TODO: lead in

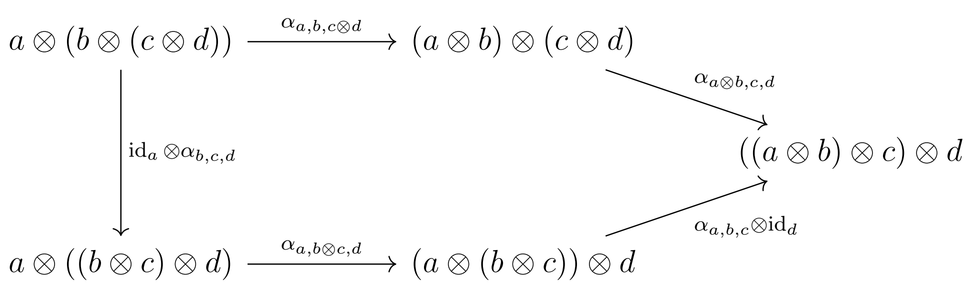
3.1. Monoidal categories.

Definition 3.1.1. A *monoidal category* is a category $\mathcal{C}$ together with the following additional data:

- A functor $-\otimes- : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$,
- A distinguished object $1_{\mathcal{C}}\in\mathcal{C}$,
- *associator* isomorphisms $\alpha_{a,b,c} : a\otimes(b\otimes c)\xrightarrow{\cong}(a\otimes b)\otimes c$ for all $a,b,c\in\mathcal{C}$, separately natural in all components, and
- *unitor* natural isomorphisms $\lambda_a : 1_{\mathcal{C}}\otimes a\xrightarrow{\cong} a$ and $\rho_a : a\otimes 1_{\mathcal{C}}\xrightarrow{\cong} a$ for all $a\in\mathcal{C}$,

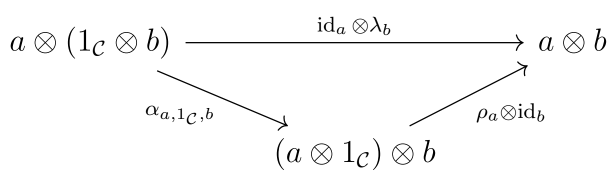
and this data must satisfy the following axioms:

- (pentagon) for all $a,b,c,d\in\mathcal{C}$, the following diagram commutes:



(⬠)

- (triangle) for all $a,b\in\mathcal{C}$, the following diagram commutes:



(△)

A monoidal category $(\mathcal{C},\otimes,1_{\mathcal{C}},\alpha,\lambda,\rho)$ is called *strict* if for every $a,b,c\in\mathcal{C}$, $a\otimes(b\otimes c)=(a\otimes b)\otimes c$ and $1_{\mathcal{C}}\otimes a=a\otimes 1_{\mathcal{C}}=a$, and the natural isomorphisms $\alpha_{a,b,c}$, λ_a, and ρ_a are all identity morphisms.

A *linear monoidal category* is a linear category with a monoidal structure such that $\otimes : \mathcal{C}\times\mathcal{C}\to\mathcal{C}$ is bilinear.

Example 3.1.2. For G a finite group, we endow $\mathsf{Vec}(G)$ with the structure of a tensor category by

$$(V\otimes W)_g := \bigoplus_{hk=g} V_h\otimes W_k$$

where the associator just moves parentheses.

Example 3.1.3. Let G be a finite group. A *3-cocycle* $\omega\in Z^3(G;\mathbb{C}^\times)$ is a function $\omega : G\times G\times G\to\mathbb{C}^\times$ such that for all $g,h,k\in G$,

$$\omega(h,k,\ell)\omega(gh,k,\ell)^{-1}\omega(g,hk,\ell)\omega(g,h,k\ell)^{-1}\omega(g,h,k)=1.$$

The linear monoidal category $\mathsf{Vec}(G,\omega)$ has the same underlying linear category and monoidal product as $\mathsf{Vec}(G)$, but we twist the associator and unitors by $\alpha_{g,h,k} := \omega(g,h,k)\,\mathrm{id}_{ghk}$, $\lambda_g := \omega(g,1,1)\,\mathrm{id}_g$, and $\rho_g := \omega(1,1,g)^{-1}$. (Prove that the pentagon (⬠) and triangle (△) axioms hold!)

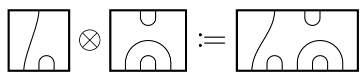
Example 3.1.4. We endow $\mathsf{Rep}(G)$ with the structure of a monoidal category by

$$(V,\pi)\otimes(W,\rho) := (V\otimes W, \pi\otimes\rho),$$

where again the associator just moves parentheses.

Example 3.1.5. If $\mathcal{C}$ is a category, then $\mathrm{End}(\mathcal{C}) = \mathsf{Fun}(\mathcal{C}\to\mathcal{C})$ is a strict monoidal category with tensor product given by composition of functors.

Example 3.1.6. We endow $\mathsf{TLJ}(d)$ from the previous chapter with the structure of a strict tensor category as follows. On objects, we define $m\otimes n := m+n$. For string diagrams $x\in\mathsf{TLJ}(d)(m\to n)$ and $y\in\mathsf{TLJ}(d)(p\to q)$, we define $x\otimes y\in\mathsf{TLJ}(d)(m+p\to n+q)$ to be the horizontal concatenation of x and y. For an explicit example,



We then extend $-\otimes- : \mathsf{TLJ}(d)\times\mathsf{TLJ}(d)\to\mathsf{TLJ}(d)$ bilinearly in each argument. **TODO: more detail**

Exercise 3.1.7 (Exchange relation)**.** Suppose $\mathcal{C}$ is a monoidal category and $f\in\mathcal{C}(a\to c)$ and $g\in\mathcal{C}(b\to d)$. Prove that

$$(f\otimes\mathrm{id}_d)\cdot(\mathrm{id}_a\otimes g) = (\mathrm{id}_b\otimes g)\cdot(f\otimes\mathrm{id}_b) \tag{3.1.8}$$

Facts 3.1.9. In a monoidal category,

- We may view the unitors λ and ρ as natural isomorphisms $\lambda : 1_{\mathcal{C}}\otimes - \Rightarrow \mathrm{id}_{\mathcal{C}}$ and $\rho : -\otimes 1_{\mathcal{C}} \Rightarrow \mathrm{id}_{\mathcal{C}}$ respectively, witnessing that $1_{\mathcal{C}}\otimes -$ and $-\otimes 1_{\mathcal{C}}$ are equivalences.
- We always have $\lambda_{1_{\mathcal{C}}} = \rho_{1_{\mathcal{C}}} : 1_{\mathcal{C}}\otimes 1_{\mathcal{C}}\to 1_{\mathcal{C}}$. (This is a rather tricky exercise; see [HV19, Ex. 1.13].)
- The existence of $(1_{\mathcal{C}},\lambda,\rho)$ is a *property* and not additional structure. **TODO: get this right**

