TODO: lead in

5.1. **2-categories.**

Definition 5.1.1. A *2-category* $\mathcal{C}$ consists of

- A collection of *objects*, a.k.a. *0-morphisms*; we write $c \in \mathcal{C}$ to denote c is an object of $\mathcal{C}$;
- For each $a, b \in \mathcal{C}$, a hom category $\mathcal{C}(a \to b)$. Objects of $\mathcal{C}(a \to b)$ are called *1-morphisms*. We write ${}_aX_b \in \mathcal{C}(a \to b)$ or $X : a \to b$ to denote that X is a 1-morphism from a to b. Morphisms in $\mathcal{C}(a \to b)$ are called *2-morphisms*. We write $f \in \mathcal{C}({}_aX_b \Rightarrow {}_aY_b)$ or $f : X \Rightarrow Y$ to denote that f is a 2-morphism from X to Y.
- For each $a, b, c \in \mathcal{C}$, a *1-composition* functor. Based on the 2-category, the typical convention for this 1-composition may vary between *left-to-right* and *right-to-left*. We use two different notations depending on our choice of convention, so that the reader may infer the direction of composition directly from the notation.
$$\otimes = \otimes_b : \mathcal{C}(a \to b) \times \mathcal{C}(b \to c) \to \mathcal{C}(a \to c) \qquad \text{(left-to-right)}$$
$$\circ = \circ_b : \mathcal{C}(b \to c) \times \mathcal{C}(a \to b) \to \mathcal{C}(a \to c) \qquad \text{(right-to-left)}$$
We will typically choose the first, which is usually used for algebras, bimodules, and intertwiners, and we will use the second for 2-categories of categories, functors, and natural transformations.
This functor necessarily satisfies the *exchange relation*
$$(f \otimes \mathrm{id}_Z) \circ (\mathrm{id}_W \otimes g) = (\mathrm{id}_X \otimes g) \circ (f \otimes \mathrm{id}_Y) \qquad \forall\, f \in \mathcal{C}({}_aW_b \to {}_aX_b),\ \forall\, g \in \mathcal{C}({}_bY_c \to {}_bZ_c).$$
- For each ${}_aX_b, {}_bY_c, {}_cZ_d$, an *associator* isomorphism
$$\alpha_{X,Y,Z} : X \otimes_b (Y \otimes_c Z) \Rightarrow (X \otimes_b Y) \otimes_c Z.$$
These associator isomorphisms must be *natural* in each variable and satisfy the obvious *pentagon axiom*.
- For each $c \in \mathcal{C}$, there is a *unit* 1-morphism $1_c \in \mathcal{C}(c \to c)$, along with *unitor* isomorphisms $\rho^c_Y : Y \otimes_c 1_c \Rightarrow Y$ for all $Y \in \mathcal{C}(b \to c)$ for all $b \in \mathcal{C}$, and $\lambda^c_Z : 1_c \otimes_c Z \Rightarrow Z$ for all $Z \in \mathcal{C}(c \to d)$ for all $d \in \mathcal{C}$. Again, these unitors must be *natural* in each variable and satisfy the obvious *triangle axiom*.

A 2-category is called *strict* if all associators and unitors are identity 2-morphisms. A 2-category is called *linear* if all 2-morphism spaces $\mathcal{C}({}_aX_b \Rightarrow {}_aY_b)$ are finite dimensional complex vector spaces, and all composition functors are bilinear.

Warning 5.1.2. Sometimes in the literature, 2-category means *strict* 2-category, and the fully weak notion is called a *bicategory*. However, 2-category is clearly the better name, and the fully weak notion is clearly the better notion, so the better notion should get the better name. We can then add adjectives for more strict notions.

Example 5.1.3. The 2-category Cat, the strict 2-category of categories, functors, and natural transformations. In these notes, we will usually write Cat for the strict 2-category of linear categories and linear functors.

Example 5.1.4. The 2-category $2\mathsf{Vec}$ is the strict 2-category of finite semisimple categories, linear functors, and natural transformations.

Example 5.1.5. There is a 2-category MonCat of monoidal categories, monoidal functors, and monoidal natural transformations.

Example 5.1.6. There is a 2-category of topological spaces, continuous maps, and homotopy classes of homotopies between continuous maps.

Example 5.1.7. There is a 2-category Alg of complex algebras, bimodules, and intertwiners. Usually, we will write Alg for the 2-category of semisimple finite dimensional complex algebras.

Example 5.1.8. Given a monoidal category $\mathcal{C}$, we get a 2-category $\mathrm{B}\mathcal{C}$ with exactly one object $*$ called the *delooping* of $\mathcal{C}$. We simply define the hom category $\mathrm{B}\mathcal{C}(* \to *) := \mathcal{C}$ with the obvious 1-composition functor, associator, unit, and unitors.

Conversely, given a 2-category $\mathcal{C}$, picking any object $c \in \mathcal{C}$, the *loop space* $\Omega_c\mathcal{C} := \mathrm{End}_{\mathcal{C}}(c)$ is a monoidal category with the obvious tensor product functor, associator, unit 1_c, and unitors.

Corollary 5.1.9. *If $\mathcal{C}$ is a 2-category, the monoid $\mathrm{End}_{\mathcal{C}}(1_{\mathcal{C}})$ is always commutative.*

Definition 5.1.10. Suppose $\mathcal{C}$ is a 2-category and $a, b \in \mathcal{C}$. The *linking category* is the monoidal category

$$\mathcal{L}_\otimes(a,b) := \begin{pmatrix} \mathcal{C}(a \to a) & \mathcal{C}(a \to b) \\ \mathcal{C}(b \to a) & \mathcal{C}(b \to b) \end{pmatrix} \qquad\qquad \mathcal{L}_\circ(a,b) := \begin{pmatrix} \mathcal{C}(a \to a) & \mathcal{C}(b \to a) \\ \mathcal{C}(a \to b) & \mathcal{C}(b \to b) \end{pmatrix}$$

whose objects are formal matrices of objects and whose morphisms are formal matrices of morphisms between objects respectively. When 1-composition is left-to-right, the first definition must be used, and when 1-composition is right-to-left, the second definition must be used. The unit is the formal matrix

$$\begin{pmatrix} 1_a & \\ & 1_b \end{pmatrix},$$

and we leave it to the reader to write down formulas for the associator and unitors. Similarly, we can define the *n-fold linking category* $\mathcal{L}(a_1, \ldots, a_n)$ for any $a_1, \ldots, a_n \in \mathcal{C}$.

Definition 5.1.11. A 2-category $\mathcal{C}$ is called *linear* if every 2-morphism space $\mathcal{C}({}_aX_b \Rightarrow {}_aY_b)$ is a finite dimensional vector space, i.e., an object in Vec.

Generally speaking, if a certain property 'P' holds for every hom 1-category of $\mathcal{C}$, we call $\mathcal{C}$ *locally 'P.'* For example, we call a linear 2-category *locally Cauchy complete* (or *locally (finitely) semisimple*) if all hom 1-categories are Cauchy complete (respectively (finitely) semisimple).

Construction 5.1.12. Suppose $\mathcal{C}$ is a linear 2-category. We define a new locally Cauchy complete linear 2-category called $¢^1(\mathcal{C})$ by replacing all hom 1-categories by their Cauchy completions. In more detail, $¢^1(\mathcal{C})$ has the same objects as $\mathcal{C}$, and we define the hom 1-categories by

$$¢^1(\mathcal{C})(a \to b) := ¢(\mathcal{C}(a \to b)).$$

To define the composition functor, first observe that we may view 1-composition as a linear functor on the Delinge product (as opposed to a bilinear functor on the product)

$$\mathcal{C}(a \to b) \boxtimes \mathcal{C}(b \to c) \xrightarrow{-\otimes-} \mathcal{C}(a \to c).$$

Now observe that we have an equivalence of categories from the Deligne product of Cauchy complete categories (which is the Cauchy completion of the usual Deligne product) to the Cauchy completion of the Deligne product of linear categories

$$¢^1(\mathcal{C})(a \to b) \boxtimes ¢^1(\mathcal{C})(b \to c) \cong ¢(\mathcal{C}(a \to b) \boxtimes \mathcal{C}(b \to c)).$$

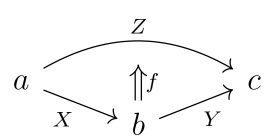
By the universal property of Cauchy completion, 1-composition thus extends to $¢^1(\mathcal{C})$:

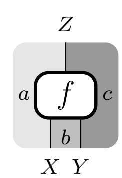
$$\begin{array}{ccccc} ¢(\mathcal{C}(a \to b) \boxtimes \mathcal{C}(b \to c)) & & \xrightarrow{\quad\cong\quad} & & ¢^1(\mathcal{C})(a \to b) \boxtimes ¢^1(\mathcal{C})(b \to c) \\ \uparrow & & & & \downarrow{\scriptstyle \exists ¢(-\otimes-)} \\ \mathcal{C}(a \to b) \boxtimes \mathcal{C}(b \to c) & \xrightarrow{-\otimes-} & \mathcal{C}(a \to c) & \hookrightarrow & ¢(\mathcal{C}(a \to c)). \end{array}$$

The associators and unitors for $¢^1(\mathcal{C})$ are defined similarly to those built for $\mathsf{Add}(\mathcal{C})$ and $\mathsf{Idem}(\mathcal{C})$ [[]], as $¢ = \mathsf{Idem} \circ \mathsf{Add}$ on linear categories.

Exercise 5.1.13. Define analogous notions of $\mathsf{Add}^1(\mathcal{C})$ for a linear 2-category which replaces each hom 1-category with its additive envelope and $\mathsf{Idem}^1(\mathcal{C})$ for a 2-category which replaces each hom 1-category with its idempotent completion.

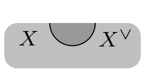
5.2. **Graphical calculus for 2-categories and adjoints.** Similar to monoidal categories, 2-categories admit a graphical calculus of string diagrams which are dual to pasting diagrams. In a pasting diagram, one represents objects as vertices, 1-morphisms as arrows, and 2-morphisms as 2-cells. In the string diagram calculus, we represent objects by shaded regions, 1-morphisms by (oriented) strands between these regions, and 2-morphisms by coupons.

$f : {}_aX \otimes_b Y_c \Rightarrow {}_aZ_c \quad \rightsquigarrow$ 

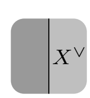
$\rightsquigarrow$ 

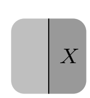
As before, we suppress all associators and unitors; 1-composition is denoted by horizontal juxtaposition, and 2-composition is denoted by stacking of diagrams.

As in a monoidal category, we can define the notion of dual and predual for 1-morphisms. If $a, b \in \mathcal{C}$ and ${}_aX_b$ is a 1-morphism, a dual is a 1-morphism ${}_bX^\vee_a$ together with maps

$\mathrm{coev}_X =$ 

and $\epsilon =$ 

where 

$= X^\vee$, 

$= X$

satisfying the zig-zag/snake equations. There is a similar notion of predual. However, as we continue our exploration into higher categories, we will reserve the word 'dual' for objects, and we will use the term *adjoint* for a dual of a 1-morphism. This term is borrowed from the 2-category Cat of categories.

Definition 5.2.1. Suppose $\mathcal{C}, \mathcal{D}$ are categories and $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors. We say that F is *left adjoint* to G, equivalently G is *right adjoint* to F, denoted $F \dashv G$, if there is a family of isomorphisms

$$\mathcal{D}(F(c) \to d) \cong \mathcal{C}(c \to G(d)). \tag{5.2.2}$$

which is natural in $c \in \mathcal{C}$ and $d \in \mathcal{D}$.

We will show in Proposition 5.2.8 below that the unit and counit witness that G is the dual of F in Cat, and F is the predual of G. (Since 1-composition in Cat is left-to-right, the convention for duals switches!)

