)$ such that $q \leq p$ and $\operatorname{tr}(q) < 2\delta$ by the definition of the inf. Since $M$ is not type I, $q$ is not minimal, so there is a non-zero projection $r \leq q$ with $0 \neq r \neq q$. Then $\delta \leq \operatorname{tr}(r)$, but

$$\operatorname{tr}(q - r) = \operatorname{tr}(q) - \operatorname{tr}(r) \leq \operatorname{tr}(q) - \delta < 2\delta - \delta = \delta,$$

a contradiction. □

**Proposition 4.6.3.** *Suppose $M$ is a $\mathrm{II}_1$ factor with a faithful trace. Then* $\operatorname{tr}(P(M)) = [0, 1]$.

*Proof.* Fix $r \in (0, 1)$, and consider $\{p \in P(M) | 0 < \operatorname{tr}(p) \leq r\}$ which is non-empty by Lemma 4.6.2. Ordering this set by $\leq$, every ascending chain has an upper bound, so by Zorn's Lemma, there is a maximal element $p$. Suppose for contradiction that $\operatorname{tr}(p) < r$. Again by Lemma 4.6.2, there is a projection $q \leq 1 - p$ with $0 < \operatorname{tr}(q) < r - \operatorname{tr}(p)$. But then $p + q$ is a projection such that $\operatorname{tr}(p) < \operatorname{tr}(p) + \operatorname{tr}(q) < r$, a contradiction. □

**Exercise 4.6.4.** Give a better description of a projection of arbitrary trace in $[0, 1]$ in $L\mathbb{F}_2$ and $LS_\infty$.

**Exercise 4.6.5.** Let $M$ be a $\mathrm{II}_1$ factor with $\sigma$-WOT continuous tracial state tr.

(1) Show that if $p \in M$ is a non-zero projection, then for every $0 < r < \operatorname{tr}(p)$, there is a projection $q \in M$ with $q \leq p$ and $\operatorname{tr}(q) = r$.
(2) For every $n \in \mathbb{N}$, there is a unital subfactor $N \subseteq M$ with $N \cong M_n(\mathbb{C})$.
(3) $M$ is algebraically simple, i.e., $M$ has no 2-sided ideals.

**Proposition 4.6.6.** *A finite von Neumann algebra $M$ with a* faithful *$\sigma$-WOT continuous tracial state* tr *is a $\mathrm{II}_1$ factor if and only if for any other $\sigma$-WOT continuous tracial state $\varphi$, $\varphi = $* tr.

*Proof.* Suppose $M$ is a $\mathrm{II}_1$ factor. It suffices to prove both traces agree on projections. By Exercise 4.6.5(2), the traces must agree on every subfactor $N \cong M_n(\mathbb{C})$ for all

$n \in \mathbb{N}$. For an arbitrary projection $p \in M$, we can build a sequence $(p_i)$ of mutually orthogonal projections such that $p = \sum p_i$ SOT (and thus also $\sigma$-WOT) and $\operatorname{tr}(p_i) = \frac{1}{n_i}$ for some $n_i \in \mathbb{N}$ for every $i$ using Exercise 4.6.5(1).
Suppose now $M$ is not a factor, and choose projection $z \in Z(M)$ such that $0 \neq z \neq 1$. Then $\varphi(x) := \frac{1}{\operatorname{tr}(z)} \operatorname{tr}(xz)$ is a $\sigma$-WOT continuous tracial state distinct from tr as $\varphi(1-z) = 0 \neq \operatorname{tr}(1-z)$. □

4.6.1. *The hyperfinite* $\mathrm{II}_1$ *factor.* We now use Proposition 4.6.6 to construct a $\mathrm{II}_1$ factor $R$ which can be well approximated by finite dimensional subalgebras.

For $n \in \mathbb{N}$, let $A_n := \bigotimes^n M_2(\mathbb{C})$. Include $A_n \hookrightarrow A_{n+1}$ by $x \mapsto x \otimes 1$, and let $A_\infty := \varinjlim A_n = \bigotimes^\infty M_2(\mathbb{C})$. Since $A_n \cong M_{2^n}(\mathbb{C})$ has a unique normalized faithful tracial state $\operatorname{tr}_n$, $\operatorname{tr}_\infty := \varinjlim \operatorname{tr}_n$ is the unique faithful trace on $A_\infty$, and it is positive definite in that $\operatorname{tr}_\infty(x^*x) \geq 0$ for all $x \in A_\infty$ with equality if and only if $x = 0$. We can thus attempt to apply the GNS construction, where there are several things we must check along the way. We define $H$ to be the completion of $A_\infty$ in $\|\cdot\|_2$ under the sesquilinear form $\langle x, y\rangle := \operatorname{tr}_\infty(y^*x)$. We write $\Omega \in H$ for the image of $1 \in A_\infty$ and $a\Omega \in H$ for the image of $a = a1 \in A_\infty$.

(R1) $A_\infty$ acts faithfully on the left of $H$ by bounded operators by $x(a\Omega) = xa\Omega$. We can thus define $R := (A_\infty)'' \subset B(H)$.