Proposition 3.1.10. *In a monoidal category,* $\mathrm{End}_{\mathcal{C}}(1_{\mathcal{C}})$ *is a commutative moniod.*

Proof. Observe that the following diagram commutes for all $f,g : 1_{\mathcal{C}}\to 1_{\mathcal{C}}$:

$$\begin{array}{ccccccc}
1_{\mathcal{C}} & \xrightarrow{\lambda_{1_{\mathcal{C}}}^{-1}} & 1_{\mathcal{C}}\otimes 1_{\mathcal{C}} & = & 1_{\mathcal{C}}\otimes 1_{\mathcal{C}} & \xrightarrow{\lambda_{1_{\mathcal{C}}}} & 1_{\mathcal{C}} \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle \mathrm{id}_{1_{\mathcal{C}}}\otimes f} & & \downarrow{\scriptstyle \mathrm{id}_{1_{\mathcal{C}}}\otimes g} & & \downarrow{\scriptstyle g} \\
1_{\mathcal{C}} & \xrightarrow{\lambda_{1_{\mathcal{C}}}^{-1}=\rho_{1_{\mathcal{C}}}^{-1}} & 1_{\mathcal{C}}\otimes 1_{\mathcal{C}} & (3.1.8) & 1_{\mathcal{C}}\otimes 1_{\mathcal{C}} & \xrightarrow{\lambda_{1_{\mathcal{C}}}=\rho_{1_{\mathcal{C}}}} & 1_{\mathcal{C}} \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle g\otimes\mathrm{id}_{1_{\mathcal{C}}}} & & \downarrow{\scriptstyle f\otimes\mathrm{id}_{1_{\mathcal{C}}}} & & \downarrow{\scriptstyle f} \\
1_{\mathcal{C}} & \xrightarrow{\rho_{1_{\mathcal{C}}}^{-1}} & 1_{\mathcal{C}}\otimes 1_{\mathcal{C}} & = & 1_{\mathcal{C}}\otimes 1_{\mathcal{C}} & \xrightarrow{\rho_{1_{\mathcal{C}}}} & 1_{\mathcal{C}}
\end{array}$$

Going right and then down is $f\cdot g$, and going down and then right is $g\cdot f$. □

Exercise 3.1.11. Consider the category $\mathrm{B}M$ for a monoid M. Show that $\mathrm{B}M$ can be given a tensor product if and only if M is commutative.

Definition 3.1.12. A *monoidal functor* $F : \mathcal{C} \to \mathcal{D}$ between monoidal categories is a functor $F : \mathcal{C} \to \mathcal{D}$ together with a family of *tensorator* isomorphisms $F^2_{a,b} : F(a) \otimes F(b) \to F(a \otimes b)$ and a *unitor* isomorphism $F^0 : 1_{\mathcal{D}} \to F(1_{\mathcal{C}})$ satisfying the following axioms:

- (associative) for all $a, b, c \in \mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccccc} F(a) \otimes (F(b) \otimes F(c)) & \xrightarrow{\mathrm{id}_{F(a)} \otimes F^2_{b,c}} & F(a) \otimes F(b \otimes c) & \xrightarrow{F^2_{a,b\otimes c}} & F(a \otimes (b \otimes c)) \\ \downarrow \alpha^{\mathcal{D}}_{F(a),F(b),F(c)} & & & & \downarrow F(\alpha^{\mathcal{C}}_{a,b,c}) \\ (F(a) \otimes F(b)) \otimes F(c) & \xrightarrow{F^2_{a,b} \otimes \mathrm{id}_{F(c)}} & F(a \otimes b) \otimes F(c) & \xrightarrow{F^2_{a\otimes b,c}} & F((a \otimes b) \otimes c) \end{array}$$

- (unital) for all $c \in \mathcal{C}$, the following diagrams commute:

$$\begin{array}{ccc} 1_{\mathcal{D}} \otimes F(c) & \xrightarrow{\lambda^{\mathcal{D}}_{F(c)}} & F(c) \\ \downarrow F^0 \otimes \mathrm{id}_{F(c)} & & \uparrow F(\lambda^{\mathcal{C}}_c) \\ F(1_{\mathcal{C}}) \otimes F(c) & \xrightarrow{F^2_{1_{\mathcal{C}},c}} & F(1_{\mathcal{C}} \otimes c) \end{array} \qquad \begin{array}{ccc} F(c) \otimes 1_{\mathcal{D}} & \xrightarrow{\rho^{\mathcal{D}}_{F(c)}} & F(c) \\ \downarrow \mathrm{id}_{F(c)} \otimes F^0 & & \uparrow F(\rho^{\mathcal{C}}_c) \\ F(c) \otimes F(1_{\mathcal{C}}) & \xrightarrow{F^2_{c,1_{\mathcal{C}}}} & F(c \otimes 1_{\mathcal{C}}) \end{array}$$

We call F *strict* if $F^2_{a,b}$ and F^0 are all identities.

A *monoidal natural transformation* $\theta : F \Rightarrow G$ between monoidal functors is a natural transformation satisfying the following axioms:

- (unital) The following diagram commutes:

$$\begin{array}{ccc} & 1_{\mathcal{D}} & \\ {}^{F^0}\swarrow & & \searrow^{G^0} \\ F(1_{\mathcal{C}}) & \xrightarrow{\theta_{1_{\mathcal{C}}}} & G(1_{\mathcal{C}}) \end{array} \tag{3.1.13}$$

  Observe that this means $\theta_{1_{\mathcal{C}}} = G^0 \cdot (F^0)^{-1}$ is completely determined and invertible.

- (monoidal) For all $a, b \in \mathcal{C}$, the following diagram commutes:

$$\begin{array}{ccc} F(a) \otimes F(b) & \xrightarrow{\theta_a \otimes \theta_b} & G(a) \otimes G(b) \\ \downarrow F^2_{a,b} & & \downarrow G^2_{a,b} \\ F(a \otimes b) & \xrightarrow{\theta_{a \otimes b}} & G(a \otimes b) \end{array} \tag{3.1.14}$$

Two monoidal categories $\mathcal{C}, \mathcal{D}$ are *equivalent* if there are monoidal functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ together with monoidal natural isomorphisms $F \circ G \cong \mathrm{id}_{\mathcal{D}}$ and $G \circ F \cong \mathrm{id}_{\mathcal{C}}$.

Exercise 3.1.15. Show that a monoidal functor $F : \mathcal{C} \to \mathcal{D}$ whose underlying functor is an equivalence of categories can be augmented to an equivalence of monoidal categories.