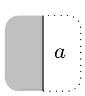
Example 5.2.3. There are many free/forget adjoints across mathematics. Examples include:

- Forget : $\mathsf{Vec} \to \mathsf{Set}$ which forgets the vector space structure and Free : $\mathsf{Set} \to \mathsf{Vec}$ defined by $S \mapsto \mathbb{C}[S]$
- **TODO: more examples**

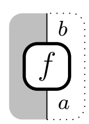
Example 5.2.4. When $b \in \mathcal{C}$ is dualizable, $b^\vee \otimes - : \mathcal{C} \to \mathcal{C}$ is left adjoint to $b \otimes - : \mathcal{C} \to \mathcal{C}$ by Frobenius reciprocity [[]]. Similarly, $- \otimes b$ is left adjoint to $- \otimes b^\vee$.[1]

Notation 5.2.5. Let $*$ denote the terminal category, which has one object $*$ and one 1-morphism id_*. When considering linear categories, $*$ denotes the category with one object $*$ and $\mathrm{End}(*) = \mathbb{C}\,\mathrm{id}_*$, which is no longer terminal (the terminal linear category satisfies $\mathrm{End}(*) = 0$).

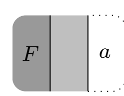
There is a canonical equivalence $\mathcal{C} \cong \mathsf{Fun}(* \to \mathcal{C})$ given by $a \mapsto (* \mapsto a)$ and $(f : a \to b) \mapsto (f_* : (* \mapsto a) \Rightarrow (* \mapsto b))$. Denoting $*$ by the empty shaded region in the graphical calculus for Cat, we see that we can represent objects $c \in \mathcal{C}$ by strings and morphisms by coupons with a $\mathcal{C}$-shading on the right (recall we compose left-to-right in Cat). We then apply the functor $F : \mathcal{C} \to \mathcal{D}$ by adding an F strand to the left of the c-strand.



$\leftrightarrow c \in \mathcal{C}$

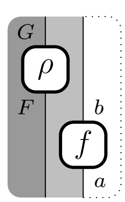


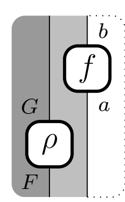
$\leftrightarrow (f : a \to b)$



$\leftrightarrow F(a) \in \mathcal{D}$

Natural transformations $\rho : F \Rightarrow G$ are the represented by coupons between the F, G strings, and naturality is represented by the exchange relation:



$= \rho_b \circ F(f) = G(f) \circ \rho_a =$


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Definition 5.2.6. Suppose $\mathcal{C}, \mathcal{D}$ are categories and $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ are functors with $F \dashv G$. We say $f : F(c) \to d$ and $g : c \to G(d)$ are *mates* if they map to each

[1] Some people call a dual a *left* (or *right*) dual and a predual a *right* (or *left*) dual (respectively). However, it is not clear whether a dual should be a left or a right dual, as tensoring on the left with $b^\vee$ is a left adjoint and tensoring on the right with $b^\vee$ is a right adjoint.

other under the natural isomorphism (5.2.2). The *unit* of the adjunction is the natural transformation $\eta : \mathrm{id}_{\mathcal{D}} \Rightarrow GF$ given by

$$\eta_c \coloneqq \mathrm{mate}(\mathrm{id}_{F(c)}) \in \mathcal{C}(c \to GF(c)) \cong \mathcal{D}(F(c) \to F(c)),$$

and the *counit* of the adjunction is the natural transformation $\epsilon : FG \Rightarrow \mathrm{id}_{\mathcal{C}}$ given by

$$\epsilon_d \coloneqq \mathrm{mate}(\mathrm{id}_{G(d)}) \in \mathcal{D}(FG(d) \to d) \cong \mathcal{C}(G(d) \to G(d)).$$

Lemma 5.2.7. *The operations of taking mate are natural with respect to pre-composition and post-composition by another morphism:*

(mate1) $\mathrm{mate}(f_2 \circ f_1) = G(f_2) \circ \mathrm{mate}(f_1)$ *for all* $f_1 : F(c) \to d_1$ *and* $f_2 : d_1 \to d_2$*.*
(mate2) $\mathrm{mate}(g_2 \circ g_1) = \mathrm{mate}(g_2) \circ F(g_1)$ *for all* $g_1 : c_1 \to c_2$ *and* $g_2 : c_2 \to G(d)$*.*

Proof. We prove the first, and the second is similar. By naturality of the adjunction isomorphism (5.2.2), the following diagram commutes.

$$\begin{array}{ccc} \mathcal{D}(F(c) \to d_1) & \xrightarrow{\cong} & \mathcal{C}(c \to G(d_1)) \\ \downarrow{\scriptstyle f_2 \circ -} & & \downarrow{\scriptstyle G(f_2) \circ -} \\ \mathcal{D}(F(c) \to d_2) & \xrightarrow{\cong} & \mathcal{C}(c \to G(d_2)). \end{array}$$

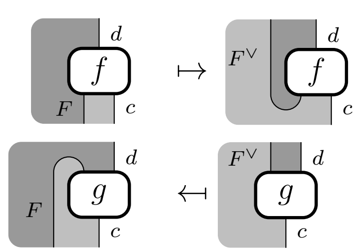
Going down and then left gives $\mathrm{mate}(f_2 \circ f_1)$, and goin right and then down gives $G(f_2) \circ \mathrm{mate}(f_1)$. $\square$

The next proposition shows that adjoints are the same thing as duals for 1-morphisms (functors) in Cat.

Proposition 5.2.8. *Suppose* $\mathcal{C}, \mathcal{D}$ *are categories and* $F : \mathcal{C} \to \mathcal{D}$ *and* $G : \mathcal{D} \to \mathcal{C}$ *are functors.*

(1) If there exist natural transformations $\mathrm{ev}_F : F \circ G \Rightarrow \mathrm{id}_{\mathcal{D}}$ *and* $\mathrm{coev}_F : \mathrm{id}_{\mathcal{C}} \Rightarrow G \circ F$ *satisfying the snake equations (so* $G \cong F^\vee$*), then* $F \dashv G$ *with adjunction natural isomorphism* (5.2.2) *given by the Frobenius reciprocity isomorphisms from* (**??**) *for 2-categories.*

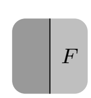
$$\mathcal{D}(F(c) \to d) \cong \mathcal{C}(c \to F^\vee(d))$$

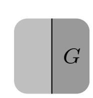


(2) If $F \dashv G$*, then the unit and counit satisfy the snake equations and thus can be represented by*

$\eta =$ 

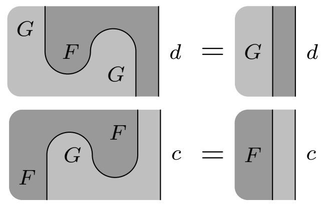
and $\epsilon =$ 

where 

$= F$, 

$= G.$

Proof. Statement (1) is immediate from naturality of ev_F and coev_F. To prove statement (2), we must prove that



$$\iff G(\epsilon_d)\circ \eta_{G(d)} = \mathrm{id}_{G(d)} \qquad \forall\, d\in\mathcal{D}$$

$$\iff \epsilon_{F(c)}\circ F(\eta_c) = \mathrm{id}_{F(c)} \qquad \forall\, c\in\mathcal{C}.$$

The first is (mate1) with $f_1 = \mathrm{id}_{FG(d)}$ and $f_2 = \epsilon_d$, and the second is (mate2) with $g_1 = \eta_c$ and $g_2 = \mathrm{id}_{GF(c)}$. □

Exercise 5.2.9. Modify [[]] to prove that every equivalence $a \simeq b$ in a 2-category can be modified to an adjoint equivalence, i.e., there are 1-morphisms ${}_aX_b$ and ${}_bY_a$ and isomorphisms $\eta : 1_a \Rightarrow X \otimes_b Y$ and $\epsilon : Y \otimes_b X \to 1_b$ which satisfy the snake equations.

Deduce that every equivalence of categories $F : \mathcal{C} \to \mathcal{D}$ can be augmented to an *adjoint equivalence*, i.e., there is an equivalence $G : \mathcal{D} \to \mathcal{C}$ such that $F \dashv G$.

Definition 5.2.10. A linear 2-category $\mathcal{C}$ is called *pre-semisimple* if it is rigid and locally semisimple, equivalently every n-fold linking category is a (semisimple) multitensor category.[2]

5.3. **Higher morphisms between 2-categories.**

Definition 5.3.1. A *2-functor* $F : \mathcal{C} \to \mathcal{D}$ between 2-categories consists of

- an assignment of an object $F(c)$ to each object $c \in \mathcal{C}$,
- a functor $F_{a,b} : \mathcal{C}(a \to b) \to \mathcal{D}(F(a) \to F(b))$,
- for all objects $c \in \mathcal{C}$, a *unitor* 2-isomorphism $F^0_c \in \mathcal{D}(1_{F(c)} \to F(1_c))$, and
- for all 1-morphisms ${}_aX_b, {}_bY_c \in \mathcal{C}$, a *compositor/tensorator* 2-isomorphism $F^2_{X,Y} \in \mathcal{D}(F(X) \otimes_{F(b)} F(Y) \Rightarrow F(X \otimes_b Y))$

subject to the following axioms:

- (naturality) $F^2_{X,Y}$ is natural in X and Y,
- (associativity) For all ${}_aX_b$, ${}_bY_c$, and ${}_cZ_d$ in $\mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccccc}
F(X)\otimes(F(Y)\otimes F(Z)) & \xrightarrow{\mathrm{id}_{F(X)}\otimes F^2_{Y,Z}} & F(X)\otimes F(Y\otimes Z) & \xrightarrow{F^2_{X,Y\otimes Z}} & F(X\otimes(Y\otimes Z)) \\
\downarrow{\scriptstyle \alpha^{\mathcal{D}}_{F(X),F(Y),F(Z)}} & & & & \downarrow{\scriptstyle F(\alpha^{\mathcal{C}}_{X,Y,Z})} \\
(F(X)\otimes F(Y))\otimes F(Z) & \xrightarrow{F^2_{X,Y}\otimes \mathrm{id}_{F(Z)}} & F(X\otimes Y)\otimes F(Z) & \xrightarrow{F^2_{X\otimes Y,Z}} & F((X\otimes Y)\otimes Z)
\end{array}$$

- (unitality) for all $a, b \in \mathcal{C}$ and ${}_aX_b \in \mathcal{C}(a \to b)$,

$$\begin{array}{ccc}
1_{F(a)}\otimes F(X) & \xrightarrow{\lambda^{F(a)}_{F(X)}} & F(X) \\
\downarrow{\scriptstyle F^1_a\otimes \mathrm{id}_{F(X)}} & & \uparrow{\scriptstyle F(\lambda^a_X)} \\
F(1_a)\otimes F(X) & \xrightarrow{F^2_{1_a,X}} & F(1_a\otimes X)
\end{array}
\qquad
\begin{array}{ccc}
F(X)\otimes 1_{F(b)} & \xrightarrow{\rho^{\mathcal{D}}_{F(X)}} & F(X) \\
\downarrow{\scriptstyle \mathrm{id}_{F(X)}\otimes F^1_b} & & \uparrow{\scriptstyle F(\rho^b_X)} \\
F(X)\otimes F(1_b) & \xrightarrow{F^2_{X,1_b}} & F(X\otimes 1_b)
\end{array}$$

A 2-functor is called:

[2]Our definition of a pre-semisimple 2-category differs slightly from that of presemisimple 2-category in [DR18, Def. 1.2.7], but our easier definition will still complete to a semisimple 2-category later on in §[[]].

- *fully faithful* if each functor $F_{a,b}$ is an equivalence, and
- *essentially surjective* if every object $d \in \mathcal{D}$ is equivalent to an object of the form $F(c)$ for some $c \in \mathcal{C}$.
- an *equivalence* if it is fully faithful and essentially surjective (cf. [JY21, Thm. 7.4.1]).

