1. Banach algebras

1.1. **Spectrum.** Let A be a unital Banach algebra. The spectrum of $a \in A$ is

$$\operatorname{sp}(a) = \left\{\lambda \in \mathbb{C} \middle| a - \lambda 1 \notin A^\times\right\},$$

which is a non-empty compact subset of $B_{r(a)}(0)$. Here, $r(a)$ is the spectral radius:

$$r(a) = \lim \|a^n\|^{1/n}.$$

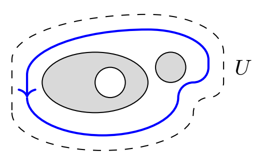
Fact 1.1.1. Suppose $\phi : A \to B$ is a unital algebra map between Banach algebras. If $a \in A^\times$, then $\phi(a) \in B^\times$, so $\operatorname{sp}_B(\phi(a)) \subseteq \operatorname{sp}_A(a)$.

Corollary 1.1.2. *Suppose $1 \in A \subset B$ is a unital inclusion of Banach algebras. For all $a \in A$, $\operatorname{sp}_B(a) \subseteq \operatorname{sp}_A(a)$ and $\partial \operatorname{sp}_A(a) \subseteq \partial \operatorname{sp}_B(a)$.*

Proof. By Fact 1.1.1, $\operatorname{sp}_B(\phi(a)) \subseteq \operatorname{sp}_A(a)$, so it suffices to prove $\partial \operatorname{sp}_A(a) \cap \operatorname{sp}_B(a)^c = \emptyset$. Suppose for contradiction that $\lambda \in \partial \operatorname{sp}_A(a) \cap \operatorname{sp}_B(a)^c$. Pick a sequence $(\lambda_n) \subset \operatorname{sp}_A(a)^c$ such that $\lambda_n \to \lambda$, so $a - \lambda_n \to a - \lambda$. Then $a - \lambda_n \in A^\times$, so $a - \lambda_n \in B^\times$, and thus $\lambda_n \notin \operatorname{sp}_B(a)$ for all n. Since we assumed $\lambda \notin \operatorname{sp}_B(a)$ and inversion is continuous on $B^\times$, we have $(a - \lambda_n)^{-1} \to (a - \lambda)^{-1} \in B$. But A is complete, so $(a - \lambda)^{-1} \in A$, a contradiction. □

1.2. **Holomorphic functional calculus.** For each $a \in A$, the holomorphic functional calculus (HFC) gives a unital algebra homomorphism $\mathcal{O}(\operatorname{sp}(a)) \to A$ given by

$$f \longmapsto f(a) := \frac{1}{2\pi i} \int_\gamma \frac{f(z)}{a - z}\, dz$$



where γ is a simple closed contour in $U \setminus \operatorname{sp}(a)$ such that

$$\operatorname{ind}_\gamma(z) = \begin{cases} 1 & \text{if } z \in \operatorname{sp}(a) \\ 0 & \text{if } z \notin U. \end{cases}$$

The HFC satisfies the following two properties, which characterize this ring homomorphism:

- If $\operatorname{sp}(a) \subset U$, and $f_n \to f$ locally uniformly on U, then $f_n(a) \to f(a)$ in A, and
- If $f(z) = \sum \alpha_k z^k$ is a power series with radius of convergence greater than $r(a)$, then $f(a) = \sum \alpha_k a^k$.

The HFC also satisfies:

(1) If $f(z) = \prod (z - z_j)^{m_j}$ is rational, then $f(a) = \prod (a - z_j)^{m_j}$.
(2) (spectral mapping) $\operatorname{sp}(f(a)) = f(\operatorname{sp}(a))$, and
(3) if $g \in \mathcal{O}(\operatorname{sp}(a))$, then $g(f(a)) = (g \circ f)(a)$.