Theorem 3.1.16 (MacLane Coherence)**.** *Every monoidal category $\mathcal{C}$ is equivalent to a strict monoidal category.*

Proof. For each $a \in \mathcal{C}$, we have a functor $a \otimes - : \mathcal{C} \to \mathcal{C}$ by $c \mapsto a \otimes c$. We get a monoidal functor $\mathcal{C} \to \mathrm{End}(\mathcal{C})$ by $a \mapsto a \otimes -$, where the tensorator is given by $F^2_{a,b} := \alpha_{a,b,-} : a \otimes (b \otimes -) \Rightarrow (a \otimes b) \otimes -$. We see that $\mathcal{C}$ is equivalent as a monoidal category to the monoidal subcategory of $\mathrm{End}(\mathcal{C})$ whose objects are the functors $a \otimes -$ and whose morphisms are those coming from $\mathcal{C}$. This latter category is strict. □

Remark 3.1.17. We will see later on that if we view $a \otimes -$ as a *right $\mathcal{C}$-module functor*, then the above proof gives a monoidal equivalence $\mathcal{C} \cong \operatorname{End}(\mathcal{C}_{\mathcal{C}})$.

Example 3.1.18. Suppose $\mathcal{C}$ is a linear monoidal category. The additive envelope $\mathsf{Add}(\mathcal{C})$ can be endowed with a monoidal structure by $(a_i)_{i=1}^m \otimes (b_j)_{j=1}^n = (a_i \otimes b_j)_{(i,j)=(1,1)}^{(m,n)}$ using the lexicographic ordering, and similarly for morphisms. The associator is given by $(\alpha_{(a_i),(b_j),(c_k)})_{i,j,k} := \alpha_{a_i,b_j,c_k}$. The unit is $(1_{\mathcal{C}})$, and the unitors are given by $(\lambda_{(a_i)})_i = \lambda_{a_i}$ and $(\rho_{(a_i)})_i = \rho_{a_i}$.

The canonical inclusion $\iota : \mathcal{C} \to \mathsf{Add}(\mathcal{C})$ is strict monoidal, and the universal property also holds for monoidal functors, i.e., For every linear monoidal category $\mathcal{D}$ which admits direct sums, precomposition with $\iota : \mathcal{C} \hookrightarrow \mathsf{Add}(\mathcal{C})$ gives an equivalence of categories

$$\mathsf{Fun}_{\otimes}(\mathsf{Add}(\mathcal{C}) \to \mathcal{D}) \xrightarrow{\iota^*} \mathsf{Fun}_{\otimes}(\mathcal{C} \to \mathcal{D}).$$

Example 3.1.19. Suppose $\mathcal{C}$ is a monoidal category. The idempotent completion $\mathsf{Idem}(\mathcal{C})$ can be endowed with a monoidal structure by $(a,e) \otimes (b,f) := (a \otimes d, b \otimes f)$, and similarly for morphisms. The associator is given by

$$\alpha_{(a,e),(b,f),(c,g)} := ((e \otimes f) \otimes g)\alpha_{a,b,c} = \alpha_{a,b,c} \cdot (e \otimes (f \otimes g)).$$

The unit is $(1_{\mathcal{C}}, \mathrm{id}_{1_{\mathcal{C}}})$, and the unitors are given by $\lambda_{(c,e)} := e \cdot \lambda_c = \lambda_c \cdot (1_{\mathcal{C}} \otimes e)$ and $\rho_{(c,e)} := e \cdot \rho_c = \rho_c \cdot (e \otimes 1_{\mathcal{C}})$.

Again, the canonical inclusion is strict monoidal and satisfies the obvious universal property.

Example 3.1.20. Suppose $\mathcal{C}$ is a linear monoidal category. the Cauchy completion $¢(\mathcal{C}) = \mathsf{Idem}(\mathsf{Add}(\mathcal{C})$ is also monoidal by combining the previous two examples. As usual, the canonical inclusion is strict monoidal and satisfies the obvious universal property.

Exercise 3.1.21. A category is called *skeletal* if whenever two objects are isomorphic, they are in fact equal. Show that every (monoidal) category is equivalent to a skeletal (monoidal) category.

Remark 3.1.22. For a general linear monoidal category $\mathcal{C}$, we can ask for at most 2 out of 3 of the properties strict, skeletal, and Cauchy complete.

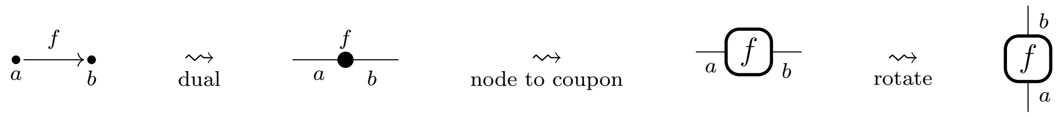
Exercise 3.1.23. For a monoidal category $\mathcal{C}$, there are three notions of opposite one might take:

- The category $\mathcal{C}^{\mathrm{op}}$ is the arrow opposite.
- The category $\mathcal{C}^{\mathrm{mp}}$ is the *monoidal* opposite, where $a \otimes_{\mathrm{mp}} b := b \otimes a$. The associator is given by $\alpha^{\mathrm{mp}}_{a,b,c} = \alpha^{-1}_{c,b,a}$.
- The category $\mathcal{C}^{\mathrm{mop}}$ is both the arrow and monoidal opposite.

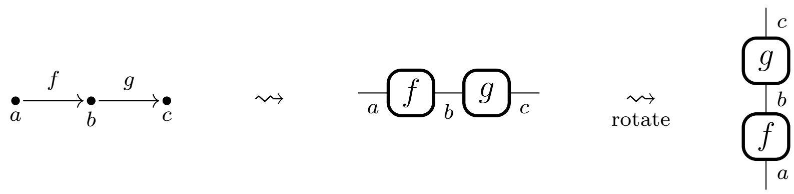
Show that all three of these notions of opposite give monoidal categories.

3.2. **Graphical calculus for monoidal categories and dualizability.** The graphical calculus is a powerful and elegant formalism for calculations in a monoidal category. Essentially, string diagrams are *dual* to pasting diagrams. We first illustrate for ordinary categories, after which we illustrate for monoidal categories.

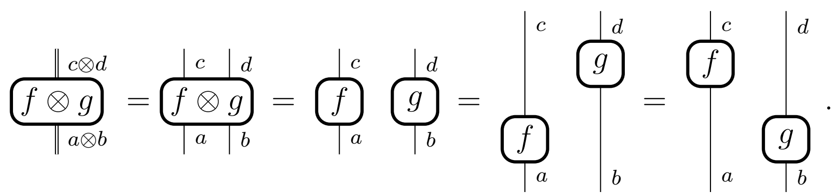
Morphisms are often first viewed as 1D arrows between 0D objects.