A 2-functor $F : \mathcal{C} \to \mathcal{D}$ between linear 2-categories is called *linear* if each functor $F_{a,b}$ is linear.

Example 5.3.2. The additive envelope Add is a 2-functor on the 2-category of linear categories. **TODO:**

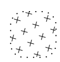
Example 5.3.3. Idempotent completion Idem is a 2-functor on the 2-category Cat categories which preserves the 2-subcategory of linear 2-categories. **TODO:**

Example 5.3.4. Composing the last two examples, Cauchy completion ¢ is a 2-functor on the 2-category of linear categories.

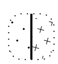
Exercise 5.3.5. A 2-functor is called *strict* if the unitors and tensorators are identities. Show that strict 2-categories and strict 2-functors form a 1-category.

Exercise 5.3.6 ($\star$)**.** Prove that every 2-category is equivalent to a strict 2-category.
Hint: Find a fully faithful 2-functor よ $: \mathcal{C} \to \mathrm{Hom}(\mathcal{C}^{\mathrm{mp}} \to \mathsf{Cat})$*. (See* [JY21, §8] *for more details.)*

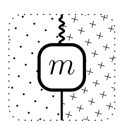
Notation 5.3.7. For 2-categories $\mathcal{C}, \mathcal{D}$ we have a *strict* 2-category of 2-functors $\mathrm{Hom}(\mathcal{C} \to \mathcal{D})$. We represent objects (2-functors) in this 2-category by unshaded regions with *textures*, e.g.:

 $= F$  $= G$  $= H$

We represent 2-transformations (see [[]] below) by *textured* strands between these textured regions:

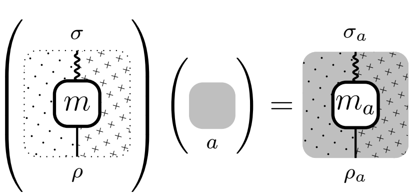
 $= \rho : F \Rightarrow G$  $= \sigma : G \Rightarrow H$

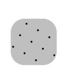
We represent 2-modifications (see [[]] below) by coupons between textured strands.

 $= m : {}_F\rho_G \Rrightarrow {}_F\sigma_G$

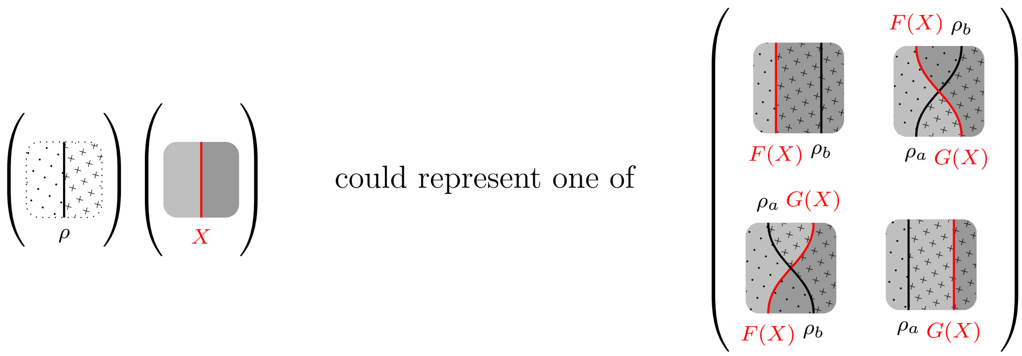
Remark 5.3.8. We use the $\otimes$ convention for 1-composition in the 2-category $\mathsf{Fun}(\mathcal{C} \to \mathcal{D})$, as the 1-morphisms are 2-transformations. Note that we do not compose functors in $\mathsf{Fun}(\mathcal{C} \to \mathcal{D})$ as they are objects, so no confusion can arise with the $\circ$ convention for 1-composition.

Notation 5.3.9. We represent a 2-morphism in $\mathcal{D}$ in the image of a 2-functor $F : \mathcal{C} \to \mathcal{D}$ using an *overlay graphical calculus*, which was described in [CP22, §2.1]. We apply a 2-morphism $m : {}_F\rho_G \Rrightarrow {}_F\sigma_G$ from $\mathsf{Fun}(\mathcal{C} \to \mathcal{D})$ to an object $a \in \mathcal{C}$ to obtain the 2-morphism $m_a : \rho_a \Rightarrow \sigma_a$ in $\mathcal{D}$:



 $= F(a)$,  $= G(a)$.

Given a 1-morphism ${}_aX_b$ in $\mathcal{C}$, there are 4 basic diagrams which one could obtain from overlaying m:



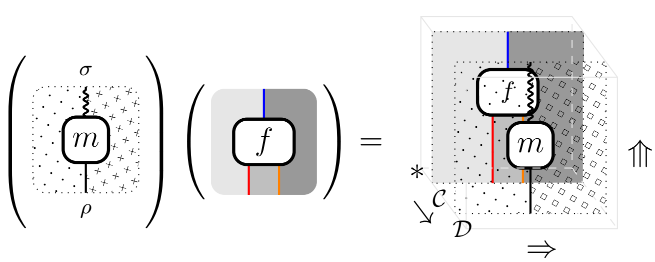
(5.3.10)

These crossings on the right hand side are data of a *2-transformation* $\rho : F \Rightarrow G$, which must satisfy various coherence conditions. For example, the two crossings on the off-diagonal should be each others' inverses under vertical composition, making these 4 diagrams into a system of matrix units. We refer the reader to [[]] below for further details.

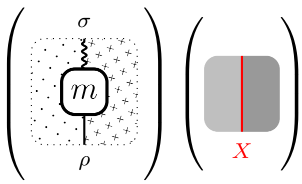
Remark 5.3.11. We will see in [[]] that 2-categories form a 3-category called 2Cat. While this 3-category is not strict, it does have the nice property that 1-composition is strictly associative. There is a semistrict notion of 3-category called a Gray-*category*, and every 3-category is equivalent to a Gray-category by [Gur13].

Now there is a 3D graphical calculus for Gray-categories [BMS12, Bar14], which can be applied to any 3-category by [Gut19]. Our overlay graphical calculus is an example of this 3D graphical calculus in 2Cat. We sketch this below, and we leave a rigorous proof to the interested reader.

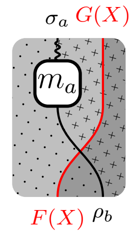
Indeed, given a 2-category $\mathcal{C} \in$ 2Cat, similar to Notation 5.2.5 above, we may identify $\mathcal{C} = \mathsf{Fun}(* \to \mathcal{C})$ where $*$ is the terminal 2-category with one object $*$, one 1-morphism 1_*, and one 2-morphism id_{1_*} (In the linear setting, $\mathrm{Hom}(1_* \to 1_*) = \mathbb{C}\,\mathrm{id}_{1_*}$). This identification allows us to identify the *internal* 2D string diagrammatic calculus for $\mathcal{C}$ with the *external* 2D string diagrammatic calculus for $\mathsf{Fun}(* \to \mathcal{C})$ as a hom 2-category of 2Cat. Since we may identify a 2-functor $F : \mathcal{C} \to \mathcal{C}$ with the 2-functor $F \circ - : \mathsf{Fun}(* \to \mathcal{C}) \to \mathsf{Fun}(* \to \mathcal{D})$ given by post-composition with F, and similarly for transformations and modifications, our overlay graphical calculus is exactly stacking of 2D sheets in the 3D graphical calculus for 2Cat.



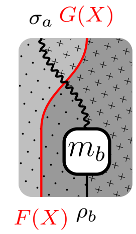
In order to interpret each overlay diagram as a 2-morphism in $\mathcal{D}$, one must require the strings and coupons of our $\mathcal{C}$-diagram and our $\mathsf{Fun}(\mathcal{C} \to \mathcal{D})$ diagram not overlap, except at finitely many points where strings can cross transversely. The axioms of 2-functor, 2-transformation, and 2-modification then ensure than any two ways of resolving non-generic intersections agree. We saw one such example in (5.3.10) above. If we also included a modification $m : \rho \Rrightarrow \sigma$ there, we would



could represent



or

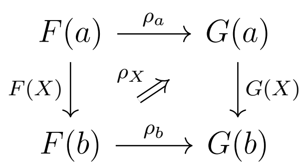


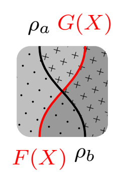
,

and requiring these diagrams to be equal produces the modification coherence axiom in [[]] below. The data of the crossing ρ_X for the 2-transformation $\rho : F \Rightarrow G$ may be interpreted as an *interchanger* in 2Cat, which arise from resolving two stacked 2D diagrams in 2Cat.

Definition 5.3.12. Suppose $\mathcal{C}, \mathcal{D}$ are 2-categories and $F, G : \mathcal{C} \to \mathcal{D}$ are 2-functors. A *2-transformation* $\rho : F \Rightarrow G$ consists of

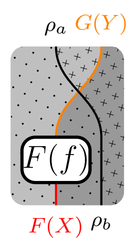
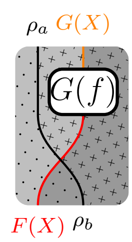
- for every $c \in \mathcal{C}$, a 1-morphism $\rho_c : F(c) \to G(c)$, and
- for every 1-morphism ${}_aX_b \in \mathcal{C}$, an invertible 2-morphism $\rho_X : F(X) \otimes_{F(b)} \rho_b \Rightarrow \rho_a \otimes_{G(a)} G(X)$

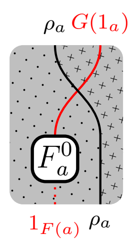
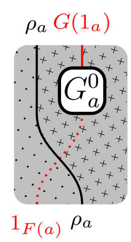


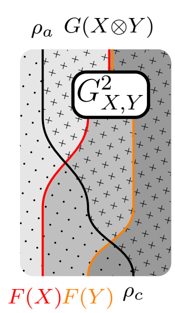
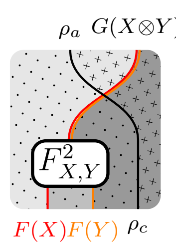


$= \rho_X$

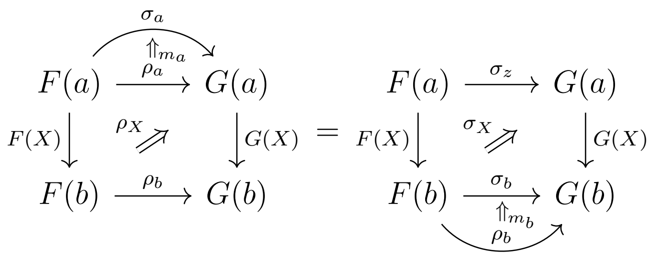
such that the following coherence axioms holds.

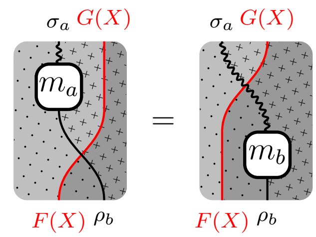
- (naturality) for all $f \in \mathcal{C}({}_aX_b \Rightarrow {}_aY_b)$,  $=$ 

- (unitality) for all $a \in \mathcal{C}$,  $=$ 

- (monoidality) for all ${}_aX_b, {}_bY_c$ in $\mathcal{C}$,  $=$ 

Definition 5.3.13. Suppose $\mathcal{C}, \mathcal{D}$ are 2-categories, $F, G : \mathcal{C} \to \mathcal{D}$ are 2-functors, and $\rho, \sigma : F \Rightarrow G$ are 2-transformations. A *2-modification* $m : \rho \Rrightarrow \sigma$ consists of a 2-morphism $m_c : \rho_c \Rightarrow \sigma_c$ for all $c \in \mathcal{C}$ such that for all 1-morphisms $X \in \mathcal{C}(a \to b)$,





5.4. **Direct sums.**

Definition 5.4.1. Suppose $\mathcal{C}$ is a locally additive linear 2-category. A *zero object* in $\mathcal{C}$ is an object $0 \in \mathcal{C}$ such that $\mathrm{End}_{\mathcal{C}}(0)$ is the terminal category, i.e., it has exactly one object 1_0 and one morphism id_{1_0}.