Corollary 1.2.1. *If $\phi : A \to B$ and $f \in \mathcal{O}(\operatorname{sp}_A(a))$, then $f(\phi(a)) = \phi(f(a))$.*

Proof. By Fact 1.1.1, $\operatorname{sp}_B(\phi(a)) \subseteq \operatorname{sp}_A(a)$, so $\mathcal{O}(\operatorname{sp}_A(a)) \subseteq \mathcal{O}(\operatorname{sp}_B(\phi(a)))$. Observe that $f(\phi(a)) = \phi(f(a))$ whenever f is a polynomial, and whenever f is a rational function with poles outside of $\operatorname{sp}_A(a)$. The result now follows by Runge's Theorem, since every $f \in \mathcal{O}(\operatorname{sp}_A(a))$ can be approximated by such rational functions. □

1.3. **Gelfand transform.** If A is unital and commutative, the Gelfand transform gives a norm-contractive unital algebra homomorphism $A \to C(\widehat{A})$ given by

$$a \mapsto [\operatorname{ev}_a \colon \varphi \mapsto \varphi(a)],$$

where $\widehat{A}$ is the set of algebra homomorphisms from $A \to \mathbb{C}$, also called characters or multiplicative linear functionals. The image of the Gelfand transform is a subalgebra of $C(\widehat{A})$ which separates points of $\widehat{A}$.

Lemma 1.3.1. *If A is unital and $a \in A$, then for all $\varphi \in \widehat{A}$, $\varphi(a) \in \operatorname{sp}(a)$.*

Proof. Observe $\varphi\big(a - \varphi(a)\big) = 0$, so $a - \varphi(a) \notin A^\times$ and thus $\varphi(a) \in \operatorname{sp}(a)$. □

2. C*- algebras

Let A be a unital C*-algebra, i.e., a unital Banach algebra with an involution satisfying $\|a^*a\| = \|a\|^2$ for all $a \in A$.

2.1. **Operators.** We call $a \in A$:

- self-adjoint if $a = a^*$,
- positive if $a = b^*b$ for some $b \in A$,
- normal if $aa^* = a^*a$,
- a projection if $a = a^* = a^2$,
- an isometry if $a^*a = 1$,
- a unitary if $a^*a = 1 = aa^*$ (equivalently, an invertible isometry),
- a partial isometry if a^*a is a projection.

Here are some elementary properties:

(C*1) Each a can be written as $a = \operatorname{Re}(a) + i\operatorname{Im}(a)$ where $\operatorname{Re}(a) = \frac{a+a^*}{2}$ and $\operatorname{Im}(a) = \frac{a-a^*}{2i}$ are self-adjoint.

(C*2) If $\lambda \in \operatorname{sp}(a)$, then $\overline{\lambda} \in \operatorname{sp}(a^*)$.

(C*3) If a is normal, then $\|a\| = r(a)$.

Proof. Observe $\|a^2\|^2 = \|(a^2)^*a^2\| = \|(a^*a)^2\| = \|a^*a\|^2 = \|a\|^4$. Thus $r(a) = \lim \|a^{2^n}\|^{2^{-n}} = \|a\|$. □

(C*4) If u is unitary, then $\operatorname{sp}(u) \subset \partial\mathbb{D} = \mathbb{T} = S^1$.

Proof. Since $u^* = u^{-1}$, by (C*2), $\lambda \in \operatorname{sp}(u)$ if and only if $\overline{\lambda}^{-1} \in \operatorname{sp}(u)$. Since $\|u\| = 1$, both $|\lambda|, |\lambda^{-1}| \leq 1$, so $\lambda \in \mathbb{T}$. □

(C*5) If $a = a^*$, then e^{ia} is unitary (defined by the HFC).

Proof. Observe $(e^{ia})^* = \left(\sum \frac{(ia)^n}{n!}\right)^* = \sum \frac{(-ia)^n}{n!} = e^{-ia} = (e^{ia})^{-1}$. □

(C*6) If $a = a^*$, then $\mathrm{sp}(a) \subset \mathbb{R}$.