In the string diagrammatic calculus, we dualize these pictures, representing the objects as 1D strings and morphisms as 0D coupons. Strings are always *oriented*, but by convention, we will suppress this orientation using the convention that we always read either *left to right* or *bottom to top*. Composition is given by stacking coupons.

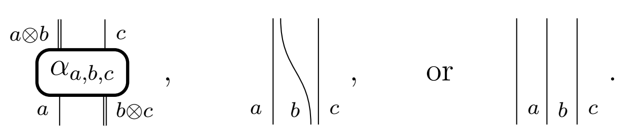


In a monoidal category, we represent identity morphisms as strands without coupons, and we represent the monoidal product by horizontal juxtaposition. Observe that the exchange relation means we can perform vertical isotopy without changing the morphism. Typically we will write a doubled strand to indicate a monoidal product of two objects. For example, for $f \in \mathcal{C}(a \to c)$ and $g \in \mathcal{C}(b \to d)$,

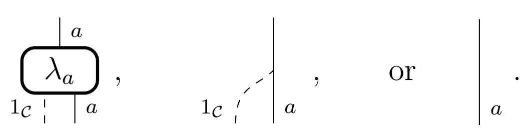


Remark 3.2.1. Above, we saw that we can think of a monoidal category as a 2-category with one object via delooping. Thus dualizing a 2D pasting diagram yields a 2D string diagram, where 0-morphisms are presented by regions, 1-morphisms are represented by 1D strands, and 2-morphisms are represented by 0D coupons. Since we have only one object, we have only one region, which is unlabelled. When we generalize the graphical calculus to 2-categories, our regions will obtain shadings.

Associators can then be represented by: coupons with three strands, three strands with no coupons where we change the horizontal position of the middle strand, or my personal favorite: omitted entirely!



Similarly, we may represent a unitor by coupon with a dashed strand labelled for $1_\mathcal{C}$, a dashed strand which terminates on another strand, or we may simply choose to never draw the unit object $1_\mathcal{C}$ and suppress all unitors, e.g.,

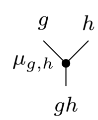


While the reader may worry that omitting associators and unitors could lead to some kind of problem, in fact, it is not by the following theorem.

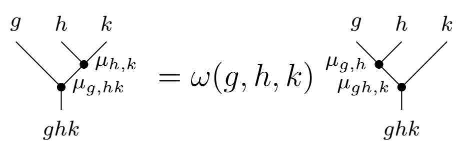
Theorem 3.2.2 (Correctness and completeness of graphical calculus [HV19, Thm. 1.8])**.** *A well-typed equation between morphisms in a monoidal category follows from the axioms if and only if it holds in the graphical calculus up to isotopy. (Here, isotopy must fix lower and upper boundaries, and may not create any local maxima/minima of strands.)*

Basically, this theorem tells us that every string diagram can be interpreted in *infinitely many* ways as an algebraic expression in the morphisms of $\mathcal{C}$ (e.g., we may put in an arbitrary number of copies of $1_{\mathcal{C}}$, and an arbitrary number of α and α^{-1} which cancel), but given particular parenthesizations for the source and target objects, any one of these infinite algebraic expressions yields the same morphism in $\mathcal{C}$. Using this convention, we will suppress the associator whenever possible; the reader who is uncomfortable with this is welcome to work in an equivalent strict monoidal category.

Example 3.2.3. In this example, we give a strict diagrammatic presentation of $\mathsf{Vec}(G, \omega)$ where $\omega \in Z^3(\mathbb{C}^\times)$. First, for each pair $g, h \in G$, pick an isomorphism $\mu_{g,h} : \mathbb{C}_{gh} \to \mathbb{C}_g \otimes \mathbb{C}_h$, which we denote by a trivalent vertex. To ease the notation, we denote $\mathbb{C}_g$ by g.



Since $\mathrm{Hom}(g \otimes (h \otimes k) \to (g \otimes h) \otimes k))$ is 1-dimensional, the left and right associated tree diagrams must be equal up to a non-zero scalar $\omega(g, h, k) \in \mathbb{C}$; this is exactly the suppressed associator in the graphical calculus for the non-strict model of $\mathsf{Vec}(G, \omega)$.



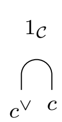
Observe now that any other choice of isomorphism $\mu'_{g,h}$ differs from $\mu_{g,h}$ by a non-zero scalar, say $\mu'_{g,h} = \beta(g, h)\mu_{g,h}$. We then see that replacing the μ with μ' changes the scalar $\omega(g, h, k)$ by multiplying by

$$\beta(h, k)^{-1}\beta(gh, k)\beta(g, hk)^{-1}\beta(g, h) = \frac{1}{(d\beta)(g, h, k)}.$$

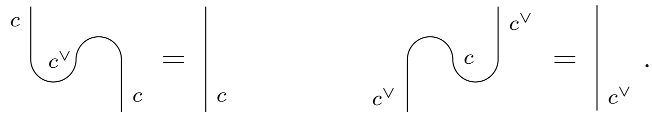
This has the effect of multiplying ω by a 3-coboundary, resulting in a cohomologous cocycle.

Definition 3.2.4. Suppose $\mathcal{C}$ is a monoidal category. A *dual* of an object $c \in \mathcal{C}$ consists of:

- an object $c^\vee \in \mathcal{C}$ called *a dual* of c, and
- *evaluation* and *coevaluation* morphisms $\mathrm{ev}_c : c^\vee \otimes c \to 1_{\mathcal{C}}$ and $\mathrm{coev}_c : 1_{\mathcal{C}} \to c \otimes c^\vee$ which we represent graphically by a cap and a cup respectively

$\mathrm{ev}_c =$  $\mathrm{coev}_c =$ 

satisfying the *snake/zig-zag equations*, which are best expressed graphically:



(3.2.5)

(As this is the first time we have done so, we remark that the diagrams in (3.2.5) suppress associators and unitors. We leave it as an exercise to the reader to write the corresponding pasting diagrams. We will not make this kind of remark below.)

The object $c \in \mathcal{C}$ is called *dualizable* if c has a dual, and there is a *predual* object $c_\vee \in \mathcal{C}$ which has a dual $(c_\vee)^\vee$ such that there exists an isomorphism $c \cong (c_\vee)^\vee$.

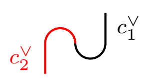
We say $\mathcal{C}$ *has duals* or is *rigid* if every object is dualizable.

Exercise 3.2.6. Consider the linear monoidal category Vec_∞ of (possibly infinite dimensional) complex vector spaces. Prove that $V \in \mathsf{Vec}_\infty$ is dualizable if and only if V is finite dimensional. Repeat this exercise for Hilb_∞.