Given $a, b \in \mathcal{C}$, a *direct sum* is an object $a \boxplus b \in \mathcal{C}$ and 1-morphisms $I_a : a \to a \boxplus b$, $I_b : b \to a \boxplus b$, $P_a : a \boxplus b \to a$, and $P_b : a \boxplus b \to b$, satisfying the following axioms (in right-to-left convention):

($\boxplus$1) $P_a \circ I_a \cong 1_a$ and $P_b \circ I_b \cong 1_b$,
($\boxplus$2) $I_a \circ P_a \oplus I_b \circ P_b \cong 1_{a \boxplus b}$, and
($\boxplus$3) $P_b \circ I_a$ and $P_a \circ I_b$ are both zero objects in their respective hom 1-categories.

Note that when we write 1-composition from left-to-right, these relations will appear transposed.

Remark 5.4.2. When $\mathcal{C}$ is locally semisimple, ($\boxplus$3) is superfluous. Indeed, when $\mathcal{S}$ is a semisimple category, $s \cong s \oplus t$ implies $t \cong 0$. Hence

$$P_b \cong P_b \circ I_a \circ P_a \oplus P_b \circ I_b \circ P_b \cong P_b \circ I_a \circ P_a \oplus P_b$$

implies $P_b \circ I_a \circ P_a \cong 0$. Now pre-compose with I_a to see that $P_b \circ I_a \cong 0$. Similarly, $P_a \circ I_b \cong 0$.

Note here that the data of the isomorphisms at height 2 are not part of the data of the direct sum. Direct sums of $a, b \in \mathcal{C}$ form a contractible space. Indeed, given another direct sum $a \boxplus' b$, we have a canonical equivalence 1-morphism $I_a \circ P'_a \oplus I_b \circ P'_b : a \boxplus' b \to a \boxplus b$ with inverse $I'_a \circ P_a \oplus I'_b \circ P_b : a \boxplus b \to a \boxplus' b$. This equivalence is unique up to canonical natural isomorphism, so there is a contractible 2-groupoid (2-category with all morphisms invertible) of direct sums.

Definition 5.4.3. We call a locally additive linear 2-category *additive* if it admits a zero object and all direct sums.

Exercise 5.4.4. Show that linear 2-functors preserve direct sums (when they exist).

We now construct the additive envelope of a 2-category.

Construction 5.4.5. Suppose $\mathcal{C}$ is a linear 2-category. By replacing $\mathcal{C}$ with $\cancel{c}^1(\mathcal{C})$ via Construction 5.1.12, we may assume $\mathcal{C}$ is locally Cauchy complete (or at least locally additive). We define a 2-category $\mathsf{Add}(\mathcal{C})$ as follows.

- Objects are formal tuples $(a_j)_{j=1}^n$ of objects in $\mathcal{C}$.
- Hom 1-categories are formal matrix categories which are defined similar to linking categories.

$$\mathsf{Add}(\mathcal{C})((a_j)_{j=1}^n \to (b_i)_{i=1}^m) = \begin{pmatrix} \mathcal{C}(a_1 \to b_1) & \cdots & \mathcal{C}(a_1 \to b_n) \\ \vdots & & \vdots \\ \mathcal{C}(a_m \to b_1) & \cdots & \mathcal{C}(a_n \to b_m) \end{pmatrix}$$

- 2-composition happens component-wise.
- 1-composition is given by the usual matrix product formula, only taking direct sums. If $X = (X_{ij} \in \mathcal{C}(a_i \to b_j))$ and $Y = (Y_{jk} \in \mathcal{C}(b_j \to c_k))$ where $i = 1, \ldots, m$, $j = 1, \ldots, n$, and $k = 1, \ldots, p$, then

$$(X \otimes_{(b_j)} Y)_{ik} \coloneqq \bigoplus_{j=1}^n X_{ij} \circ Y_{jk} \in \mathcal{C}(a_i \to c_k).$$

  A similar formula is used for 2-morphisms, and one can see $- \otimes -$ is a functor.
- The associator $\alpha_{X,Y,Z} : X \otimes_{(b_j)} (Y \otimes_{(c_k)} Z) \Rightarrow (X \otimes_{(b_j)} Y) \otimes_{(c_k)} Z$ is a formal matrix of associators whose i, ℓ-entry is given by

$$\bigoplus_{j=1}^n \bigoplus_{k=1}^p \alpha_{X_{ij}, Y_{jk}, Z_{k\ell}} : \bigoplus_{j=1}^n \bigoplus_{k=1}^p X_{ij} \otimes_{b_j} (Y_{jk} \otimes_{c_k} Z_{k\ell}) \Longrightarrow (X_{ij} \otimes_{b_j} Y_{jk}) \otimes_{c_k} Z_{k\ell}.$$

  Unitors are defined similarly.

Observe that $\mathsf{Add}(\mathcal{C})$ comes equipped with a fully faithful 2-functor $\iota : \mathcal{C} \hookrightarrow \mathsf{Add}(\mathcal{C})$ given by $a \mapsto (a)$, ${}_aX_b \mapsto (X)$, and $(f : {}_aX_b \Rightarrow {}_aY_b) \mapsto (f)$.

Lemma 5.4.6. $\mathsf{Add}(\mathcal{C})$ *is additive.*

Proof. Since each $\mathcal{C}(a_j \to b_i)$ is additive, so is

$$\mathsf{Add}(\mathcal{C})((a_j)_{j=1}^n \to (b_i)_{i=1}^m) = \bigoplus_{i,j} \mathcal{C}(a_j \to b_i),$$

so $\mathsf{Add}(\mathcal{C})$ is locally additive.

Now consider objects $c_1 = (a_j)_{j=1}^m$ and $c_2 = (a_j)_{j=m+1}^n$ in $\mathsf{Add}(\mathcal{C})$. We define their direct sum $c_1 \boxplus c_2$ as the tuple $(a_j)_{j=1}^n$. For $i = 1, 2$, we define $I_i : c_i \to c_1 \boxplus c_2$ and $P_i : c_1 \boxplus c_2 \to c_i$ as matrices of objects whose entries are either 1_{a_j} or 0 depending on the indices similar to Lemma [[]] for 1-categories. The rest of the verification is left to the reader. □

Our next task is to prove the universal property of the additive envelope.

Proposition 5.4.7. *For every additive 2-category $\mathcal{D}$, precomposition with the canonical inclusion $\mathcal{C} \hookrightarrow \mathsf{Add}(\mathcal{C})$ gives an equivalence*

$$\iota^* : \mathsf{Fun}(\mathsf{Add}(\mathcal{C}) \to \mathcal{D}) \xrightarrow{\cong} \mathsf{Fun}(\mathcal{C} \to \mathcal{D}).$$

Proof. Suppose $F : \mathcal{C} \to \mathcal{D}$ is a linear 2-functor. We define a linear 2-functor $\mathsf{Add}(\mathcal{C}) \to \mathcal{D}$ by setting

$$\begin{aligned} \mathsf{Add}(F)((a_j)_{j=1}^n) &:= \boxplus_{j=1}^n F(a_j) \\ \mathsf{Add}(F)((X_{ij})) &:= \bigoplus_{i,j} P_i \otimes_{a_i} F(X_{ij}) \otimes_{a_j} I_j \\ \mathsf{Add}(F)((f_{ij})) &:= \sum_{i,j} \iota_{ij} \cdot F(f_{ij}) \cdot \pi_{ij} \end{aligned}$$

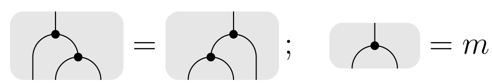
where the P_i, I_j witness $\boxplus_{j=1}^n F(a_j)$ as a 2-direct sum and ι_{ij}, π_{ij} witness $\mathsf{Add}(F)((X_{ij}))$ as an ordinary 1-direct sum. The compositor $\mathsf{Add}(F)^2_{(X_{ij}),(Y_{jk})}$ is made from the structure isomorphisms $I_j \otimes P_j \cong \mathrm{id}_{F(a_j)}$ and associators and unitors. Clearly $F = \mathsf{Add}(F) \circ \iota$, so ι^* is essentially surjective.

TODO: 2-transformations and 2-modifications □

5.5. **Algebras: higher idempotents.** For this section, we fix a 2-category $\mathcal{C}$.

Definition 5.5.1. Suppose $a \in \mathcal{C}$. A pair $({}_aA_a, m : A \otimes_a A \Rightarrow A)$ is called an *algebra* if the following *associativity axiom* is satisfied:

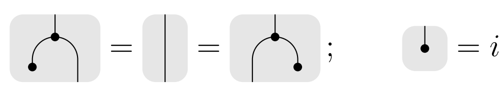
$$\begin{array}{ccccc} A \otimes (A \otimes A) & \xrightarrow{\mathrm{id}_A \otimes m} & A \otimes A & \overset{m}{\searrow} & \\ \downarrow{\scriptstyle \alpha_{A,A,A}} & & & & A \\ (A \otimes A) \otimes A & \xrightarrow{m \otimes \mathrm{id}_A} & A \otimes A & \underset{m}{\nearrow} & \end{array} \quad \rightsquigarrow$$

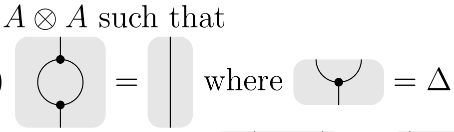


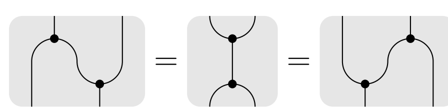
An algebra $({}_aA_a, m)$ is called

- *unital* if there is a 2-morphism $i : 1_a \Rightarrow A$ such that the following *unitality axioms* are satisfied:

$$\begin{array}{ccccc} 1_a \otimes A & \xleftarrow{(\lambda^c_A)^{-1}} & A & \xrightarrow{(\rho^c_A)^{-1}} & A \otimes 1_a \\ \downarrow{\scriptstyle i \otimes \mathrm{id}_A} & & \downarrow{\scriptstyle \mathrm{id}_A} & & \downarrow{\scriptstyle \mathrm{id}_A \otimes i} \\ A \otimes A & \xrightarrow{m} & A & \xleftarrow{m} & A \otimes A \end{array} \quad \rightsquigarrow$$



  A unital algebra is called *connected* if $\dim(\mathcal{C}(1_a \Rightarrow A)) = 1$.
- *separable* if the multiplication map splits as an $A - A$ bimodule map, i.e., there is a map $\Delta : A \to A \otimes A$ such that
  - (m splits) 

  - (as an $A - A$ bimodule) 

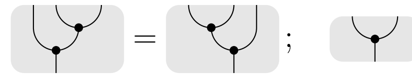
A triple (A, m, Δ) consisting of a separable algebra (A, m) equipped with a separator Δ is called a *condensation algebra* [GJF19].

A quintuple $(A, m, i, \Delta, \varepsilon)$ is called a *Frobenius algebra* if

- (A, m, i) is a unital algebra,

- (A, Δ, ε) is a co-unital coalgebra. Coassociativity is the folllowing axiom:

$$\begin{array}{ccccc} A \otimes (A \otimes A) & \xleftarrow[\mathrm{id}_A \otimes \Delta]{} & A \otimes A & \nwarrow^{\Delta} & \\ \downarrow \alpha_{A,A,A} & & & & A \\ (A \otimes A) \otimes A & \xleftarrow[\Delta \otimes \mathrm{id}_A]{} & A \otimes A & \swarrow_{\Delta} & \end{array} \quad \rightsquigarrow$$

 $= \Delta.$

  We leave the co-unital axiom to the reader.
- Δ is an $A - A$ bimodule map.