Proof. By (C*4), $\mathrm{sp}(e^{ia}) \subset \mathbb{T}$, and by the Spectral Mapping Theorem, $\mathrm{sp}(e^{ia}) = e^{i\,\mathrm{sp}(a)}$. Hence $\mathrm{sp}(a) \subset \mathbb{R}$. □

2.2. **Continuous functional calculus.**

Lemma 2.2.1. *If A is commutative, then every $\varphi \in \widehat{A}$ is a $*$-homomorphism.*

Proof. Let $a \in A$ and $\varphi \in \widehat{A}$. Recall from (C*1) that $a = \mathrm{Re}(a) + i\,\mathrm{Im}(a)$ where $\mathrm{Re}(a), \mathrm{Im}(a)$ are self-adjoint. From Lemma 1.3.1 and (C*6) we see that $\varphi(\mathrm{Re}(a)) \in \mathrm{sp}(\mathrm{Re}(a)) \subset \mathbb{R}$ and $\varphi(\mathrm{Im}(a)) \in \mathrm{sp}(\mathrm{Im}(a)) \subset \mathbb{R}$. Thus

$$\varphi(a^*) = \varphi\big(\mathrm{Re}(a)\big) - i\varphi\big(\mathrm{Im}(a)\big) = \overline{\varphi\big(\mathrm{Re}(a)\big) + i\varphi\big(\mathrm{Im}(a)\big)} = \overline{\varphi(a)}.$$ □

Theorem 2.2.2. *The Gelfand transform affords an equivalence of categories*

$$\{\text{Unital commutative C}^*\text{-algebras}\} \quad \cong \quad \{\text{Compact Hausdorff spaces}\}^{\mathrm{op}}.$$

Question 2.2.3. *What happens for non-unital* C*-*algebras?*

Lemma 2.2.4 (Spectral permanence)**.** *Suppose $1 \in A \subset B$ is a unital inclusion of* C*-*algebras. Then $\mathrm{sp}_A(a) = \mathrm{sp}_B(a)$ for all $a \in A$.*

Proof. By Corllary 1.1.2, $\mathrm{sp}_B(a) \subseteq \mathrm{sp}_A(a)$, so it suffices to prove $b \in A \cap B^\times$ implies $b \in A^\times$. Suppose $b \in A \cap B^\times$. Then $b^* \in A \cap B^\times$ and $b^*b \in A \cap B^\times$. By (C*6), $\mathrm{sp}_A(b^*b), \mathrm{sp}_B(b^*b) \subset \mathbb{R}$. By Corollary 1.1.2,

$$\mathrm{sp}_A(b^*b) = \partial\,\mathrm{sp}_A(b^*b) \subseteq \partial\,\mathrm{sp}_B(b^*b) = \mathrm{sp}_B(b^*b) \subseteq \mathrm{sp}_A(b^*b),$$

so equality holds. Notice that this shows that b admits a left inverse in A, since

$$(b^*b)^{-1}b^*b = 1.$$

A similar argument for bb^* shows b has a right inverse, and the result follows. □

Given $a \in A$ normal, the continuous functional calculus (CFC) is a unital $*$-isomorphism from $\Phi_a\colon C(\mathrm{sp}(a)) \to \mathrm{C}^*(a)$, the smallest unital C*-subalgebra of A containing a, which extends the HFC. It is characterized by the properties:

- $\Phi_a(1) = 1$ and $\Phi_a(\mathrm{id}\colon z \mapsto z) = a$, and
- for all $f \in \mathcal{O}(\mathrm{sp}(a))$, $\Phi_a(f) = f(a)$ from the HFC.

Thus it makes sense to denote $\Phi_a(f) = f(a)$.

Exercise 2.2.5. Show that every a in a unital C*-algebra A is a linear combination of 4 unitaries.
Hint: Show every self-adjoint a with $\|a\| \leq 1$ is a linear combination of 2 unitaries by considering $f(t) := t + i\sqrt{1-t^2}$ on $\operatorname{sp}(a)$.

Exercise 2.2.6. Suppose V is an inner product space (not necessarily complete) and $\pi : A \to \operatorname{End}(V)$ is a unital $*$-homomorphism such that $\langle \pi(a)u, v\rangle = \langle u, \pi(a)^*v\rangle$ for all $u, v \in V$. Prove that π induces a unital $*$-homomorphism $\overline{\pi} : A \to B(\overline{V})$.
Hint: First show for every unitary $u \in A$, $\|\pi(u)\| = 1$.