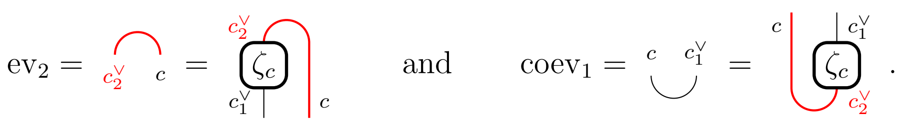
Facts 3.2.7. Here are some elementary properties about duals in a monoidal category $\mathcal{C}$.

($\vee$1) Suppose $c \in \mathcal{C}$ is dualizable. If $\mathrm{coev}'_c : 1_\mathcal{C} \to c \otimes c^\vee$ satisfies (3.2.5), then $\mathrm{coev}'_c = \mathrm{coev}_c$. In this sense, we say coev_c is completely determined by ev_c. Similarly, ev_c is completely determined by coev_c.

($\vee$2) Duals are unique up to canonical isomorphism; that is, the space of duals of an object $c \in \mathcal{C}$ is contractible. Indeed, by the previous property, if $c \in \mathcal{C}$ with duals $(c_i^\vee, \mathrm{coev}_i, \mathrm{ev}_i)$ for $i = 1, 2$, the isomorphism

$\zeta_c :=$  $= (\mathrm{ev}_2 \otimes \mathrm{id}_{c^\vee{}_1}) \cdot (\mathrm{id}_{c^\vee{}_2} \otimes \mathrm{coev}_1)$

is the unique isomorphism $c_2^\vee \to c_1^\vee$ satisfying

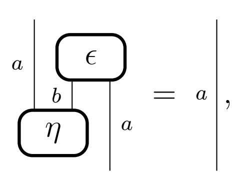


($\vee$3) (Frobenius reciprocity) If $b \in \mathcal{C}$ is dualizable, there are canonical isomorphisms

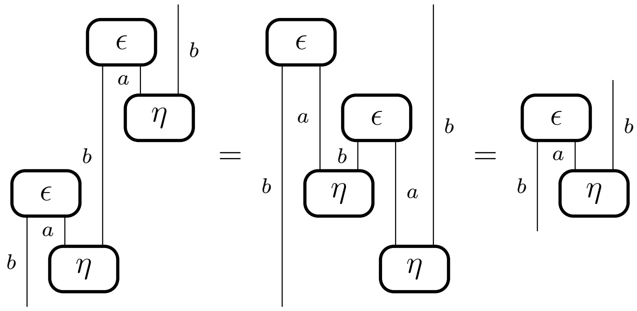
$$\mathcal{C}(a \otimes b \to c) \cong \mathcal{C}(a \to c \otimes b^\vee) \qquad \text{and} \qquad \mathcal{C}(a \to b \otimes c) \cong \mathcal{C}(b^\vee \otimes a \to c) \tag{3.2.8}$$

which are natural in both a and c.

($\vee$4) If $a, b \in \mathcal{C}$, and $\eta : 1_\mathcal{C} \to a \otimes b$ and $\epsilon : b \otimes a \to 1_\mathcal{C}$ satisfy one zig-zag relation, say



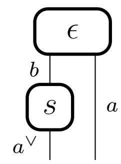
then the other zig-zag is an idempotent. Indeed,

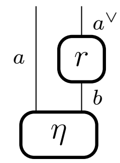


.

Additionally,

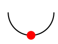
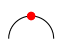
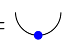
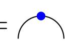
- If η, ϵ are isomorphisms, then so is the above zig-zag (what is its inverse?). Since the only invertible idempotent is the identity, the above zig-zag must equal id_b.
- If $\mathcal{C}$ is idempotent complete, a splitting $(a^\vee, r, s)$ for this other zig-zag idempotent gives a dual with

$\mathrm{ev}_a :=$ 

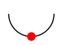
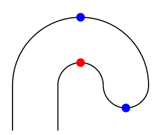
and $\mathrm{coev}_a :=$ 

.

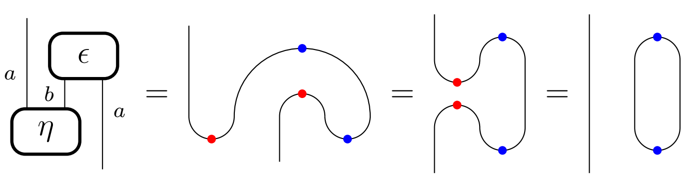
(∨5) (Adjoint equivalences) We call an object $a \in \mathcal{C}$ *invertible* if there is an object $b \in \mathcal{C}$ and isomorphisms $\eta_0 : 1_\mathcal{C} \to a \otimes b$ and $\epsilon_0 : b \otimes a \to 1_\mathcal{C}$. We abbreviate

$\eta_0 =$ , $\eta_0^{-1} =$ , $\epsilon_0^{-1} =$ , and $\epsilon_0 =$ .

Setting

$\eta := \eta_0 =$  and $\epsilon :=$ 

we see that η and ϵ are isomorphisms satisfying one zig-zag:



$= \mathrm{id}_a$.

Hence by the first bullet point in (∨4), η, ϵ also satisfy the other zig-zag. The tuple (a, b, η, ϵ) is called an *adjoint equivalence* for reasons which will be made clear in [[]] in the next chapter.

(∨6) If $F : \mathcal{C} \to \mathcal{D}$ is a monoidal functor and $c \in \mathcal{C}$ is dualizable, then so is $F(c)$ with dual $F(c^\vee)$ and evaluation and coevaluation given by

$$\mathrm{ev} : F(c^\vee) \otimes F(c) \xrightarrow{F^2_{c^\vee, c}} F(c^\vee \otimes c) \xrightarrow{F(\mathrm{ev}_c)} 1_\mathcal{C}$$
$$\mathrm{coev} : 1_\mathcal{C} \xrightarrow{F(\mathrm{coev}_c)} F(c \otimes c^\vee) \xrightarrow{(F^2_{c,c^\vee})^{-1}} F(c) \otimes F(c^\vee).$$