*Proof.* Since $x^*x \leq \|x^*x\|_{A_n}$ for all $x \in A_n$, and since the inclusions $A_n \hookrightarrow A_{n+k}$ are all injective and thus norm-preserving, we have
$$\|xa\Omega\|^2 = \operatorname{tr}_\infty(a^*x^*xa) \leq \|x^*x\|_{A_n} \cdot \operatorname{tr}_\infty(a^*a) = \|x\|_{A_n}^2 \cdot \|a\Omega\|^2.$$
Faithfulness of the action follows as $\Omega$ is separating for $A_\infty$ by faithfulness of $\operatorname{tr}_\infty$ on $A_\infty$. □

(R2) $\operatorname{tr}_R(x) := \langle x\Omega, \Omega\rangle$ is a $\sigma$-WOT continuous tracial state on $R$ such that $\operatorname{tr}_R|_{A_\infty} = \operatorname{tr}_\infty$.

*Proof.* For $x \in A_\infty$, $\operatorname{tr}_R(x) = \langle x\Omega, \Omega\rangle = \operatorname{tr}_\infty(x)$. Since $\operatorname{tr}_R$ is a vector state, it is both SOT-continuous and $\sigma$-WOT continuous. For $x, y \in R$, by the Kaplansky Density Theorem, we may pick bounded nets $(x_i), (y_i) \subset A_\infty$ with $x_i \to x$ and $y_i \to y$ SOT. Since multiplication is jointly SOT-continuous on bounded sets, $x_i y_i \to xy$ and $y_i x_i \to yx$ SOT. We thus have
$$\operatorname{tr}_R(xy) = \lim{}^{\mathrm{SOT}} \operatorname{tr}_\infty(x_i y_i) = \lim{}^{\mathrm{SOT}} \operatorname{tr}_\infty(y_i x_i) = \operatorname{tr}_R(yx). \qquad \square$$

(R3) $A_\infty$ acts on the *right* of $H$ by bounded operators by $x(a\Omega) = ax\Omega$.

*Proof.* This is the step that uses that tr is a trace:
$$\begin{aligned}\|ax\Omega\|^2 &= \operatorname{tr}_\infty(x^*a^*ax) = \operatorname{tr}_\infty(axx^*a^*) \leq \|xx^*\|_{A_n} \cdot \operatorname{tr}_\infty(aa^*)\\ &= \|x^*x\|_{A_n} \cdot \operatorname{tr}_\infty(a^*a) = \|x\|_{A_n}^2 \cdot \|a\Omega\|^2.\end{aligned} \qquad \square$$

(R4) $\operatorname{tr}_R$ is faithful on $R$ so that $R$ is a $\mathrm{II}_1$ factor by Proposition 4.6.6.

*Proof.* Suppose $\operatorname{tr}_R(x^*x) = 0$. Since the right $A_\infty$-action is bounded and commutes with the left $A_\infty$-action on $H$ and thus also commutes with $R$, for all $a \in A_\infty$,

$$\|xa\Omega\|^2 = \|xR_a\Omega\|^2 = \|R_a x\Omega\|^2 \leq \|R_a\|^2 \cdot \|x\Omega\|^2 = \|R_a\|^2 \cdot \operatorname{tr}_R(x^*x) = 0.$$

Since $A_\infty\Omega$ is dense in $H$, $x = 0$. □

**Exercise 4.6.7.** Build a projection of arbitrary trace in $[0, 1]$ in $R$.

4.7. **Useful results on comparison of projections.** Our next task is to prove every finite von Neumann algebra admits a tracial state. We begin with some general results on projections in a von Neumann algebra. For this section, unless stated otherwise, $M$ is a von Neumann algebra and $p, q \in P(M)$.

**Facts 4.7.1.** Here are some basic facts about comparison of projections.

($\preccurlyeq$1) (Kaplansky's formula) $p \vee q - p \approx q - p \wedge q$.

*Proof.* Consider $x = (1-p)q$. Then $\ker(x) = \ker(q) \oplus (p \wedge q)H$, so

$$p_{\ker(x)} = (1-q) + p \vee q \qquad \text{and} \qquad \operatorname{range}(x^*) = 1 - p_{\ker(x)} = q - p \wedge q.$$

Since $x = [(1-(1-q))(1-p)]^*$, the above argument also tells us that

$$\operatorname{range}(x) = (1-p) - (1-p) \wedge (1-q) = (1 - p - (1 - p \vee q) = p \vee q - p.$$

Since $\operatorname{range}(x^*) = \operatorname{supp}(x)$, these projections are equivalent by Example 4.4.2. □

($\preccurlyeq$2) If $p_1 \preccurlyeq q_1$, $p_2 \preccurlyeq q_2$, and $q_1 q_2 = 0$, then $p_1 \vee p_1 \preccurlyeq q_1 + q_2$.