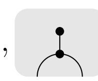
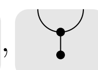
There is also a notion of algebra object in a monoidal category $\mathcal{C}$; it is equivalent to an algebra in $\mathrm{B}\mathcal{C}$.

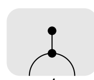
Remark 5.5.2. The most natural settings in which to work are condensation algebras, Frobenius algebras, and unital separable algebras. Working with separable algebras which are non-unital, but not equipped with a particular splitting, is not a well-behaved notion.

Facts 5.5.3. We now give a list of basic facts about algebras.

(A1) If an algebra (A, m) is unital, then its unit is unique.

(A2) If (A, m, Δ) is a condensation algebra, then Δ is automatically co-associative. **TODO:**

(A3) If $(A_a, m, i, \Delta, \varepsilon)$ is a Frobenius algebra, then $\Big(A,$  ,  $\Big)$ is a dual for A.

(A4) Suppose (A, m, i) is an algebra in a multitensor category and $\varepsilon : A \to 1_{\mathcal{C}}$ is a fixed non-zero map. The following are equivalent:
- The pairing  is *non-degenerate*, i.e., **TODO:** is invertible
- There is a $\Delta : A \to A \otimes A$ giving a Frobenius algebra structure

Indeed, **TODO:**

The proof above proves that $\Delta : A \to A \otimes A$ above is unique as it can be expressed in terms of **TODO:**

(A5) When $\mathcal{C}$ is a pivotal multitensor category, we call a Frobenius algebra *symmetric* if **TODO:**

(A6) If $\mathcal{C}$ is a pivotal tensor category and (A, m, i) is a connected separable algebra such that $\dim_L(A) \neq 0 \neq \dim_R(A)$, then A can be canonically equipped with the structure of a symmetric Frobenius algebra. **TODO:**

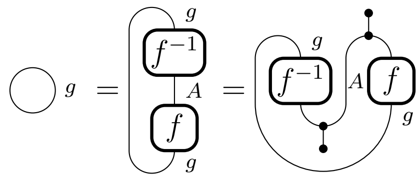
(A7) Given an algebra (A, m), $(A^\vee, m^\vee)$ is also an algebra. Moreover, (A, m) is unital if and only if $(A^\vee, m^\vee)$ is with unit $i^\vee$. If Δ makes (A, m) into a condensation algebra, then $\Delta^\vee$ makes $(A^\vee, m^\vee)$ into a condensation algebra. An analogous statement holds for Frobenius structure as well.

Exercise 5.5.4. Find a complete characterization of unital algebras in Set and in Cat.

Exercise 5.5.5. Show that a unital algebra in Vec is separable if and only if it is semisimple.

Example 5.5.6. We classify connected unital separable algebras in $\mathsf{Vec}(G, \omega)$. Suppose $A = \bigoplus A_g \in \mathsf{Vec}(G, \omega)$ is a connected unital algebra with unit $i : \mathbb{C} \cong A_e \subset A$ and multiplication $m : A \otimes A \to A$. We will use the spherical structure on $\mathsf{Vec}(G, \omega)$ in which all quantum dimensions of simples are equal to 1.

First, we claim that each non-zero $f \in A_g$ is left invertible. Indeed, we may view $A_g \cong \mathrm{Hom}(\mathbb{C}_g \to A)$ by Yoneda, and f non-zero means it has a left-inverse $f^{-1} : A \to \mathbb{C}_g$ such that $f^{-1} \cdot f = \mathrm{id}_{\mathbb{C}_g}$. Since A is connected and separable and $d_A \neq 0$, there is a canonical symmetric Frobenius algebra structure on A by (A6) which satisfies $\varepsilon \cdot i = d_A$. We calculate

$1 =$ 

The diagram on the right is the product of the element $(f^{-1})^\vee \in A_{g^{-1}}$ times the element $f \in A_g$ in A viewed as an element in $A_e \cong \mathrm{Hom}(\mathbb{C}_e \to A)$, post-composed with ε. Since it is non-zero, we conclude that a scalar multiple of $(f^{-1})^\vee$ is a left-inverse for f in A.

We now see that each A_g is either 0 or 1-dimensional, and $\dim(A_g) = \dim(A_{g^{-1}})$ for all $g \in G$. Indeed, if $a, b \in A_g$ are non-zero and $a^{-1} \in A_{g^{-1}}$ is the left inverse of $a \in A$ which exists by the above argument, then $ba^{-1} \in A_e \cong \mathbb{C}$, so $ba^{-1} = \lambda i$ for some $\lambda \in \mathbb{C}$. Right multiplying by a, we have $b = ba^{-1}a = \lambda a$ as claimed.

We have just seen that to each connected separable algebra, there is a subgroup $H \leq G$ on which A is supported, i.e., $H = \{h \in G | A_h \neq 0\}$. Pick a basis element $a_h \in A_h$ for each $h \in H$. The multiplication map m is then the same data as a scalar $\mu_{g,h} \in \mathbb{C}^\times$ satisfying $a_g a_h = \mu_{g,h} a_{gh}$ for each $g, h \in H$. Associativity of the algebra implies that

$$\mu_{gh,k}\mu_{h,k}a_{ghk} = a_g(a_h a_k) = \omega(g,h,k)(a_g a_h)a_k = \omega(g,h,k)\mu_{g,h}\mu_{gh,k}$$

which implies that

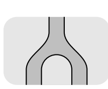
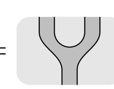
$$\omega(g,h,k) = \mu_{h,k}\mu_{gh,k}^{-1}\mu_{g,hk}\mu_{g,h}^{-1} = (d\mu)(g,h,k).$$

Thus a connected separable algebra is the data of a subgroup $H \leq G$ and a 2-cochain $\mu \in C^2(G, \mathbb{C}^\times)$ which trivializes the 3-cocycle ω, i.e., A is exactly the twisted group algebra $\mathbb{C}[H, \mu]$.

If we chose different bases $(b_h)_{h \in H}$, then for each $h \in H$, we have a scalar $\lambda_h \in \mathbb{C}^\times$ such that $b_h = \lambda_h a_h$. This scalar changes μ by a 1-coboundary, so connected separable algebras up to algebra isomorphism correspond to subgroups $H \leq G$ on which ω trivializes together with a 2-cocycle $[\mu] \in H^2(H, \mathbb{C}^\times)$.

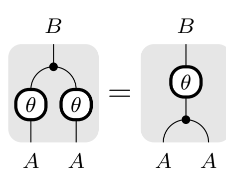
Example 5.5.7. Suppose ${}_aX_b \in \mathcal{C}(a \to b)$. A *separable dual* for ${}_aX_b$ is a dual ${}_bX_a^\vee \in \mathcal{C}(b \to a)$ with maps $\mathrm{coev}_X \in \mathcal{C}(1_a \to {}_aX \otimes_b X_a^\vee)$ and $\mathrm{ev}_X \in \mathcal{C}({}_bX^\vee \otimes_a X_b \Rightarrow 1_b)$ such that ev_X admits a right inverse $\epsilon_X \in \mathcal{C}(1_b \to {}_bX^\vee \otimes_a X_b)$.

Given a separable dual for ${}_aX_b$, we can canonically endow $X \otimes_b X^\vee$ with the structure of a unital condensation algebra. Indeed, we define

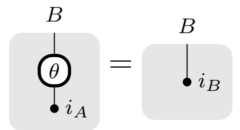
$m :=$  $= \mathrm{id}_X \otimes \mathrm{ev}_X \otimes \mathrm{id}_{X^\vee}$ $\quad i :=$  $= \mathrm{coev}_X$ $\quad \Delta :=$  $= \mathrm{id}_X \otimes \epsilon_X \otimes \mathrm{id}_{X^\vee}$

We leave the rest of the straightforward verification to the reader.

Definition 5.5.8. Suppose $({}_aA_a, m_A), ({}_aB_a, m_B)$ are algebras and $\theta : A \Rightarrow B$. We call θ an *algebra map* if



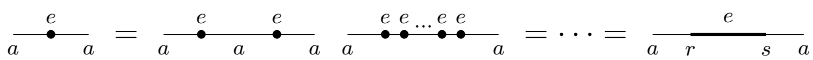
If A, B are unital, we call θ a *unital* algebra map if in addition



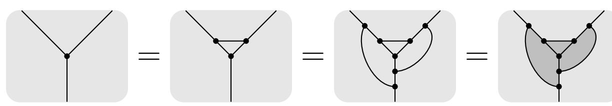
Observe that algebra objects in $\Omega_a\mathcal{C}$ and algebra maps form a 1-category.

Definition 5.5.9. A unital separable algebra ${}_aA_a$ *splits* if it is isomorphic via an algebra map to a unital separable algebra of the form ${}_aX \otimes_b X^\vee_a$ from Example 5.5.7 where ${}_bX^\vee_a$ is a separable dual of ${}_aX_b$.

Remark 5.5.10. A condensation algebra is the 2-categorical analog of an idempotent. An idempotent in a 1-category can replicate freely on a line, and replicating arbitrarily many times leads to the notion of splitting for an idempotent.



Similarly, a condensation algebra can replicate freely in a 2D mesh, and replicating arbitrarily many times leads to the notion of splitting for a separable algebra.

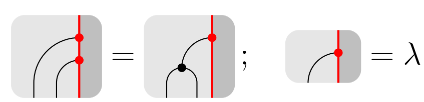


Definition 5.5.11. A locally Cauchy complete 2-category is called *2-idempotent complete* if every unital separable algebra splits.

5.6. **Bimodules and intertwiners.** One gets the notion of a module M for an algebra A by taking the axioms for a module and changing the appropriate instance of A to M.

Definition 5.6.1. Suppose $({}_aA_a, m)$ is an algebra in $\mathcal{C}$. A *left A-module* is a pair $({}_aM_b, \lambda : A \otimes_a M \to M)$ for some $b \in \mathcal{C}$ such that the following associativity axiom holds:

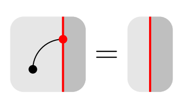
$$\begin{array}{ccccc} A\otimes(A\otimes M) & \xrightarrow{\mathrm{id}_A\otimes\lambda} & A\otimes M & & \\ \big\downarrow{\scriptstyle \alpha_{A,A,M}} & & & \searrow^{\lambda} & M \\ (A\otimes A)\otimes M & \xrightarrow{m\otimes \mathrm{id}_M} & A\otimes M & \nearrow_{\lambda} & \end{array} \quad \rightsquigarrow$$



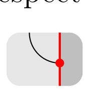
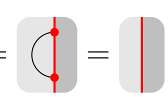
We leave the definition of right module to the reader.

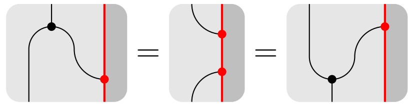
If $({}_aA_a, m)$ is unital, we call $({}_aM_b, \lambda)$ *unital* if the following unitality axiom is saitsfied:

$$\begin{array}{ccc} 1_a\otimes M & \xleftarrow{(\lambda^c_A)^{-1}} & M \\ \big\downarrow{\scriptstyle i\otimes \mathrm{id}_M} & & \big\downarrow{\scriptstyle \mathrm{id}_M} \\ A\otimes M & \xrightarrow{\lambda} & M \end{array} \quad \rightsquigarrow$$



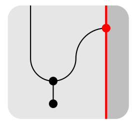
If $({}_aA_a, m)$ is separable, we call $({}_aM_b, \lambda)$ *separable* if λ splits as a left A-module map, and λ is also a co-module map with respect to the splitting, i.e.,

- (λ splits) there is a map  $= \delta : M \to A \otimes M$ such that $\lambda \cdot \delta =$  $= \mathrm{id}_M$,

- (as an A-(co)module map) 

Exercise 5.6.2. Suppose $({}_aA_a, m, i)$ is a unital separable algebra and $({}_aM_b, \lambda)$ is a left A-module. Prove that the following are equivalent: **TODO: fix this**

(1) (M, λ) is unital, and
(2) For any choice of separator $\Delta : A \to A \otimes A$, the map $\delta : M \to A \otimes_a M$ given by

$$\delta :=$$ 

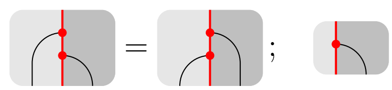
witnesses the separability of M.