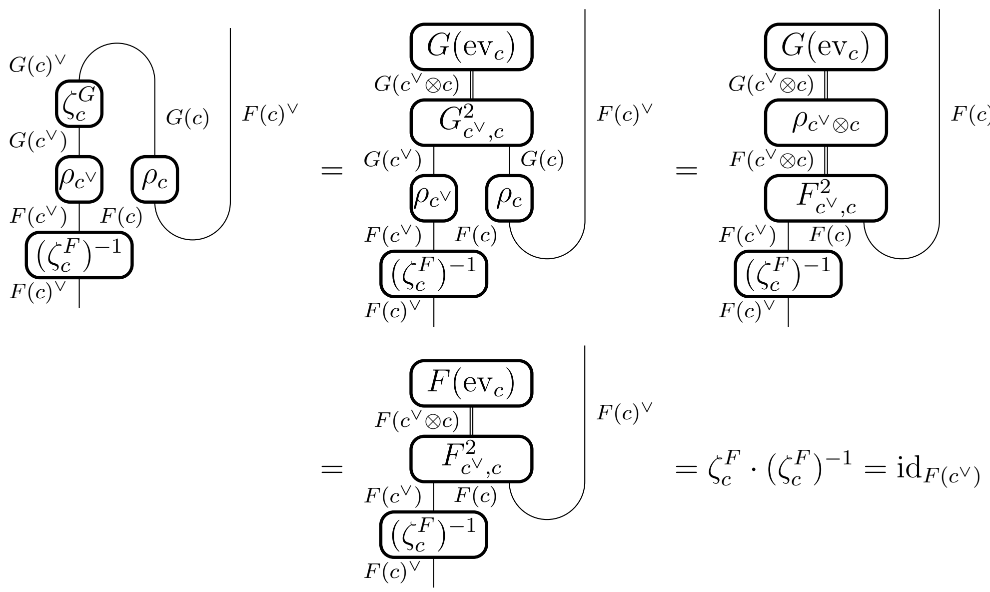
Thus if $F(c)^\vee$ is any other dual of $F(c) \in \mathcal{D}$, the unique isomorphism $F(c^\vee) \to F(c)^\vee$ compatible with the (co)evaluations is given by

$$\zeta_c : F(c^\vee) \xrightarrow{\cong} F(c^\vee) \otimes 1_{\mathcal{C}} \xrightarrow{\mathrm{id} \otimes \mathrm{coev}_{F(c)}} F(c^\vee) \otimes F(c) \otimes F(c)^\vee \xrightarrow{F^2_{c^\vee,c} \otimes \mathrm{id}} 1_{\mathcal{C}} \otimes F(c)^\vee \xrightarrow{\cong} F(c)^\vee. \tag{3.2.9}$$

($\vee$7) If $\mathcal{C}$ is rigid and $F, G : \mathcal{C} \to \mathcal{D}$ are two monoidal functors, then every monoidal natural transformation $\rho : F \Rightarrow G$ is invertible. More precisely, if $c \in \mathcal{C}$ is dualizable, then $\rho_{c^\vee}$ is invertible with inverse $(\rho_c)^\vee$, up to the canonical isomorphism (3.2.9), i.e., the following diagram commutes.

$$\begin{array}{ccc} F(c^\vee) & \xrightarrow{\rho_{c^\vee}} & G(c^\vee) \\ \downarrow{\scriptstyle \zeta_c^F} & & \downarrow{\scriptstyle \zeta_c^G} \\ F(c)^\vee & \xleftarrow{\rho_c^\vee} & G(c)^\vee \end{array}$$

One uses monoidality and naturality of ρ in the following string diagrammatic proof.



(While one might think the naturality axiom for a monoidal natural isomorphism should leave a floating coupon labelled $\rho_{1_{\mathcal{C}}} = G^0 \cdot (F^0)^{-1}$ by (3.1.13), the diagrammatic calculus suppresses a $(G^0)^{-1}$ at the top after the $G(\mathrm{ev}_c)$ coupon on the first line, and the $(F^0)^{-1}$ is then left over to be suppressed on the top of the $F(\mathrm{ev}_c)$ at the end of the calculation.)

Composition in the other direction is treated similarly.

Exercise 3.2.10. Give a direct proof that η, ϵ satisfy the other zig-zag in ($\vee$5).

3.3. **Semisimple multitensor categories.**

Definition 3.3.1. A *multitensor category* is a semisimple[1] monoidal category in which all objects are dualizable. A multitensor category is called a *tensor category* if $1_{\mathcal{C}}$ is simple, i.e., $\dim(\mathrm{End}(1_{\mathcal{C}})) = 1$.

[1]In [EGNO15], semisimple is replaced with locally finite abelian.

A *multifusion category* is a finite multitensor category. A multifusion category is called a *fusion category* if $1_\mathcal{C}$ is simple.

	finite	infinite
$1_\mathcal{C}$ simple	fusion	tensor
$1_\mathcal{C}$ not simple	multifusion	multitensor

Remark 3.3.2. Multitensor categories admit a graphical calculus where strings may have local maxima and minima, and we may perform isotopies which create or annihilate finitely many of such minima.

As we saw in [[]], a finitely semisimple category $\mathcal{C}$ is equivalent as a category to $\mathsf{Vec}^{\oplus r}$, where r is the *rank* of $\mathcal{C}$, which is the number of isomorphism classes of simple objects. Thus the only interesting part of a fusion category is the tensor product, which tells us how simple objects (thought of as *elementary particles*) merge and split.

Definition 3.3.3 (Fusion rules)**.** Suppose $\mathcal{C}$ is a semisimple linear monoidal category, and let $\mathrm{Irr}(\mathcal{C})$ be a set of representatives for the simple objects of $\mathcal{C}$ which includes $1_\mathcal{C}$. For $a, b, c \in \mathrm{Irr}(\mathcal{C})$, we define the *fusion rules* as the non-negative integers

$$N_{ab}^c := \dim(\mathrm{Hom}(a \otimes b \to c)) = \dim(\mathrm{Hom}(c \to a \otimes b)).$$

The equality above follows by semisimplicity; indeed, by the Yoneda Lemma [[]], we have an isomorphism

$$a \otimes b \xrightarrow{\upsilon_{a\otimes b}} \bigoplus_{c\in\mathrm{Irr}(\mathcal{C})} \mathcal{C}(a \otimes b \to c) \otimes c \cong \bigoplus_{c\in\mathrm{Irr}(\mathcal{C})} c^{\oplus N_{a,b}^c}.$$

The *fusion graph* Γ_a of $a \in \mathcal{C}$ has vertices the set $\mathrm{Irr}(\mathcal{C})$ and $\dim(\mathrm{Hom}(a \otimes b \to c))$ oriented edges between the vertices $b, c \in \mathrm{Irr}(\mathcal{C})$.

We call $\mathcal{C}$ *multiplicity free* if $N_{ab}^c \in \{0, 1\}$ for all $a, b, c \in \mathrm{Irr}(\mathcal{C})$.

Exercise 3.3.4. Suppose G is a finite group.

- Show that the fusion graph of $g \in \mathsf{Vec}(G)$ is a dijoint union of $|g|$-cycles.
- Show that the fusion graph of $\mathbb{C}[G] \in \mathsf{Vec}(G)$ is the graph with vertices labelled by $g \in G$ and one edge from g to h for every $g, h \in G$.

Exercise 3.3.5. Consider the fusion category $\mathsf{Rep}(S_3)$. Compute the fusion graph for the standard 2-dimensional representation of S_3.