*Proof.* By ($\preccurlyeq$1), $p_1 \vee p_2 - p_2 \approx p_1 - p_1 \wedge p_2 \preccurlyeq q_1$ so $p_1 \vee p_2 = (p_1 \vee p_2 - p_2) + p_2 \preccurlyeq q_1 + q_2$. □

($\preccurlyeq$3) (Comparison Theorem) There is a $z \in P(Z(M))$ such that $pz \preccurlyeq qz$ and $q(1-z) \preccurlyeq p(1-z)$.

*Proof.* By Zorn's Lemma, there are maximal families of mutually orthogonal projections $\{p_i\}, \{q_i\}$ such that $\sum p_i \leq p$, $\sum q_i \leq q$, and $p_i \approx q_i$ for all $i$. Set $z_1 := z\left(p - \sum p_i\right)$ and $z_2 := z\left(q - \sum q_i\right)$. By maximailty, $z_1 z_2 = 0$, so

$$\left(p - \sum p_i\right) \leq z_1 \leq 1 - z_2 \qquad \Longrightarrow \qquad z_2\left(p - \sum p_i\right) = 0$$
$$\left(q - \sum q_i\right) \leq z_2 \qquad \Longrightarrow \qquad (1-z_2)\left(q - \sum q_i\right) = 0.$$

Since $\sum p_i \approx \sum q_i$, we see

$$z_2 p = z_2 \sum p_i \approx z_2 \sum q_i \leq z_2 q$$
$$(1-z_2)q = (1-z_2)\sum q_i \approx (1-z_2)\sum p_i \leq (1-z_2)p.$$ □

($\precsim$4) If $p, q$ are finite, so is $p \vee q$.

We omit the proof, which is quite techinical. There is a much simpler proof when $p, q$ are central in addition, which you will do on homework.

($\precsim$5) If $p, q$ are finite and $p \approx q$, then $1-p \approx 1-q$. Hence there is a $u \in U(M)$ such that $u^*pu = q$.

**Remark 4.7.2.** The proof below only uses ($\precsim$4) to reduce to the case that $M$ is finite. Since we will only use ($\precsim$5) for finite von Neumann algebras, the rest of these notes is still self-contained without a proof of ($\precsim$4) above.

*Proof.* By ($\precsim$4), $p \vee q$ is finite, so replacing $M$ by $(p \vee q)M(p \vee q)$, we may assume $M$ is finite. By ($\precsim$3), there is a central projection $z \in P(Z(M))$ such that $(1-p)z \precsim (1-q)z$ and $(1-q)(1-z) \precsim (1-p)(1-z)$. Since we can consider $Mz$ and $M(1-z)$ separately, we may assume $1-p \approx r \leq 1-q$. Since $1 = (1-p)+p \approx r+q$, and $M$ is finite, $r+q = 1$, so $1-p \approx r = 1-q$. Now if $vv^* = p$, $v^*v = q$ and $ww^* = 1-p$, $w^*w = 1-q$, then $u = v+w$ is a unitary satisfing $u^*pu = q$. □

($\precsim$6) Suppose $p, q \in P(M)$ finite with $p, q \leq r$.

($\precsim$6a) If $p \approx q$, then $r-p \approx r-q$.

($\precsim$6b) If $p \precsim q$, then $r-q \precsim r-p$.

**Remark 4.7.3.** Again, in the proof below, we will only use ($\precsim$4) to pass to the case $M$ is finite and $r = 1$.

*Proof.* Since $p, q \leq r$ implies $p \vee q \leq r$, passing to $(p \vee q)M(p \vee q)$, we may assume $M$ is finite and $r = 1$ by ($\precsim$4). Now ($\precsim$6a) follows immediately from ($\precsim$5). For ($\precsim$6b), let $s \in P(M)$ with $p \approx s \leq q$. By ($\precsim$5) $1-p \approx 1-s \geq 1-q$. □

($\precsim$7) If $(q_n)$ is an inrcreasing sequence of finite projections and $p \in P(M)$ such that $q_n \precsim p$ for all $n$, then $\bigvee q_n \precsim p$.

*Proof.* We inductively define a sequence of mutually orthogonal projections $p_n \leq p$ such that $p_0 = q_1$ and for all $n \in \mathbb{N}$, $p_n \approx q_{n+1} - q_n$. Then

$$\bigvee_{n=1}^{\infty} q_n = q_1 + \sum_{n=1}^{\infty}(q_{n+1} - q_n) \approx \sum_{0}^{\infty} p_n \leq p.$$

By assumption, $q_1 \precsim p$, so there is a $p_0 \leq p$ such that $q_1 \approx p_0$. Suppose we have $p_0, p_1, \ldots, p_n$.