Repeat the above exercise for right B-modules and $A - B$ bimodules.

Exercise 5.6.3. Prove that for every $c \in \mathcal{C}$, 1_c is canonically a unital condensation algebra in $\mathcal{C}$. Then prove that for every ${}_aX_b \in \mathcal{C}(a \to b)$, the only separable $1_a - 1_b$ bimodule structure on X is given by the unitors.

Definition 5.6.4. Suppose now $({}_aA_a, m_A)$ and $({}_bB_b, m_B)$ are algebras. an $A - B$ *bimodule* is a triple $({}_aM_b, \lambda : A \otimes_a M \to M, \rho : M \otimes_b B \to M)$ such that (M, λ) is a left A-module, (M, ρ) is a right B-module, and the additional associativity axiom holds:

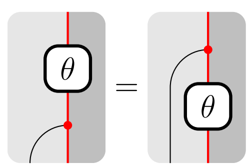
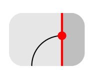
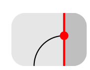
$$\begin{array}{ccccc} A \otimes (M \otimes A) & \xrightarrow{\mathrm{id}_A \otimes \rho} & A \otimes M & & \\ \downarrow{\scriptstyle \alpha_{A,M,A}} & & & \overset{\lambda}{\searrow} & M \\ (A \otimes M) \otimes A & \xrightarrow{\lambda \otimes \mathrm{id}_A} & M \otimes A & \underset{\rho}{\nearrow} & \end{array} \quad \rightsquigarrow$$

 $= \rho.$

As before, there is a notion of a (bi)module object in a multitensor category $\mathcal{C}$ for an algebra object; it is a (bi)module for the corresponding algebra in B$\mathcal{C}$.

Definition 5.6.5. Suppose $({}_aA_a, m)$ is an algebra. Given two left A-modules $({}_aM_b, \lambda_M)$ and $({}_aN_b, \lambda_N)$, a 2-morphism $\theta \in \mathcal{C}({}_aM_b \Rightarrow {}_aN_b)$ is called a *left A-module map* if the following diagram commutes:

$$\begin{array}{ccc} A \otimes M & \xrightarrow{\lambda_M} & M \\ \downarrow{\scriptstyle \mathrm{id}_A \otimes \theta} & & \downarrow{\scriptstyle \theta} \\ A \otimes N & \xrightarrow{\lambda_N} & N \end{array} \quad \rightsquigarrow$$

 ;  $= \lambda_M,$  $= \lambda_N$

We leave the definition of a right B-module map and an $A - B$ bimodule map to the reader.

Construction 5.6.6. Given an algebra object (A, m) in $\mathcal{C}$, we can take the category $\mathcal{C}_A$ whose:

- objects are unital right A-module objects (M, ρ) in $\mathcal{C}$, and
- 1-morphisms are right A-module maps.

Given $c \in \mathcal{C}$, we get a *free A-module* given by $c \otimes A$ with right action map $\mathrm{id}_c \otimes \rho$. Observe that we get a functor $\mathcal{C} \to \mathcal{C}_A$ given by $c \mapsto c \otimes A$ and $(f : c_1 \to c_2) \mapsto f \otimes \mathrm{id}_A$; we call this the *free module functor*.

There is also a similar category ${}_A\mathcal{C}$ of left A-modules in $\mathcal{C}$ and a category of left free A-modules.

Exercise 5.6.7. Suppose $({}_aA_a, m, i)$ is a unital algebra in $\mathcal{C}$, $({}_bM_a, \rho)$ is a right A-module, and ${}_bX_a \in \mathcal{C}(b \to a)$. Find an isomorphism

$$\mathcal{C}_A({}_bX \otimes_a A_a \Rightarrow {}_bM_a) \cong \mathcal{C}({}_bX_a \Rightarrow {}_bM_a)$$

which is natural in both M and X. Deduce that the forgetful functor $\mathcal{C}_A \to \mathcal{C}$ which forgets the right A-module structure is right-adjoint to the free module functor $\mathcal{C} \to \mathcal{C}_A$.

Exercise 5.6.8. Suppose (A, m) is an algebra object in $\mathcal{C}$. Show that every idempotent right A-module map $e : M_A \to N_A$ splits in $\mathcal{C}_A$. Deduce that $\mathcal{C}_A$ is Cauchy complete, as is ${}_A\mathcal{C}_A$, the category of $A - A$ bimodules in $\mathcal{C}$ with $A - A$ bimodule maps.

Exercise 5.6.9. Suppose (A, m) is an algebra in a locally semisimple 2-category $\mathcal{C}$. Show that any right A-module sits in a co-equalizer diagram

$$M \otimes A \otimes A \underset{\mathrm{id}_M \otimes m}{\overset{\rho_M \otimes A}{\Rrightarrow}} M \otimes A \overset{\rho_M}{\Longrightarrow} M.$$

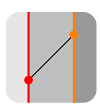
Deduce the following facts.

(1) Every free module is projective.
(2) If (A, m, Δ) is a condensation algebra in $\mathcal{C}$, then every right condensation module M_A is a summand of the free module $M \otimes A$.
(3) $\mathcal{C}_A$ is the Cauchy completion of $\mathsf{FreeMod}_{\mathcal{C}}(A)$.

Definition 5.6.10. Given a locally semisimple 2-category $\mathcal{C}$, we have a 2-category $\mathsf{Alg}(\mathcal{C})$ where

- objects are unital separable algebras in $\mathcal{C}$,
- 1-morphisms $A \to B$ are separable $A - B$ bimodules in $\mathcal{C}$, and
- 2-morphisms ${}_AM_B \Rightarrow {}_AN_B$ are intertwiners.

We now define 1-composition. Suppose $A, B, C \in \mathsf{Alg}(\mathcal{C})$ and ${}_AM_B$ and ${}_BN_C$ are separable bimodules. Observe that $M \otimes_b N$ is organically an $A - C$ bimodule, and by Exercise 5.6.8, the category $\mathsf{Bim}_{\mathcal{C}}(A \to C)$ of $A - C$ bimodules in $\mathcal{C}$ is Cauchy complete. We define the *relative tensor product* $M \otimes_B N$ by splitting the idempotent

$p_{M,N} :=$  $:= (\mathrm{id}_M \otimes \lambda_N) \circ (\Delta_M \otimes \mathrm{id}_N)$ (5.6.11)

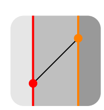
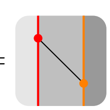
in $\mathsf{Bim}_{\mathcal{C}}(A \to C)$. Observe that $M \otimes_B N$ is only defined up to unique isomorphism.

Given $f : {}_AK_B \Rightarrow {}_AM_B$ and $g : {}_BL_C \Rightarrow {}_BN_C$, the map $f \otimes g : K \otimes_b L \Rightarrow M \otimes_b N$ is an $A - C$ bimodule map such that

$$p_{M,N} \cdot (f \otimes g) = (f \otimes g) \cdot p_{K,L}.$$

We thus define $f \otimes_B g : {}_AL \otimes_B K_C \Rightarrow {}_AM \otimes_B N_C$ as the above composite morphism. It is straightforward to verify the interchange axiom, so we have a 1-composition functor $- \otimes_B - : \mathsf{Bim}_{\mathcal{C}}(A \to B) \times \mathsf{Bim}_{\mathcal{C}}(B \to C) \to \mathsf{Bim}_{\mathcal{C}}(A \to C)$.

TODO: associator

Exercise 5.6.12. Verify (5.6.11) is an idempotent. Then show  $=$  .

Lemma 5.6.13. $\mathsf{Alg}(\mathcal{C})$ *is 2-idempotent complete.*

Proof. **TODO:** □

Definition 5.6.14. Two algebras $A, B \in \mathcal{C}$ are called *Morita equivalent* if their categories of right modules in $\mathcal{C}$ are equivalent, i.e., $\mathcal{C}_A \cong \mathcal{C}_B$.

Proposition 5.6.15. *Two unital separable algebras are Morita equivalent if and only if they are equivalent in the 2-category* $\mathsf{Alg}(\mathcal{C})$.

Proof. First, suppose $A \sim B$ in $\mathsf{Alg}(\mathcal{C})$, i.e., there is an $A - B$ bimodule ${}_AM_B$ and a $B - A$ bimodule ${}_BN_A$ and bimodule isomorphisms ${}_AM \otimes_B N_A \to {}_AA_A$ and ${}_BN \otimes_A M_B \to {}_BB_B$. **TODO:** □

5.7. **Module categories.** In this section, $\mathcal{C}$ denotes a multitensor category unless stated otherwise.

For $(M, \rho) \in \mathcal{C}_A$, observe that $c \otimes M$ also has a right A-module structure with action map $\mathrm{id}_c \otimes \rho$. Thus the category $\mathsf{Mod}_{\mathcal{C}}(A)$ has the structure of a left $\mathcal{C}$-*module category*. (Below, we require module categories to be semisimple, but this is not necessary in the definition.)

Definition 5.7.1. A left $\mathcal{C}$-module category for a multitensor category $\mathcal{C}$ consists of a semisimple category $\mathcal{M}$ together with a left $\mathcal{C}$-action functor $\triangleright : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ a family of natural unitor isomorphisms $\lambda_m : 1_{\mathcal{C}} \triangleright m \to m$, and a family of natural associator isomorphisms $\alpha_{a,b,m} : a \triangleright (b \triangleright m) \to (a \otimes b) \triangleright m$ which satisfy the pentagon and triangle axioms.

Proposition 5.7.2. *The following are equivalent for a unital algebra* (A, m, i) *in a semisimple tensor category* $\mathcal{C}$.

(1) A *is separable,*
(2) $\mathcal{C}_A$ *is semisimple,*
(3) ${}_A\mathcal{C}$ *is semisimple, and*
(4) ${}_A\mathcal{C}_A$ *is semisimple.*

Proof.
(1) ⇒ (2): **TODO: add abelian category section in earlier chapter** If A is separable, then $\mathcal{C}_A$ is abelian by [[]]. We show every object $M_A \in \mathcal{C}_A$ is projective. **TODO:**
(1) ⇒ (3): Similar to (1) ⇒ (2) and omitted.
(2) ⇒ (4): **TODO:**
(3) ⇒ (4): Similar to (2) ⇒ (4) and omitted.
(4) ⇔ (1): **TODO:** □

Definition 5.7.3. Suppose $\mathcal{M}, \mathcal{N}$ are two left $\mathcal{C}$-module categories. A $\mathcal{C}$-*module functor* $\mathcal{F} : \mathcal{M} \to \mathcal{N}$ is a functor equipped with a family of natural actionator isomorphisms $\mathcal{F}^2_{c,m} : c \triangleright \mathcal{F}(m) \to \mathcal{F}(c \triangleright m)$ satisfying an associative condition. Given two $\mathcal{C}$-module functors $\mathcal{F}, \mathcal{G} : \mathcal{M} \to \mathcal{N}$, a $\mathcal{C}$-module natural transformation $\theta : \mathcal{F} \Rightarrow \mathcal{G}$ is a natural transformation $\mathcal{F} \Rightarrow \mathcal{G}$ such that the following compatibility axiom is satisfied with the actionators:

$$\begin{array}{ccc} c \triangleright \mathcal{F}(m) & \xrightarrow{\mathcal{F}^2_{c,m}} & \mathcal{G}(c \triangleright m) \\ \downarrow{\scriptstyle \mathrm{id}_c \triangleright \theta_m} & & \downarrow{\scriptstyle \theta_m} \\ c \triangleright \mathcal{G}(m) & \xrightarrow{\mathcal{G}^2_{c,m}} & \mathcal{G}(c \triangleright m). \end{array} \tag{5.7.4}$$

Exercise 5.7.5. Show how to endow the left $\mathcal{C}$-modules, $\mathcal{C}$-module functors, and $\mathcal{C}$-module natural transformations with the structure of a 2-category. We call this 2-category $\mathsf{Mod}(\mathcal{C})$.