Lemma 2.2.7. *If $\phi : A \to B$, $a \in A$ is normal, and $f \in C(\operatorname{sp}_A(a))$, then $f(\phi(a)) = \phi(f(a))$.*

Proof. By Fact 1.1.1, $\operatorname{sp}_B(\phi(a)) \subseteq \operatorname{sp}_A(a)$, so there is a canonical surjection $C(\operatorname{sp}_A(a)) \twoheadrightarrow C(\operatorname{sp}_B(\phi(a)))$. Observe that $f(\phi(a)) = \phi(f(a))$ whenever f is a polynomial in z and $\overline{z}$. The result now follows by the Stone-Weierstrass Theorem. □

Proposition 2.2.8. *Every $*$-homomorphism $\phi\colon A \to B$ of unital C*-algebras is norm-contractive. If ϕ is injective, then*

(1) $\operatorname{sp}_B(\phi(a)) = \operatorname{sp}_A(a)$ for all normal $a \in A$, and
(2) $\|\phi(a)\| = \|a\|$ for all $a \in A$.

Proof. Since $a \in A^\times$ implies $\phi(a) \in B^\times$, we have $\operatorname{sp}_B(\phi(a)) \subseteq \operatorname{sp}_A(a)$, and thus $r(\phi(a)) \leq r(a)$ for all $a \in A$. Then

$$\|\phi(a)\|^2 = \|\phi(a)^*\phi(a)\| = \|\phi(a^*a)\| \underset{(C^*3)}{=} r(\phi(a^*a)) \leq r(a^*a) \underset{(C^*3)}{=} \|a^*a\| = \|a\|^2.$$

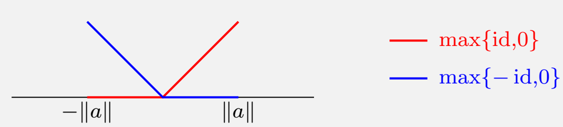
(1) Suppose $\lambda \in \operatorname{sp}_A(a) \setminus \operatorname{sp}_B(\phi(a))$ for some normal $a \in A$. We will show ϕ is not injective. Since $\operatorname{sp}_A(a)$ is compact Hausdorff, it is normal. By Urysohn's Lemma, there is a continuous $f\colon \operatorname{sp}_A(a) \to [0,1]$ such that $f|_{\operatorname{sp}_B(\phi(a))} = 0$ and $f(\lambda) = 1$. Then $f(a) \neq 0$, but by Lemma 2.2.7, $\phi(f(a)) = f(\phi(a)) = 0$, so ϕ is not injective.
(2) This follows by (1), (C*3), and the C*-identity. □

2.3. **Positivity.** Let A be a unital C*-algebra. Recall that $a \in A$ is called positive, denoted $a \geq 0$, if $a = b^*b$ for some $b \in A$. We write $a \geq b$ if $a - b \geq 0$. You will show some of the following facts in the homework.

Facts 2.3.1.

($\geq$1) If $a = a^*$, there are positive a_+ and a_- in $C^*(a)$ such that $a = a_+ - a_-$ and $a_+a_- = 0$.

Proof. Use the CFC to set $a_+ := \max\{\operatorname{id}, 0\}(a)$ and $a_- := \max\{-\operatorname{id}, 0\}(a)$. □



(≥2) If $a = a^*$, then $a \le \|a\|$.

> *Proof.* Observe that the absolute value function dominates the identity function on $\mathbb{R}$, and apply the CFC. □

(≥3) If $a \le b$, then for all $c \in A$, $c^*ac \le c^*bc$.

> *Proof.* Write $b - a = d^*d$, and observe $c^*bc - c^*ac = c^*(b-a)c = c^*d^*dc$. □

(≥4) $a \ge 0$ if and only if $a = a^*$ and $\mathrm{sp}(a) \subset [0, \infty)$.