**Claim.** $q_{n+2} - q_{n+1} \precsim p - \sum_{i=0}^{n} p_i$.

*Proof of Claim.* Observe $q_{n+2} \precsim p$, so there is a partial isometry $v$ such that $vv^* = q_{n+2}$ and $e_{n+2} := v^*v \leq p$. Since $q_{n+2} \geq q_{n+1}$,

$$e_{n+1} := v^*q_{n+1}v \leq v^*q_{n+2}v = v^*vv^*v = v^*v \leq p$$

and $e_{n+1} \approx q_{n+1}$. Then

$$v^*(q_{n+2}-q_{n+1})v = e_{n+2}-e_{n+1} \quad \text{and} \quad (q_{n+2}-q_{n+1})vv^*(q_{n+2}-q_{n+1}) = q_{n+2}-q_{n+1},$$

so $q_{n+2} - q_{n+1} \approx e_{n+2} - e_{n+1}$. By the induction hypothesis,

$$e_{n+1} \approx q_{n+1} = (q_{n+1} - q_n) + (q_n - q_{n-1}) + \cdots + (q_2 - q_1) + q_1 \approx \sum_{i=0}^{n} p_i \leq p.$$

Since $q_{n+2}, q_{n+1}$ are finite, so are $e_{n+2}, e_{n+1} \approx \sum_{i=0}^{n} p_i$. We calculate

$$q_{n+2} - q_{n+1} \approx e_{n+2} - e_{n+1} = (p - e_{n+1}) - (p - e_{n+1}) \leq p - e_{n+1} \underset{(\preccurlyeq 6b)}{\approx} p - \sum_{i=0}^{n} p_i,$$

proving the claim. □

By the claim, we can find a projection $q_{n+2} - q_{n+1} \approx p_{n+1} \leq p - \sum_{i=0}^{n} p_i$, so we can inductively build the sequence as claimed. □

($\preccurlyeq$8) Suppose $M$ is a finite von Nuemann algebra and $(p_n)$ is an infinite sequence of mutually orthogonal projections. Suppose $(q_n)$ is another sequence of projections with $p_n \approx q_n$ for each $n$. Then $q_n \to 0$ SOT.

*Proof.* By induction using ($\preccurlyeq$2), for all $m \leq n$,

$$\bigvee_{i=m}^{n} q_i \preccurlyeq \sum_{i=m}^{n} p_i \leq \sum_{i \geq m} p_i.$$

Since $\bigvee_{i=m}^{n} q_i$ is increasing in $n$, $\bigvee_{i \geq m} q_i \preccurlyeq \sum_{i \geq m} p_i$ by ($\preccurlyeq$7). Let $p_0 = 1 - \sum_{i=0}^{\infty} p_i$. By ($\preccurlyeq$6b),

$$p_0 + \sum_{i=1}^{m-1} p_i = 1 - \sum_{i \geq m} p_i \preccurlyeq 1 - \bigvee_{i \geq m} q_i \leq 1 - \bigwedge_{m=1}^{\infty} \bigvee_{i \geq m} q_i.$$

Again by ($\preccurlyeq$7), we can conclude that

$$1 = p_0 + \sum_{i=1}^{\infty} p_i \preccurlyeq 1 - \bigwedge_{m=1}^{\infty} \bigvee_{i \geq m} q_i.$$

Since $M$ is finite, we must have

$$0 = \bigwedge_{m=1}^{\infty} \underbrace{\bigvee_{i \geq m} q_i}_{\text{decreasing}} = \text{SOT} - \lim \underbrace{\bigvee_{i \geq m} q_i}_{\geq q_m}.$$

Hence for all $\xi \in H$,

$$\|q_m\|^2 = \langle q_m \xi, \xi \rangle \leq \left\langle \bigvee_{i \geq m} q_i \xi, \xi \right\rangle = \left\| \bigvee_{i \geq m} q_i \xi \right\|^2 \xrightarrow{m \to \infty} 0,$$

and thus $q_m \to 0$ SOT. □

4.8. **Existence of a trace on a finite von Neumann algebra.** For this section, $M$ is a finite von Neumann algebra. Recall that the $\sigma$-WOT on $M$ is the weak* topology induced by $M_*$. Thus we may identify $M_*$ with the $\sigma$-WOT continuous linear functionals on $M$.

**Definition 4.8.1.** Let $S(M) \subset M_*$ be the set of $\sigma$-WOT continuous states of $M$. Note that $U(M)$ acts on $S(M)$ by $u \cdot \varphi := \varphi(u^* \cdot u)$.

**Lemma 4.8.2.** *Let $M$ be a von Neumann algebra and $\varphi \in M^*$ a state. The following are equivalent.*

*(1) $\varphi$ is tracial, i.e., $\varphi(xy) = \varphi(yx)$ for all $x, y \in M$.*
*(2) For all $x \in M$, $\varphi(xx^*) = \varphi(x^*x)$.*
*(3) For all $u \in U(M)$, $\varphi(u^*xu) = \varphi(x)$.*

*Proof.*
$(1) \Rightarrow (2):$ Obvious.
$(2) \Rightarrow (3):$ For $x \geq 0$, $\varphi(u^*xu) = \varphi(u^*x^{1/2}x^{1/2}u) = \varphi(x^{1/2}uu^*x^{1/2}) = \varphi(x)$. Now use that every $x \in M$ is a linear combination of 4 positive operators.
$(3) \Rightarrow (1):$ Replacing $x$ with $ux$, we have $\varphi(xu) = \varphi(ux)$ for all $x \in M$ and $u \in U(M)$. Now use that every $y \in M$ is a linear combination of 4 unitaries. □

So to construct a trace in $S(M)$ for $M$ finite, we will find a fixed point in $S(M)$ under the $U(M)$-action. To do this, we will use the Ryll-Nardzewski Fixed Point Theorem. Our approach here follows the proof of Jacob Lurie.