Facts 3.3.6. We have the following elementary properties about semisimple multitensor categories. (Proofs are much easier if you assume $\mathcal{C}$ is unitary!)

(MT1) For all $a, b, c, d \in \mathrm{Irr}(\mathcal{C})$, the associator gives an isomorphism between the following two decompositions using the canonical isomorphism υ:

$$a \otimes (b \otimes c) \cong \bigoplus_{f\in\mathrm{Irr}(\mathcal{C})} \mathcal{C}(b \otimes c \to f) \otimes a \otimes f \cong \bigoplus_{d,f\in\mathrm{Irr}(\mathcal{C})} \mathcal{C}(b \otimes c \to f) \otimes \mathcal{C}(a \otimes f \to d) \otimes d$$

$$(a \otimes b) \otimes c \cong \bigoplus_{e\in\mathrm{Irr}(\mathcal{C})} \mathcal{C}(a \otimes b \to e) \otimes c \otimes e \cong \bigoplus_{e,d\in\mathrm{Irr}(\mathcal{C})} \mathcal{C}(a \otimes b \to e) \otimes \mathcal{C}(c \otimes e \to d) \otimes d.$$

Now applying the representable functor $\mathcal{C}(d \to -)$, post-composition with $\alpha_{a,b,c}$ gives a canonical isomorphism

$$\begin{aligned}\bigoplus_{f \in \operatorname{Irr}(\mathcal{C})} \mathcal{C}(b \otimes c \to f) \otimes \mathcal{C}(a \otimes f \to d) \cong \mathcal{C}(d \to a \otimes (b \otimes c))& \\ \xrightarrow{(\alpha_{a,b,c})_*} \mathcal{C}(d \to (a \otimes b) \otimes c)& \\ \cong \bigoplus_{e \in \operatorname{Irr}(\mathcal{C})} \mathcal{C}(a \otimes b \to e) \otimes \mathcal{C}(c \otimes e \to d)&\end{aligned}$$

called an *F-matrix*, denoted F_d^{abc}. Observe that the F-matrices for $\mathcal{C}$ completely determine the associator by the Yoneda Lemma. In §4.2 below, we will expand our discussion of F-matrices in the unitary setting.

(MT2) For all $a, b, c, d \in \operatorname{Irr}(\mathcal{C})$, $\sum_{e \in \operatorname{Irr}(\mathcal{C})} N_{ab}^e N_{ec}^d = \sum_{f \in \operatorname{Irr}(\mathcal{C})} N_{af}^d N_{bc}^f$. Indeed, this follows by taking dimensions at the end of (MT1).

(MT3) Let $1_\mathcal{C} = \bigoplus_{i=1}^n 1_i$ be a decomposition into simples. Since $\operatorname{End}_\mathcal{C}(1_\mathcal{C})$ is an abelian semisimple algebra, $1_i \otimes 1_j = \delta_{i=j} 1_i$, and thus $N_{1_i 1_j}^{1_k} = \delta_{i=j=k}$ and $1_i^\vee = 1_i$.

We write $r_i : 1_\mathcal{C} \to 1_i$ and $s_i : 1_i \to 1_\mathcal{C}$ for a splitting of the idempotent $p_i \in \operatorname{End}_\mathcal{C}(1_\mathcal{C})$ corrseponding to the summand 1_i. (In the unitary setting, p_i is orthogonal, and $s_i = v_i$ is an isometry and $r_i = v_i^\dagger$.)

(MT4) For each $c \in \operatorname{Irr}(\mathcal{C})$, there are unique $s(c), t(c) \in \{1, \ldots, n\}$ such that $c \cong 1_{s(c)} \otimes c \cong c \otimes 1_{t(c)}$ and $1_i \otimes c = 0 = c \otimes 1_j$ for all other $i \neq s(c)$ and $j \neq t(c)$. We call $1_{s(c)}$ the *source* of c and $1_{t(c)}$ the *target* of c.

Observe that $\operatorname{ev}_c : c^\vee \otimes c \to 1_\mathcal{C}$ factors through $1_{t(c)}$ and $1_{s(c^\vee)}$, so $s(c^\vee) = t(c)$. Similarly, $\operatorname{coev}_c : 1_\mathcal{C} \to c \otimes c^\vee$ factors through $1_{s(c)} = 1_{t(c^\vee)}$.

(MT5) Since $\mathcal{C}(a \to a) = \mathbb{C}\,\mathrm{id}_a$ and $\operatorname{ev}_a = \operatorname{mate}(\mathrm{id}_a)$ and $\operatorname{coev}_a = \operatorname{mate}(\mathrm{id}_a)$ under the Frobenius reciprocity isomorphisms (3.2.8), we have $N_{aa^\vee}^{1_{s(a)}} = 1 = N_{a^\vee a}^{1_{t(a)}}$. When $b \in \operatorname{Irr}(\mathcal{C})$ is any other distinct simple, $\mathcal{C}(a \to b) = 0 = \mathcal{C}(b \to a)$, and taking mates, we see $\mathcal{C}(1_\mathcal{C} \to b \otimes a^\vee) = 0 = \mathcal{C}(a^\vee \otimes b \to 1_\mathcal{C})$. Hence $N_{ab}^{1_j} = 0 = N_{ba}^{1_j}$ for all other $b \in \operatorname{Irr}(\mathcal{C})$ and $1 \leq j \leq s$.

(MT6) Each $c \in \operatorname{Irr}(\mathcal{C})$ is non-canonically isomorphic to $c^{\vee\vee}$. Just observe that $c^{\vee\vee}$ is also simple and

$$1 = N_{cc^\vee}^{1_\mathcal{C}} = \dim \mathcal{C}(c \otimes c^\vee \to 1_\mathcal{C}) = \dim \mathcal{C}(c \to c^{\vee\vee}).$$

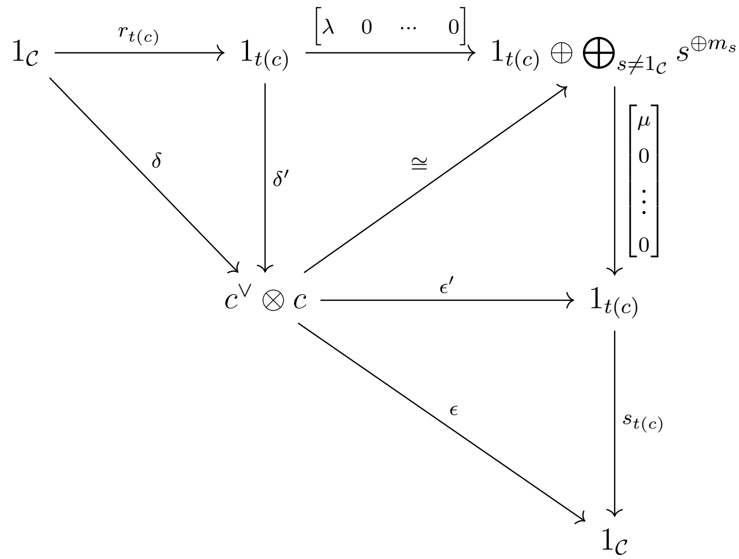
Now apply Schur's Lemma [[]].