> *Proof.* Homework. □

(≥5) The set A_+ of positive elements is a closed cone.

> *Proof.* Homework. □

(≥6) $\le$ is a partial order on A.

> *Proof.* Clearly $a \le a$.
> If $a \le b$ and $b \le a$, then $b - a \ge 0$ and $a - b = -(b-a) \ge 0$. Thus $b - a$ is self-adjoint and $\mathrm{sp}(b-a) = \{0\}$. By the CFC, $b - a = 0$, so $a = b$.
> Finally, if $a \le b$ and $b \le c$, then $b - a \ge 0$ and $c - b \ge 0$, so $c - a = (c-b) + (b-a) \ge 0$ by (≥5). □

(≥7) If $0 \le a \le b$, then $\|a\| \le \|b\|$.

> *Proof.* By (≥2), $0 \le a \le b \le \|b\|$, so by (≥6), $a \le \|b\|$. Using the CFC for a, $\mathrm{sp}(a) \subseteq [0, \|b\|]$, so $\|a\| \le \|b\|$ by (C*3). □

Definition 2.3.2. A linear functional φ on A is called positive if $\varphi(a) \ge 0$ whenever $a \ge 0$.
A state is a positive linear functional such that $\varphi(1) = 1$.

Example 2.3.3. If $\|\xi\| = 1$, $\omega_\xi(a) \coloneqq \langle a\xi, \xi\rangle$ is a state on $B(H)$.

Example 2.3.4. The unital $*$-algebra $\mathbb{C} \oplus \mathbb{C}$ with $(\alpha, \beta)^* = (\overline{\beta}, \overline{\alpha})$ has no states; its only positive linear functional is zero.

> *Proof.* The positive elements of $A \coloneqq \mathbb{C} \oplus \mathbb{C}$ are of the form $(\overline{\beta}\alpha, \overline{\alpha}\beta)$ for $\alpha, \beta \in \mathbb{C}$. Choosing $\alpha = i$ and $\beta = -i$, we see $(-1,-1)$ is positive. But choosing $\alpha = \beta = 1$, we see $(1,1)$ is positive. This means for any positive linear functional φ, we have $\pm\varphi(1,1) \ge 0$, so $\varphi(1,1) = 0$. □

Lemma 2.3.5. *If φ is positive, then $\varphi(a) \in \mathbb{R}$ whenever $a = a^*$. Moreover, for all $a \in A$, $\varphi(a^*) = \overline{\varphi(a)}$.*

Proof. If $a = a^*$, then writing $a = a_+ - a_-$ as in (≥1), we see $\varphi(a) = \varphi(a_+) - \varphi(a_-) \in \mathbb{R}$. For arbitrary $a \in A$, we have

$$\varphi(a^*) = \varphi(\mathrm{Re}(a)) - i\varphi(\mathrm{Im}(a)) = \overline{\varphi(\mathrm{Re}(a)) - i\varphi(\mathrm{Im}(a))} = \overline{\varphi(a)}. \qquad \square$$

2.4. **Representations of complex $*$-algebras and the GNS construction.** A representation of a (unital) complex $*$-algebra is a pair (H, π) where H is a Hilbert space and $\pi \colon A \to B(H)$ is a (unital) $*$-homomorphism. We call (H, π):

- nondegenerate if $\{\pi(a)\xi | a \in A \text{ and } \xi \in H\}$ is dense in H. Observe that unital representations are nondegenerate.
- cyclic if there is a vector $\Omega \in H$ such that $\pi(A)\Omega$ is dense in H. We call Ω a cyclic vector and (H, π, Ω) a cyclic representation.

Example 2.4.1. The complex $*$-algebra $C(X)$ acts on $L^2(X, \mu)$, where μ is any regular finite Borel measure.