Construction 5.7.6. Suppose $\mathcal{C}$ is a multitensor category, $A, B \in \mathcal{C}$ are separable unital algebra objects, and $M \in \mathcal{C}$ is an $A-B$ bimodule object. We get a $\mathcal{C}$-module functor $- \otimes_A M_B : \mathcal{C}_A \to \mathcal{C}_B$. Indeed, suppose $N_A \in \mathcal{C}_A$, and let L_B be an object obtained from splitting $p_{N,M}$. **TODO:**

TODO:

TODO: finitely many equivalence classes of indecomposable module cats over a multifusion. This is actually too hard for right now...

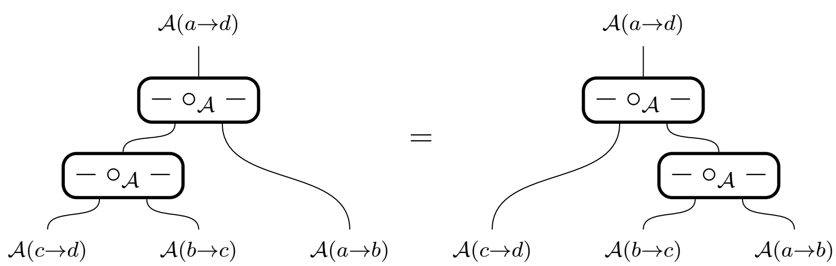
5.8. **Enriched categories and Ostrik's Theorem.** For this section, $\mathcal{V}$ denotes a monoidal category. Typically, we will take $\mathcal{V} = \mathsf{Vec}$ in applications, but sometimes we take $\mathcal{V}$ to be sVec or another multifusion category.

Definition 5.8.1 ([Kel05])**.** Given a monoidal category $\mathcal{V}$, a $\mathcal{V}$-*(enriched) category* $\mathcal{A}$ consists of the following data:

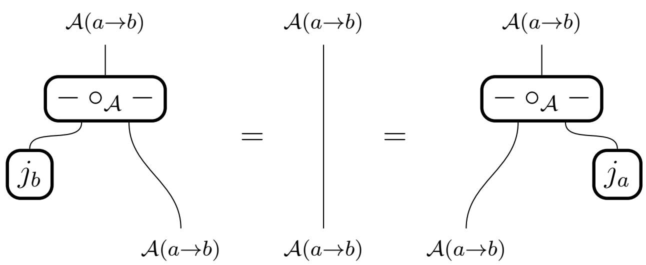
- a collection of objects $a \in \mathcal{A}$,
- for each $a, b \in \mathcal{A}$, a *hom object* $\mathcal{A}(a \to b) \in \mathcal{V}$,
- a unit map $j_c \in \mathcal{V}(1_{\mathcal{V}} \to \mathcal{A}(c \to c))$ for every $c \in \mathcal{A}$, and
- a composition morphism $- \circ_{\mathcal{A}} - \in \mathcal{V}(\mathcal{A}(b \to c) \otimes \mathcal{A}(a \to b) \to \mathcal{A}(a \to c))$ for all $a, b, c, \in \mathcal{A}$.

This data must satisfy the following axioms:

- (associativity)



- (identity)



Exercise 5.8.2. Given a $\mathcal{V}$-category $\mathcal{A}$, the *underlying category* $\mathcal{A}^{\mathcal{V}}$ has the same objects as $\mathcal{A}$, but $\mathcal{A}^{\mathcal{V}}(a \to b) := \mathcal{V}(1_{\mathcal{V}} \to \mathcal{A}(a \to b))$. Show how to endow $\mathcal{A}^{\mathcal{V}}$ with the structure of an ordinary category.

Remark 5.8.3. While we will not need this here at this time, there are notions of $\mathcal{V}$-functor and natural transformation so that $\mathcal{V}$-categories, $\mathcal{V}$-functors, and natural transformations forms a 2-category. Taking the underlying category, functor, and natural transformation gives a 2-functor $\mathcal{V}\mathsf{Cat} \to \mathsf{Cat}$. We refer the reader to [Kel05] for more details.

For the remained of this section, $\mathcal{C}$ denotes a multifusion category.

Definition 5.8.4. Suppose $\mathcal{M}$ is a finitely semisimple left $\mathcal{C}$-module category. For each $m, n \in \mathcal{M}$, $\mathcal{M}(- \triangleright m \to n) : \mathcal{C}^{\text{op}} \to \mathsf{Vec}$ is a linear functor and it is thus representable by semisimplicity of $\mathcal{C}$. We define the *internal hom* object $\widehat{\mathcal{M}}(m \to n) \in \mathcal{C}$ as the representing object for the functor $\mathcal{M}(- \triangleright m \to n)$. Moreover, internal hom is natural in each argument, giving an adjunction

$$\mathcal{C}(c \to \widehat{\mathcal{M}}(m \to n)) \cong \mathcal{M}(c \triangleright m \to n). \tag{5.8.5}$$

Construction 5.8.6. The assignment $\widehat{\mathcal{M}}(m \to n) \in \mathcal{C}$ to $m, n \in \mathcal{M}$ defines a $\mathcal{C}$-enriched category with the same objects as $\mathcal{M}$. The identity element $j_m \in \mathcal{C}(1_\mathcal{C} \to \widehat{\mathcal{M}}(m \to m))$ is the mate of $\lambda_m : 1_\mathcal{C} \triangleright m \to m$ under Adjunction (5.8.5). For each $m, n \in \mathcal{M}$, we have an *evaluation map*

$$\varepsilon_{m \to n} \in \mathcal{M}(\widehat{\mathcal{M}}(m \to n) \otimes m \to n) \underset{(5.8.5)}{\cong} \mathcal{C}(\widehat{\mathcal{M}}(m \to n) \to \widehat{\mathcal{M}}(m \to n))$$

which is the mate of the identity $\text{id}_{\widehat{\mathcal{M}}(m \to n)}$. We define the composition morphism $- \circ_{\widehat{\mathcal{M}}} - \in \mathcal{C}(\widehat{\mathcal{M}}(n \to p) \otimes \widehat{\mathcal{M}}(m \to n) \to \widehat{\mathcal{M}}(m \to p))$ as the mate of

$$\varepsilon_{n \to p} \cdot (\text{id}_{\widehat{\mathcal{M}}(n \to p)} \otimes \varepsilon_{m \to n}) \in \mathcal{M}(\widehat{\mathcal{M}}(n \to p) \otimes \widehat{\mathcal{M}}(m \to n) \triangleright m \to p)$$

under Adjunction (5.8.5).

Exercise 5.8.7. Verify the coherence axioms for the $\mathcal{C}$-enriched category $\widehat{\mathcal{M}}$ constructed above.

Remark 5.8.8. Construction 5.8.6 builds a $\mathcal{C}$-enriched category $\widehat{\mathcal{M}}$ from a left $\mathcal{C}$-module category. Conversely, given a particularly nice $\mathcal{C}$-enriched category which is *tensored* [Lin81, MPP18], we can build a left $\mathcal{C}$-module category. These two constructions are mutually inverse; indeed the 2-category of these nice tensored $\mathcal{C}$-enriched categories is equivalent to the 2-category of left $\mathcal{C}$-module categories, $\mathcal{C}$-module functors, and $\mathcal{C}$-module natural transformations. We refer the reader to [Lin81, MPP18, KYZZ21, Del21] for more details.

Exercise 5.8.9. Show that for every $m \in \mathcal{M}$, $\widehat{\mathcal{M}}(m \to m)$ has the structure of a unital algebra object in $\mathcal{C}$, and for every $n \in \mathcal{M}$, $\widehat{\mathcal{M}}(m \to n)$ is a right $\widehat{\mathcal{M}}(m \to m)$-module.

Exercise 5.8.10. Suppose $\mathcal{M}$ is a finitely semisimple left $\mathcal{C}$-module category.

(1) Find a canonical isomorphism $\widehat{\mathcal{M}}(m_1 \to c \triangleright m_2) \cong c \otimes \widehat{\mathcal{M}}(m_1 \to m_2)$.
(2) Use (1) to prove that for any $m \in \mathcal{M}$, $\widehat{\mathcal{M}}(m \to -) : \mathcal{M} \to \mathcal{C}$ is a left $\mathcal{C}$-module functor.
(3) Find a canonical isomorphism $\widehat{\mathcal{M}}(c \triangleright m_1 \to m_2) \cong \widehat{\mathcal{M}}(m_1 \to m_2) \otimes c^\vee$.

Lemma 5.8.11. *Suppose $\mathcal{M}$ is an indecomposable left $\mathcal{C}$-module category, i.e., $\mathcal{M}$ is not the direct sum of two non-zero $\mathcal{C}$-module categories. For every $m \in \mathcal{M}$, $A := \widehat{\mathcal{M}}(m \to m)$ is a unital separable algebra such that $\widehat{\mathcal{M}}(m \to -) : \mathcal{M} \to \mathcal{C}_A$ is an equivalence of left $\mathcal{C}$-module categories.*

Proof. Exercise 5.8.9 builds the unital algebra structure. Separability will follow showing $\widehat{\mathcal{M}}(m \to -) : \mathcal{M} \to \mathcal{C}_A$ is an equivalence by Proposition 5.7.2.

First, $\widehat{\mathcal{M}}(m \to -) : \mathcal{M} \to \mathcal{C}_A$ is well-defined by Exercise 5.8.9.

Second, since $\mathcal{M}$ is indecomposable, m *generates* $\mathcal{M}$ as a $\mathcal{C}$-module category, i.e., every object of $\mathcal{M}$ is a summand of a direct sum of objects of the form $c \triangleright m$. We claim this means $n \neq 0$ in $\mathcal{M}$ implies $\widehat{\mathcal{M}}(m \to n) \neq 0$ in $\mathcal{C}_A$. Indeed, by indecomposability, there is a $c \in \mathcal{C}$ such that $\mathcal{M}(c \triangleright m \to n) \neq 0$, which immediately implies that

$$\mathcal{C}(c \to \widehat{\mathcal{M}}(m \to n)) \cong \mathcal{M}(c \triangleright m \to n) \neq 0.$$

Third, since $\widehat{\mathcal{M}}(m \to -)$ is a linear functor which is non-zero on every simple object of $\mathcal{M}$, it is automatically faithful by [[]]. To see it is also full, we show it is full on objects of the form $c \triangleright m$. Indeed,

$$\begin{aligned}
\mathcal{C}_A(\widehat{\mathcal{M}}(m \to a \triangleright m) \to \widehat{\mathcal{M}}(m \to b \triangleright m)) &\underset{\text{(Exer. 5.8.10)}}{\cong} \mathcal{C}_A(a \otimes \widehat{\mathcal{M}}(m \to m) \to \widehat{\mathcal{M}}(m \to b \triangleright m)) \\
&\underset{\text{(Exer. 5.6.7)}}{\cong} \mathcal{C}(a \to \widehat{\mathcal{M}}(m \to b \triangleright m)) \\
&\underset{(5.8.5)}{\cong} \mathcal{M}(a \triangleright m \to b \triangleright m).
\end{aligned}$$

We thus see that $\widehat{\mathcal{M}}(m \to -)$ is full on the full subcategory $\mathcal{M}_0$ of $\mathcal{M}$ with objects of the form $c \triangleright m$, so it is full on all of $\mathcal{M}$ as $\mathcal{M}$ is the Cauchy completion of $\mathcal{M}_0$.