**Theorem 4.8.3** (Ryll-Nardzewski)**.** *Let $X$ be a Banach space and $K \subset X$ a weakly compact convex subset. Suppose $G \subset B(X)$ is a group of isometries with $GK \subseteq K$. Then there is an $x \in K$ such that $gx = x$ for all $g \in G$.*

For $u \in U(M)$, we define $\pi_u \in B(M_*)$ by $\pi_u\varphi := \varphi(u^* \cdot u)$. Hence for our purposes, $G = \pi(U(M)) \subset B(M_*)$.

The following theorem is the main result of this section.

**Theorem 4.8.4.** *Suppose $M$ is a finite von Neumann algebra and fix $\varphi \in S(M)$. Define*

$$K_0 := \pi(U(M))\varphi = \{\varphi(u^* \cdot u) | u \in U(M)\} \subset S(M),$$

*and let $K$ be the weakly closed convex hull of $K_0$ in $M_*$. Then $K$ is weakly compact.*

Before proving this theorem, observe that combining it with the Ryll-Nardzewski Fixed Point Theorem 4.8.3 yields the desired result.

**Corollary 4.8.5.** *There exists a $\sigma$-WOT continuous tracial state on a finite von Neumann algebra.*

*Proof.* Let $\varphi \in S(M)$. By Theorem 4.8.4, the weakly closed convex hull $K \subset S(M)$ of $\pi(U(M))\varphi$ is weakly compact. As $K$ is clearly $\pi(U(M))$-invariant, by the Ryll-Nardzewski Fixed Point Theorem 4.8.3, there is a $\pi(U(M))$-fixed point $\mathrm{tr} \in K \subset S(M)$, which is a tracial state by Lemma 4.8.2. □

**Lemma 4.8.6.** *For a positive linear functional $\varphi \in M^*$, the following are equivalent.*

*(1) $\varphi$ is $\sigma$-WOT continuous.*
*(2) $\varphi$ is normal: for all increasing nets of positive operators $x_i \nearrow x$ in $M$, $\varphi(x_i) \nearrow \varphi(x)$.*
*(3) $\varphi$ is completely additive: for every family $(p_i)$ of mutually orthogonal projections in $M$, $\varphi\left(\sum p_i\right) = \sum \varphi(p_i)$.*

*Proof.* Homework. □

**Remark 4.8.7.** Suppose $(p_i)$ is a family of mutually orthogonal projections in $M$. For all positive $\varphi \in M^*$, and for all finite subsets $F \subset I$, $\sum_{i\in F} \varphi(p_i) = \varphi\left(\sum_{i\in F} p_i\right) \le \varphi\left(\sum p_i\right)$, so $\sum \varphi(p_i) \le \varphi\left(\sum p_i\right)$. Hence $\varphi$ is completely additive if and only if for every family of mutually orthogonal projections $(p_i)$ in $M$, for all $\varepsilon > 0$, there is a finite $F \subset I$ such that $\varphi\left(\sum_{i\notin F} p_i\right) \le \varepsilon$. Indeed,

$$\sum \varphi(p_i) = \sup_{F\subset I} \sum_{i\in F} \varphi(p_i) = \sup_{F\subset I} \varphi\left(\sum_{i\in F} p_i\right) = \sup_{F\subset I} \varphi\left(\sum p_i\right) - \varphi\left(\sum_{i\notin F} p_i\right)$$
$$= \varphi\left(\sum p_i\right) - \inf_{F\subset I} \varphi\left(\sum_{i\notin F} p_i\right).$$

*Proof of Theorem 4.8.4.* Recall that the relative weak* topology on $X \subseteq X^{**}$ is the weak topology. To show $K \subset M_*$ is weakly compact, by the Banach-Alaoglu Theorem, it suffices to prove $K \subseteq M_*^{**} = M^*$ is weak* closed, as $K \subseteq (M^*)_1$ which is weak* compact.
Let $\psi \in \overline{K}$, the weak* closure of $K$ in $M^*$. We show $\psi$ is completely additive, and thus $\psi \in M_*$, so $\psi \in K$. Suppose for contradiction that $\psi$ is not completely additive. Then there is a family $(p_i)_{i\in I}$ of mutually orthogonal projections and an $\varepsilon > 0$ such that for all finite $F \subset I$, $\psi\left(\sum_{i\notin F} p_i\right) > \varepsilon$.