Example 2.4.2. Let Γ be a discrete group. Then Γ acts on $\ell^2\Gamma$ by $(\lambda_g \xi)(h) := \xi(g^{-1}h)$. Since λ_g is isometric and has inverse $\lambda_{g^{-1}}$, it is unitary. We thus get a group homomorphism $\lambda \colon \Gamma \to U(H)$, the unitary group of H. Extending by linearity, we get a unital $*$-homomorphism $\mathbb{C}[\Gamma] \to B(\ell^2\Gamma)$, where $\mathbb{C}[\Gamma]$ is the group algebra of Γ. The reduced group C*-algebra of Γ is the C*-algebra $C^*_r(\Gamma)$ generated by $\{\lambda_g | g \in \Gamma\}$.

Given a positive linear functional φ on A, define $\langle a, b\rangle_\varphi := \varphi(b^*a)$, which is a positive sesquilinear form on A. Observe that all positive sesquilinear forms satisfy the Cauchy-Schwarz inequality, which is a powerful tool.

Proposition 2.4.3. *Suppose A is a unital Banach $*$-algebra ($*$ is an isometric involution) and φ is a positive linear functional.*

(1) If $a = a^*$ and $\|a\| < 1$, there is a $b \in A$ with $b = b^*$ such that $b^2 = 1 - a$,
(2) $\varphi(a^*a) \leq \|a^*a\|\varphi(1)$ for all $a \in A$, and
(3) $\|\varphi\| = \varphi(1)$.

Proof.

(1) The function $\sqrt{1-z}$ is analytic on $B_1(0) \supset \mathrm{sp}(a)$. setting $b := \sqrt{1-a}$, we have $b^2 = 1 - a$. To see b is self-adjoint, observe $\sqrt{1-a}$ is a uniform limit of polynomials in a on $\mathrm{sp}(a)$. (Indeed, we can find an open U such that $\mathrm{sp}(a) \subset U \subset \overline{U} \subset B_1(0)$.)

(2) Let $\varepsilon > 0$. Applying (1) to $\frac{a^*a}{\|a^*a\|+\varepsilon}$, we have a $b = b^*$ such that $b^2 = 1 - \frac{a^*a}{\|a^*a\|+\varepsilon}$. Thus

$$0 \leq \varphi(b^*b) = \varphi(1) - \frac{\varphi(a^*a)}{\|a^*a\| + \varepsilon} \qquad \Longrightarrow \qquad \varphi(a^*a) \leq (\|a^*a\| + \varepsilon)\varphi(1).$$

Since $\varepsilon > 0$ was arbitrary, the result follows.

(3) Take square roots in the inequality

$$|\varphi(a)|^2 = |\langle a, 1\rangle_\varphi|^2 \underset{\mathrm{CS}}{\leq} \langle 1, 1\rangle_\varphi \langle a, a\rangle_\varphi = \varphi(1)\varphi(a^*a) \underset{(2)}{\leq} \varphi(1)^2\|a^*a\| \leq \varphi(1)^2\|a\|^2,$$

and observe the bound $\varphi(1)$ is achieved at $1 \in A$. $\square$

Proposition 2.4.4. *Suppose A is a unital* C**-algebra and φ is a linear functional. Then φ is positive if and only if φ is bounded and $\|\varphi\| = \varphi(1)$.*

Proof. Positivity implies $\|\varphi\| = \varphi(1)$ by Proposition 2.4.3. Conversely, suppose φ is bounded with $\|\varphi\| = \varphi(1)$. By normalizing φ, we may assume $\|\varphi\| = \varphi(1) = 1$. It remains to show that $\varphi(a) \geq 0$ whenever $a \geq 0$. By the CFC, it suffices to prove this for a positive linear functional on $C(X)$ where X is compact Hausdorff. Suppose $\varphi(f) = \alpha + i\beta$ for $f = \overline{f}$. Then for all $t \in \mathbb{R}$, we have

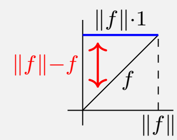
$$|\varphi(f+it)|^2 = |\alpha + i(\beta+t)|^2 = \alpha^2 + \beta^2 + 2\beta t + t^2, \qquad \text{but}$$
$$|\varphi(f+it)|^2 \leq \|f+it\|^2 = (\|f\|^2 + t^2).$$

This implies

$$\alpha^2 + \beta^2 + 2\beta t \leq \|f\|^2 \qquad \forall t \in \mathbb{R},$$

which is only possible if $\beta = 0$. Now, if $f \geq 0$,

$$|\varphi(f) - \|f\|| = |\varphi(f - \|f\| \cdot 1)| \leq \|f - \|f\| \cdot 1\| \leq \|f\|$$



which implies $\varphi(f) \geq 0$. $\square$

Definition 2.4.5. A state on a normed unital $*$-algebra is a continuous positive linear functional such that $\varphi(1) = 1$.