TODO: rewrite below: get a fully faithful linear functor from a semisimple category $\mathcal{M}$ to the Cauchy complete category $\mathcal{C}_A$ whose essential image contains enough projectives (every object of $\mathcal{C}_A$ is a quotient of a projective in the essential image). This implies the latter is semisimple. Put this in an abelian category section earlier on.

Finally, to see $\widehat{\mathcal{M}}(m \to -)$ is essentially surjective, pick $M_A \in \mathcal{C}_A$. By Exercise 5.6.9, M_A fits in a co-equalizer diagram

$$M \otimes A \otimes A \underset{\mathrm{id}_M \otimes m}{\overset{\rho_M \otimes A}{\rightrightarrows}} M \otimes A \xrightarrow{\rho_M} M. \tag{5.8.12}$$

Since $\widehat{\mathcal{M}}(m \to a \triangleright m) \cong a \otimes \widehat{\mathcal{M}}(m \to m) = a \otimes A$ for all $a \in \mathcal{C}$, by the last step, we have

$$\begin{aligned}
\mathcal{M}((M \otimes A) \triangleright m \to M \triangleright m) &\cong \mathcal{C}_A(\widehat{\mathcal{M}}(m \to (M \otimes A) \triangleright m) \to \widehat{\mathcal{M}}(m \to M \triangleright m)) \\
&\cong \mathcal{C}_A(M \otimes A \otimes \widehat{\mathcal{M}}(m \to m) \to M \otimes \widehat{\mathcal{M}}(m \to m)) \\
&= \mathcal{C}_A(M \otimes A \otimes A \to M \otimes A)
\end{aligned}$$

Since $\widehat{\mathcal{M}}(m \to -)$ is linear, it preserves kernels and cokernels by [[]]. Hence by transporting the co-equalizer (5.8.12) across the fully faithful functor $\widehat{\mathcal{M}}(m \to -)$, we see that M_A fits in a coequalizer diagram of free modules, and is thus in the essential image of $\widehat{\mathcal{M}}(m \to -)$. □

We now state Ostrik's Theorem and give a pedestrian proof following [Ost03]. (One can use the Barr-Beck Theorem to prove this as well [BZBJ18, §4].)

Theorem 5.8.13 (Ostrik's Theorem for multifusion categories)**.** *Let $\mathcal{C}$ be a multifusion category. The map $A \mapsto \mathcal{C}_A$ and ${}_AM_B \mapsto - \otimes_A M_B$ is a 2-equivalence $\mathsf{Alg}(\mathcal{C}) \to \mathsf{Mod}(\mathcal{C})$.*

Proof of Theorem 5.8.13. It is straightforward to verify the above map gives a 2-functor.

First, we check that for all unital $A-B$ bimodules ${}_AM_B, {}_AN_B$,

$${}_A\mathcal{C}_B(M \Rightarrow N) \ni \theta \mapsto - \otimes \theta \in \mathsf{Fun}_{\mathcal{C}-}(- \otimes_A M_B \Rightarrow - \otimes_A N_B)$$

is an isomorphism. Indeed, every $\mathcal{C}$-module natural transformation $\zeta : - \otimes_A M_B \Rightarrow - \otimes_A N_B$ is completely determined by ζ_A using (5.7.4) as $\mathcal{C}_A$ is the Cauchy completion of $\mathsf{FreeMod}_{\mathcal{C}}(A)$ by Exercise 5.6.9.

Thus to show our 2-functor is fully faithful, we need to prove the hom functors are essentially surjective. Suppose $\mathcal{F} : \mathcal{C}_A \to \mathcal{C}_B$ is a left $\mathcal{C}$-module functor. Then $\mathcal{F}(A) \in \mathcal{C}_B$ carries both a right B-action and a left A-action using the modulator: $\lambda_A := \mathcal{F}(m_A) \circ \mathcal{F}^2_{A,A}$. Setting ${}_AM_B := \mathcal{F}(A)$, it is straightforward to check that $\mathcal{F} \cong - \otimes_A M$.

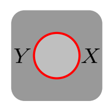
It remains to show the 2-functor is essentially surjective. To do this, we must show that every semisimple left $\mathcal{C}$-module category $\mathcal{M}$ is equivalent to $\mathcal{C}_A$ for some separable unital algebra A. We decompose $\mathcal{M}$ into indecomposable summands and apply Lemma 5.8.11 for each summand to conclude the proof. $\square$

5.9. **2-idempotent completion.** We now prove the universal property of 2-idempotent completion. The following is an adaptation of the proof from [CP22].

Definition 5.9.1. Suppose $\mathcal{C}$ is a 2-category and $a, b \in \mathcal{C}$. A *condensation* or *split surjection* $X : a \twoheadrightarrow b$ consists of 1-morphisms ${}_aX_b$ and ${}_bY_a$ and 2-morphisms $\epsilon : Y \otimes_a X \to 1_b$ and $\delta : 1_b \to Y \otimes_a X$ such that $\epsilon\delta = \mathrm{id}_{1_b}$. In diagrams: **TODO: bullets**

$\delta =$ 

and $\epsilon =$ 

such that 

$=$  $= \mathrm{id}_{1_b}$.

Definition 5.9.2. A 2-functor $F : \mathcal{C} \to \mathcal{E}$ is called

- *locally dominant* if every hom functor $F_{a\to b} : \mathcal{C}(a \to b) \to \mathcal{E}(F(a) \to F(b))$ is dominant as an ordinary functor,
- 0-*dominant* if for all $e \in \mathcal{E}$, there is a $c \in \mathcal{C}$ and a condensation $F(c) \twoheadrightarrow e$, and
- *dominant* if F is both locally dominant and 0-dominant.

Proposition 5.9.3. *Suppose $G : \mathcal{C} \to \mathcal{E}$ is a 2-functor and consider the 2-functor*

$$G^* = - \circ G : \mathsf{Fun}(\mathcal{E} \to \mathcal{D}) \to \mathsf{Fun}(\mathcal{C} \to \mathcal{D}).$$

If G is 0-dominant, then G^* is faithful on 2-morphisms. If G dominant, then G^* is fully faithful on 2-morphisms.

Proof. **TODO:** $\square$

Lemma 5.9.4. *The inclusion functor $\iota_{\mathcal{C}} : \mathcal{C} \hookrightarrow \mathsf{Alg}(\mathcal{C})$ is dominant.*

Proof. **TODO:** $\square$

Theorem 5.9.5. *Suppose $\mathcal{D}$ is a 2-idempotent complete 2-category. The 2-functor ι^* : $\mathsf{Fun}(\mathsf{Alg}(\mathcal{C}) \to \mathcal{D}) \to \mathsf{Fun}(\mathcal{C} \to \mathcal{D})$ is an equivalence.*

Proof. **TODO:** $\square$

5.10. **Yoneda for (linear) 2-categories.** Suppose $\mathcal{C}$ is a 2-category such that all hom 1-categories are finite semisimple categories. For each $c \in \mathcal{C}$, we get two representable 2-functors:

$$\mathcal{C}(c \to -) : \mathcal{C} \to 2\mathsf{Vec} \qquad \text{and} \qquad \mathcal{C}(- \to c) : \mathcal{C}^{1\,\mathrm{op}} \to 2\mathsf{Vec}.$$

Here, $\mathcal{C}^{1\,\mathrm{op}}$ means the 2-category with the same objects as $\mathcal{C}$, but $\mathcal{C}^{1\,\mathrm{op}}(a \to b) := \mathcal{C}(b \to a)$. The associators and unitors are defined similar to the definition of $\mathcal{D}^{\mathrm{mp}}$ for a monoidal category $\mathcal{D}$.

Lemma 5.10.1 (Yoneda)**.** *Given a 2-functor $F : \mathcal{C}^{1\,\mathrm{op}} \to 2\mathsf{Vec}$, evaluation at 1_c gives an equivalence of categories*

$$よ(c, F) : \mathrm{Hom}(\mathcal{C}(- \to c) \Rightarrow F) \cong F(c).$$

Moreover, $よ(c, F)$ is natural in both c and F.

Proof. **TODO:** □

TODO: Yoneda embedding, when is it an equivalence...

Corollary 5.10.2. *The Yoneda embedding 2-functor*

$$\begin{aligned} よ : \mathcal{C} &\hookrightarrow \mathsf{Fun}(\mathcal{C}^{1\,\mathrm{op}} \to 2\mathsf{Vec}) \\ c &\mapsto \mathcal{C}(- \to c) \end{aligned}$$

is fully faithful.

Proof. **TODO:** □

Definition 5.10.3. A 2-functor $F : \mathcal{C}^{1\,\mathrm{op}} \to 2\mathsf{Vec}$ is called *representable* if there is an $a \in \mathcal{C}$ and an invertible 2-transformation $\alpha : F \Rightarrow \mathcal{C}(- \to a)$. We call (a, α) a representing pair for F.

Proposition 5.10.4. *Representing pairs for F form a contractible space (when they exist).*

Proof. First, note that if (a, α) is a representing pair for F, then α^{-1} is uniquely defined up to a unique invertible 2-modification. As in the case for representable 1-functors, we thus get canonical invertible 2-transformations

$$\mathcal{C}(- \Rightarrow a) \xRightarrow{\alpha^{-1}} F \xRightarrow{\beta} \mathcal{C}(- \to b) \qquad \text{and} \qquad \mathcal{C}(- \Rightarrow b) \xRightarrow{\beta^{-1}} F \xRightarrow{\alpha} \mathcal{C}(- \to a).$$

By the Yoneda embedding, these 2-transformations must come from $\mathcal{C}(a \to b)$ and $\mathcal{C}(b \to a)$, i.e., there is an ${}_aX_b \in \mathcal{C}(a \to b)$ and an invertible 2-modification $m : \beta \circ \alpha^{-1} \Rrightarrow - \otimes_a X_b$, and similarly a ${}_bY_a \in \mathcal{C}(b \to a)$ and an invertible 2-modification $n : \alpha \circ \beta^{-1} \Rrightarrow - \otimes_b Y_a$.
TODO: □

5.11. **Semisimple 2-categories.** For 1-categories, semisimplicity is a purely algebraic property, but for 2-categories, semisimplicity also includes dualizability, which is more of a topological property.
TODO:

Definition 5.11.1. A linear 2-category is called *semisimple* if it is locally finite semisimple, additive, and idempotent complete such that all 1-morphisms admit left and right adjoints.

Proposition 5.11.2. *An additive, 2-idempotent complete linear 2-category is semisimple if and only if every endomorphism multifusion category* $\mathrm{End}_{\mathcal{C}}(c)$ *is a multifusion category for each* $c \in \mathcal{C}$.

Proof. **TODO:** □

TODO: bound on centers not what we know how to do now

Definition 5.11.3. A semisimple 2-category $\mathcal{C}$ is called *finite* if there is a global bound $K > 0$ such that for every $c \in \mathcal{C}$, the multifusion category $\mathrm{End}_{\mathcal{C}}(c)$ has at most K indecomposable summands.

Theorem 5.11.4. *A 2-category* $\mathcal{C}$ *is finite semisimple if and only it is equivalent to* $\mathsf{Mod}(\mathcal{F})$ *for a multifusion category* $\mathcal{F}$.

Proof. **TODO:** □

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