**Claim.** *If $F \subset I$ is any finite set, there is a $\phi \in K_0$ and a finite set $G \subset I \setminus F$ such that $\phi(\sum_{i\in G} p_i) > \varepsilon$.*

*Proof.* The convex hull $\operatorname{conv}(K_0)$ is weakly dense in $K$, which is weak* dense in $\overline{K}$, so $\operatorname{conv}(K_0)$ is weak* dense in $\overline{K}$. Thus for all $\delta > 0$, the weak* open neighborhood

$$\left\{\phi \in M^* \,\middle|\, \left|(\psi - \phi)\left(\sum_{i\notin F} p_i\right)\right| < \delta\right\}$$

of $\psi$ has non-empty intersection with $\operatorname{conv}(K_0)$, so pick $\phi$ in this intersection. Since $\psi(\sum_{i\notin F} p_i) > \varepsilon$, choosing $\delta$ small, we have $\phi(\sum_{i\notin F} p_i) > \varepsilon$. Now if $\phi = \sum_{k=1}^n \lambda_k \phi_k$ is a convex combination of $\phi_k \in K_0$, there must be a particular $k$ so that $\phi_k(\sum_{i\notin F} p_i) > \varepsilon$. Now since $\phi_k$ is completely additive, there is a finite $G \subset I \setminus F$ such that $\phi_k(\sum_{i\in G} p_i) > \varepsilon$. □

**Claim.** *There is a sequence* $(F_n)$ *of disjoint finite subsets of* $I$ *and a sequence of states* $(\phi_n) \subset K_0$ *such that for all* $n \in \mathbb{N}$,

$$\phi_n\left(\sum_{i\in F_n} p_i\right) > \varepsilon.$$

*Proof.* We induct on $n$. Since $\psi(\sum p_i) > \varepsilon$, by the first claim, there is a $\phi_1 \in K_0$ and a finite set $F_1 \subset I$ such that $\phi_1(\sum_{i\in F_1} p_i) > \varepsilon$. Now suppose we have disjoint sets $F_1, \ldots, F_n \subset I$ and states $\phi_1, \ldots, \phi_n \in K_0$ such that $\phi_k(\sum_{i\in F_k} p_i) > \varepsilon$ for all $k = 1, \ldots, n$. Since $\psi$ is not completely additive,

$$\psi\left(\sum_{i\notin \coprod_{j=1}^n F_j} p_i\right) > \varepsilon,$$

so again by the first claim, there is a $\phi_{n+1} \in K_0$ and a set $F_{n+1} \subset I \setminus \coprod_{j=1}^n F_j$ such that $\phi_{n+1}(\sum_{i\in F_{n+1}} p_i) > \varepsilon$. □

Now by the above claim, for each $\phi_n \in K_0$, there is a unitary $u_n \in U(M)$ such that $\phi_n = \varphi(u_n^* \cdot u_n)$. Moreover, setting $q_n := \sum_{i\in F_n} p_i$, we have a sequence $(q_n)$ of mutually orthogonal projections such that $\varphi(u_n^* q_n u_n) > \varepsilon$ for all $n$. We now have our desired contradiction. Since the $F_n$ are disjoint, the $q_n$ are mutually orthogonal. Since $u_n^* q_n u_n \approx q_n$ for all $n$, $u_n^* q_n u_n \to 0$ SOT (and thus also $\sigma$-WOT) by ($\preccurlyeq$8). But $\varphi \in S(M)$ is $\sigma$-WOT continuous and $\varphi(u_n^* q_n u_n) > \varepsilon$ for all $n$, a contradiction. □

4.9. **The proof of Ryll-Nardzewski.** In this section, we prove the Ryll-Nardzewski Fixed Point Theorem 4.8.3 following Lurie's proof.
https://www.math.ias.edu/~lurie/261ynotes/lecture26.pdf.
We begin by restating (a version of) the Ryll-Nardzewski Fix Point Theorem.

**Theorem** (Ryll-Nardzewski, Theorem 4.8.3)**.** *Let* $X$ *be a Banach space and* $K \subset X$ *a weakly compact convex subset. Suppose* $G \subset B(X)$ *is a group of isometries with* $GK \subseteq K$*. Then there is an* $x \in K$ *such that* $gx = x$ *for all* $g \in G$*.*

**Remark 4.9.1.** Without loss of generality, we may assume $G$ is finitely generated. Indeed, write $G = \bigcup G_i$ where each $G_i$ is finitely generated. Then $K^G = \bigcap K^{G_i}$. By compactness of $K$ and the finite intersection property, $\bigcap K^{G_i} \neq \emptyset$ for all $i$ implies $K^G \neq \emptyset$.

Fix a Banach space $X$ and a weakly compact convex subset $K \subset X$. We begin with the following warmup.

**Lemma 4.9.2.** *Suppose* $T \in B(X)$ *such that* $TK \subseteq K$*. There is an* $x \in K$ *such that* $Tx = x$*.*

*Proof.* For $n \in \mathbb{N}$, let $T_n := \frac{1}{n}\sum_{k=0}^{n-1} T^k$ and $K_n = T_n K \subseteq K$ as $K$ is convex. We claim that $\{K_n\}$ has the finite intersection property. Indeed,

$$K_{n_1} \cap \cdots \cap K_{n_k} \supseteq T_{n_1} \cdots T_{n_k} K$$

as $T_m T_n = T_n T_m$ for all $m, n$.