Corollary 2.4.6. *If A is a normed unital $*$-algebra and φ is positive and continuous, then $\|\varphi\| = \varphi(1)$.*

Proof. Let $\overline{A}$ be the completion of A, which is a normed unital Banach $*$-algebra. Since φ is bounded, it extends to $\overline{A}$ by Hahn-Banach. If $a \in \overline{A}$, choose a sequence (a_n) such that $a_n \to a$. Then (a_n) is norm-bounded, and $a_n^* a_n \to a^* a$. Thus the extension of φ to $\overline{A}$ is positive. Now apply Proposition 2.4.3. $\square$

The left kernel of the form is given by

$$N_\varphi := \{a \in A | \varphi(a^*a) = \langle a, a\rangle_\varphi = 0\} \underset{\text{(CS)}}{=} \{a \in A | \langle a, b\rangle_\varphi = 0 \ \forall\, b \in A\},$$

which is a left ideal of A. Thus the left regular action $L_a \colon A/N_\varphi \to A/N_\varphi$ given by $L_a(b + N_\varphi) := ab + N_\varphi$ is well-defined.

Exercise 2.4.7. Prove the assertion that N_φ is a left ideal of A.

Question 2.4.8. *When is the left regular action of A on A/N_φ bounded?*

Proposition 2.4.9. *If A is a unital normed $*$-algebra and φ is a continuous positive linear functional, then the left regular action of A on A/N_φ is bounded with $\|L_a\| \leq \|a\|$.*

Proof. Since the left regular action preserves N_φ, it suffices to prove that the left regular action of A on itself is bounded with $\|L_a\| \leq \|a\|$. For $b \in A$, define $\varphi_b(a) := \varphi(b^*ab)$, which is a continuous positive linear functional on A. By extending φ_b to $\overline{A}$ as in the proof of Corollary 2.4.6, we see that $\varphi_b(a^*a) \leq \|a^*a\|\varphi_b(1)$ for all $a \in A$ by Proposition 2.4.3 applied to $\overline{A}$. Thus

$$\|ab\|_\varphi^2 = \varphi(b^*a^*ab) = \varphi_b(a^*a) \leq \|a^*a\|\varphi_b(1) = \|a^*a\|\varphi(b^*b) = \|a^*a\|\|b\|_\varphi^2 \leq \|a\|^2\|b\|_\varphi^2.$$

The result follows. □

Define $H_\varphi := \overline{A/N_\varphi}$, which is called the GNS Hilbert space with respect to φ. Observe that the image of $1 \in A$ in H_φ, denoted Ω_φ, is a cyclic vector for the representation (H_φ, π_φ), i.e., $\pi_\varphi(A)\Omega_\varphi$ is dense in H_φ. Observe that $\varphi(a) = \langle a\Omega_\varphi, \Omega_\varphi\rangle$ for all $a \in A$, so φ is a vector state in the GNS representation.

Question 2.4.10. *When does A act on the right of H_φ by bounded operators? That is, consider the map R_a on A given by $b \mapsto ba$. When does this pass to A/N_φ? And when is it bounded?*

Exercise 2.4.11. Consider the linear functional tr on $\mathbb{C}[\Gamma]$ given by $\operatorname{tr}(\sum c_g g) := c_e$.

(1) Show that tr is positive and continuous. Here, the norm on $\mathbb{C}[\Gamma]$ is the operator norm coming from its left regular action on $\ell^2\Gamma$.
Hint: Show that $\operatorname{tr} = \langle \cdot\, \delta_e, \delta_e\rangle$, *where* $\delta_e \in \ell^2\Gamma$ *is given by* $\delta_e(g) = \delta_{g=e}$.
(2) Prove that $\operatorname{tr}(xy) = \operatorname{tr}(yx)$ for all $x, y \in \mathbb{C}[\Gamma]$.
(3) Find a unitary isomorphism $H_{\operatorname{tr}} \to \ell^2\Gamma$ which intertwines the left regular action of $\mathbb{C}[\Gamma]$ on H_{tr} with the left action $\lambda\colon \mathbb{C}[\Gamma] \to B(\ell^2\Gamma)$ from Example 2.4.2.

If (H_i) is a family of Hilbert spaces, the direct sum $\bigoplus H_i$ is the completion of the algebraic direct sum under the inner product $\langle \eta, \xi\rangle := \sum_i \langle \eta_i, \xi_i\rangle$. One can show that elements of $\bigoplus H_i$ are square-summable sequences of vectors.

Definition 2.4.12. If (H_i, π_i) is a family of representations of a unital C*-algebra A, then $\bigoplus H_i$ carries an action of A via $\bigoplus \pi_i$ defined by $(\bigoplus \pi_i)(a)_j := \pi_j(a)$. Observe $\bigoplus \pi_i(a)$ is bounded if and only if $(\|\pi_i(a)\|)$ is uniformly bounded.

Definition 2.4.13. The universal representation of a unital C*-algebra A is $\displaystyle\bigoplus_{\text{states } \varphi} L^2(A, \varphi)$, which is a direct sum of cyclic representations.

Lemma 2.4.14. *Suppose $1 \in A \subset B$ is a unital inclusion of C*-algebras. Then any state on A extends to a state on B.*

Proof. Use Hahn-Banach to extend the state φ on A to $\widetilde{\varphi}$ on B, and note

$$\widetilde{\varphi}(1) = \varphi(1) \underset{\text{(Prop. 2.4.3)}}{=} \|\varphi\| \underset{\text{(HB)}}{=} \|\widetilde{\varphi}\|.$$

So $\widetilde{\varphi}$ is positive by Proposition 2.4.3. □

Proposition 2.4.15. *Suppose A is a unital C^*-algebra and $a \in A$ is self-adjoint (or normal). For every $\lambda \in \mathrm{sp}(a)$, there is a state φ on A such that $\varphi(a) = \lambda$.*

Proof. Recall $C^*(a) \cong C(\mathrm{sp}(a))$ where a corresponds to the identity function. Use Lemma 2.4.14 to extend $\mathrm{ev}_\lambda \colon C(\mathrm{sp}(a)) \to \mathbb{C}$ (which is manifestly positive) to a state φ on A. Since $\mathrm{ev}_\lambda(\mathrm{id}) = \lambda$, $\varphi(a) = \lambda$. □

Theorem 2.4.16 (Gelfand-Naimark)**.** *The universal representation of a unital C^*-algebra is isometric. Thus every C^*-algebra is $*$-isomorphic to a closed $*$-subalgebra of bounded operators on a Hilbert space.*

Proof. Let $a \in A$. Then $\|a\|^2 \geq \|\psi(a)\|^2 = \|\psi(a^*a)\| \geq \|\pi_\psi(a^*a)\|$ for all states ψ. By Proposition 2.4.15, there is a state $\varphi \in A^*$ such that $\|a^*a\| = \varphi(a^*a)$, as $\|a^*a\| \in \mathrm{sp}(a^*a)$. We then have that

$$\|a\|^2 = \|a^*a\| = \varphi(a^*a) = \langle \pi_\varphi(a^*a)\Omega_\varphi, \Omega_\varphi\rangle_\varphi$$

Since the norm is equal to the numerical radius for normal operators, we have $\|\pi_\varphi(a^*a)\| \geq \|a\|^2$. We thus have that

$$\|a\|^2 \leq \|\pi_\varphi(a^*a)\| \leq \|\pi(a)\|^2 \leq \|a\|^2.$$

We conclude that π is isometric. □