Now let $x \in \bigcap K_n \neq \emptyset$. For each $n \in \mathbb{N}$, there is a $y \in K$ such that $x = T_n y$, so

$$Tx - x = (T-1)T_n y = \frac{1}{n}(T-1)\sum_{k=0}^{n-1} T^k y = \frac{1}{n}(T^n y - y) \in \frac{1}{n}(K-K).$$

Since $K$ is weakly compact, so is $K-K$, and in particular, $K-K$ is bounded.[a] Thus for every open neighborhood $U$ of $K-K$, there is an $n \in \mathbb{N}$ such that $\frac{1}{n}(K-K) \subset U$. But this means $Tx - x \in U$ for every open neighborhood $U$ of 0, so $Tx = x$. □

[a] If $S \subset X \subseteq X^{**}$ is weakly compact, then each $s \in S$ is pointwise bounded as a map on $X^*$ by compactness. Now apply the Uniform Boundedness Principle.

The strategy of the proof will be to take our finitely generated group $G = \langle g_1, \ldots, g_n \rangle \subseteq B(X)$ of isometries and find a candidate fixed point $x \in K$ for $G$ using Lemma 4.9.2. We will prove by contradiction that this candidate $x \in K$ satisfies $g_i x = x$ for each generator. The next lemma is the second main ingredient to achieve our contradiction.

**Lemma 4.9.3.** *Suppose $g_1, \ldots g_k \in B(X)$ are isometries and $x \in X$ such that $g_i(x) \neq x$ for all $i = 1, \ldots, k$. Let $C$ be the weak closed convex hull of $\langle g_1, \ldots, g_k \rangle x$, which is weakly compact. Let $\varepsilon > 0$ such that $\|g_i(x) - x\| > \varepsilon$ for all $i = 1, \ldots, n$. Then there is a weakly compact subset $C \subsetneq C$ such that* $\operatorname{diam}(C \setminus C') \leq \varepsilon$.

Assuming this lemma, we can now prove Theorem 4.8.3.

*Proof of Theorem 4.8.3.* Set $T = \frac{1}{n}\sum g_i \in B(X)$. By the warmup Lemma 4.9.2, there is an $x \in K$ such that $Tx = x$. If $g_i(x) = x$ for all $i$, we have our fixed point proving Theorem 4.8.3. Otherwise, relabelling the $g_i$, there is a $1 \leq k \leq n$ such that $g_i(x) \neq x$ for all $i = 1, \ldots, k$ and $g_i(x) = x$ for all $i = k+1, \ldots, n$. Then

$$x = Tx = \frac{1}{n}\sum_{i=1}^{n} g_i(x) = \frac{1}{n}\sum_{i=1}^{k} g_i(x) + \frac{n-k}{n}x,$$

which immediately implies that

$$x = \frac{1}{k}\sum_{k=1}^{k} g_i(x).$$

By Lemma 4.9.3, there is a weakly compact convex subset $C' \subsetneq C = \langle g_1, \ldots, g_k \rangle x \subseteq K$ such that $\operatorname{diam}(C - C') \leq \varepsilon$. Since $C' \neq C$, there is an $h \in G$ such that $hx \notin C'$, so

$$hx = hTx = \frac{1}{k}\sum_{i=1}^{k} hg_i(x) \notin C'.$$

Since $C'$ is convex, there must be some $1 \leq i \leq k$ such that $hg_i(x) \notin C'$, so both $hx, hg_i(x) \notin C'$. But since $h$ is an isometry, we have

$$\|x - g_i(x)\| = \|hx - hg_i(x)\| \leq \operatorname{diam}(C - C') \leq \varepsilon,$$

a contradiction. □

We now prove the lemma.

*Proof of Lemma 4.9.3.* To prove the lemma, it suffices to work in the closure of

$$\operatorname{span}\{g_{i_1}\cdots g_{i_m}x | m\in\mathbb{N} \text{ and } 1\le i_1,\dots,i_m\le k\},$$

which is a *separable* Banach space.

Let $E=\partial_{\text{ext}}C\subseteq C$ be the set of extreme points. By the Krein-Milman Theorem, $C$ is the weak closed convex hull of $E$. Let $\overline{E}\subseteq C$ be the weak closure of $E$, and let $B=\overline{B_{\varepsilon/3}(0)}$ be the closed ball of radius $\varepsilon/3$. Since $B$ is convex and norm closed, $B$ is also weakly closed as the norm and weak topology have the same closed convex sets. Since $X$ is separable, there is a sequence $(y_j)\subset X$ such that $(y_j+B)$ covers $X$. Thus $((y_j+B)\cap\overline{E})$ is a cover of the weakly compact set $\overline{E}$. By the Baire Category Theorem, there is a $j$ such that $(y_j+B)\cap\overline{E}$ has non-empty interior $U$ in $\overline{E}$ with respect to the relative weak topology on $\overline{E}$.

Now define

$$C_1 := \text{weak closed convex hull of } \overline{E}\setminus U$$

$$C_2 := \text{weak closed convex hull of } (y_j+B)\cap\overline{E},$$

which are both weakly closed convex subsets of $C$. Since $C$ is the closed convex hull of

$$E\subseteq(\overline{E}\setminus U)\cup((y_j+B)\cap\overline{E}),$$

$E$ is the *convex join* of $C_1$ and $C_2$, i.e., $C=\operatorname{im}(\theta)$ for

$$\theta: C_1\times C_2\times[0,1]\to X \qquad \text{given by} \qquad (a,b,t)\mapsto ta+(1-t)b.$$

We now consider the sets $C(\delta):=\operatorname{im}(\theta|_{C_1\times C_2\times[\delta,1]})$.

Step 1: Each $C(\delta)$ is a weakly closed convex subset of $C$.

Closed: Since $\theta$ is continuous from the (weak,weak,standard) product topology to the weak topology as $X$ with the weak topology is a topological vector space, $K(\delta)$ is weakly compact, and thus closed.

Convex: First, note that for all $0<\delta\le 1$, $\delta C_1+(1-\delta)C_2$ is convex. We claim that

$$\theta(C_1\times C_2\times[\delta,1])=\theta(C_1\times(\delta C_1+(1-\delta)C_2)\times[0,1]),$$

which is manifestly convex.

$\subseteq$: If $t\in[\delta,1]$, $ta+(1-t)b=sa+(1-s)(\delta a+(1-\delta b))$ for $s\in[0,1]$ such that $(1-s)(1-\delta)=(1-t)$. This condition is equivalent to $t=\delta+s(1-\delta)$.

$\supseteq$: If $s\in[0,1]$, then $sa_1+(1-s)[\delta a_2+(1-\delta)b]=ta+(1-t)b$ for $t=s+(1-s)\delta=\delta+s(1-\delta)\in[\delta,1]$ as before and

$$a=\frac{sa_1+(1-s)\delta a_2}{s+(1-s)\delta}\in C_1.$$

Step 2: For $\delta>0$ sufficiently small, $\operatorname{diam}(C\setminus C(\delta))\le\varepsilon$.

Since $C$ is weakly compact, it is bounded, so $C \subset B_R(0)$ for some $R > 0$. If $y, y' \in C \setminus C(\delta)$, then there are $0 \leq t, t' < \delta$, $a, a' \in C_1$, and $b, b' \in C_2$ such that

$$y = ta + (1-t)b \qquad \text{and} \qquad y' = t'a' + (1-t')b'.$$

Then

$$\begin{aligned} \|y - y'\| &= \|t(a-b) + b - t'(a'-b') - b'\| \\ &\leq t(\|a\| + \|b\|) + t'(\|a'\| + \|b'\|) + \|\underbrace{b - b'}_{b,b' \in C_2}\| \\ &\leq 4\delta R + \frac{2}{3}\varepsilon \end{aligned}$$

as $b, b' \in C_2 \subset y_j + B$ which has diameter $2/3 \cdot \varepsilon$. Now choose $\delta < \frac{\varepsilon}{12R}$.

Step 3: For $\delta$ as in Step 2 above, $C(\delta) \neq C$.

Since $U \subseteq \overline{E}$ is a non-empty open subset, there is a $y \in E \cap U$. We claim that $y \notin C(\delta)$. Since $y \in E$ is an extreme point of $C$, it suffices to prove $y \notin C_1$. (Indeed, if $y \notin C_1$ and $y = ta + (1-t)b$ for $a \in C_1$ and $b \in C_2$, since $y$ is extreme, $y = a = b$. But since $a \in C_1$ and $y \notin C_1$, we must have $t = 0$. Thus $y$ cannot be written as $ta + (1-t)b$ for $a \in C_1$, $b \in C_2$, and $t \in [\delta, 1]$.) Since $X$ with the weak topology is locally convex, there is a weakly open convex neighborhood $V$ of 0 such that the weak closure $\overline{V}$ satisfies $(y - \overline{V}) \cap \overline{E} \subseteq U$. (Indeed, we can use here that $\overline{E}$ is weakly compact and thus weakly normal.)

Now since $\overline{E} \setminus U$ is weakly compact, it admits a weakly open cover $\{z_i + V\}_{i=1}^k$ where each $z_i \in \overline{E} \setminus U$. Thus $C_1$ is contained in the closed convex hull of

$$\bigcup_{i=1}^{k} (z_i + V) \cap \overline{E} \supseteq \overline{E} \setminus U.$$

In turn, $\bigcup_{i=1}^{k} (z_i + V) \cap \overline{E}$ is contained in the convex join of the $(z_i + \overline{V}) \cap C$. If $y \in C_1$, then $y \in (z_i + \overline{V}) \cap C$ for some $i$. But then $z_i \in (y - \overline{V}) \cap \overline{E} \subseteq U$, a contradiction to $z_i \in \overline{E} \setminus U$.

Thus if $\delta > 0$ is sufficiently small, we can take $C' = C(\delta) \subsetneq C$. □