(MT7) When $\mathcal{C}$ is multitensor, for each $a, b, c \in \operatorname{Irr}(\mathcal{C})$, the Frobenius reciprocity isomorphisms (3.2.8) imply $N_{ab}^c = N_{bc^\vee}^{a^\vee} = N_{c^\vee a}^{b^\vee} = N_{cb^\vee}^a = N_{a^\vee c}^b = N_{b^\vee, a^\vee}^{c^\vee}$.

(MT8) For every simple $c \in \mathcal{C}$ and non-zero maps $\epsilon : c^\vee \otimes c \to 1_\mathcal{C}$ and $\delta : 1_\mathcal{C} \to c^\vee \otimes c$, $\epsilon \cdot \delta \neq 0$. Indeed, by (MT4), $N_{c^\vee c}^{1_{t(c)}} = 1$ and ϵ, δ both factor through $1_{t(c)}$; say $\epsilon = s_{t(c)} \cdot \epsilon'$ and $\delta = \delta' \cdot r_{t_c}$. We may decompose

$$c^\vee \otimes c \cong 1_{t(c)} \oplus \bigoplus_{s \neq 1_{t(c)}} s^{\oplus m_s},$$

and we have non-zero scalars λ, μ such that the following diagram commutes.



Hence $\epsilon \cdot \delta$ is visibly non-zero.

(MT9) As a corollary to the last property, any tensor functor $F : \mathcal{C} \to \mathcal{D}$ into a non-zero linear monoidal category ($\mathrm{End}(1_{\mathcal{D}}) \neq 0$) is automatically faithful as long as $F(1_i) \neq 0$ for all simple summands 1_i of $1_{\mathcal{C}}$. In particular, this is automatic when $\mathcal{C}$ is tensor.

Indeed, when $c \in \mathrm{Irr}(\mathcal{C})$, there is a splitting $\delta : 1_{t(c)} \to c^\vee \otimes c$ for ev_c by the last property, so

$$0 \neq \mathrm{id}_{F(1_{t(c)})} = F(\mathrm{id}_{1_{t(c)}}) = F(\mathrm{ev}_c) \cdot F(\delta),$$

which implies $F(\mathrm{id}_c) \neq 0$. Now for any map $f \in \mathcal{C}(a \to b)$, the following diagrams commute:

$$\begin{array}{ccc} a & \xrightarrow{\upsilon_a} & \bigoplus_{s\in\mathrm{Irr}(\mathcal{C})} \mathcal{C}(s \to a) \otimes s \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle (f\cdot -)\otimes \mathrm{id}_s} \\ b & \xrightarrow{\upsilon_b} & \bigoplus_{s\in\mathrm{Irr}(\mathcal{C})} \mathcal{C}(s \to b) \otimes s \end{array} \quad \Longrightarrow \quad \begin{array}{ccc} F(a) & \xrightarrow{\upsilon_a} & \bigoplus_{s\in\mathrm{Irr}(\mathcal{C})} \mathcal{C}(s \to a) \otimes F(s) \\ \downarrow{\scriptstyle F(f)} & & \downarrow{\scriptstyle (f\cdot -)\otimes \mathrm{id}_{F(s)}} \\ F(b) & \xrightarrow{\upsilon_b} & \bigoplus_{s\in\mathrm{Irr}(\mathcal{C})} \mathcal{C}(s \to b) \otimes F(s) \end{array}$$

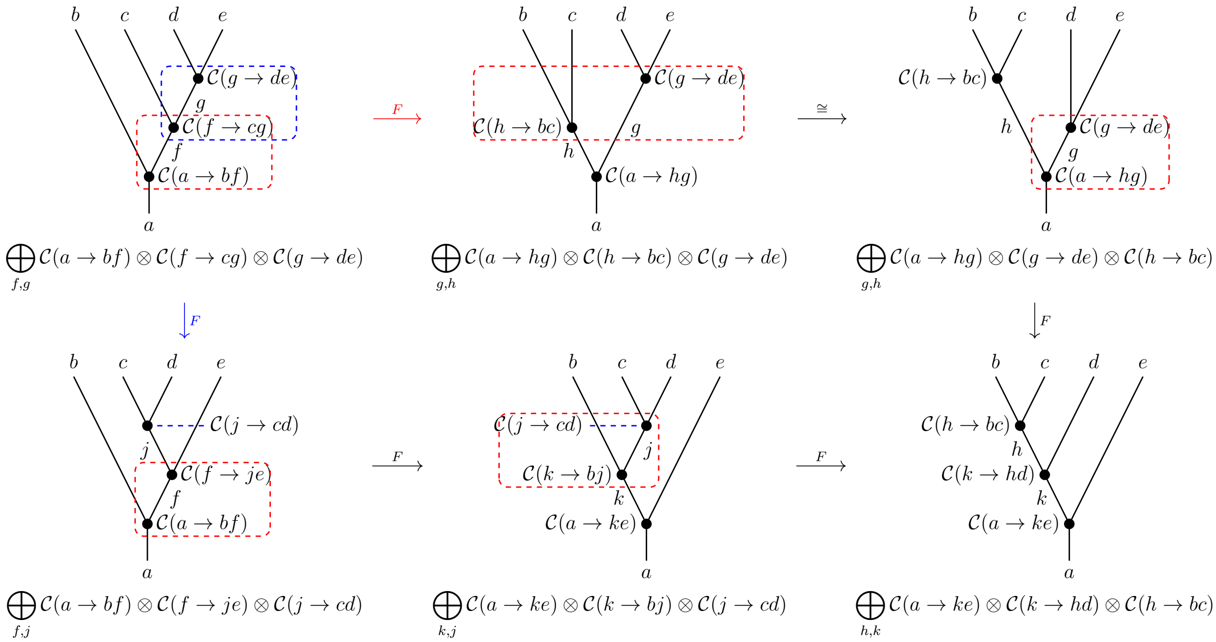
Since $\mathrm{id}_{F(s)} \neq 0$ for all simple $s \in \mathrm{Irr}(\mathcal{C})$, $F(f) \neq 0$ whenever $f \neq 0$.

3.4. ***F*-matrices of the Fibonacci fusion category.** In this section, we compute all possible F-matrices for a fusion category $\mathcal{C}$ with simple objects $1_{\mathcal{C}}$ and τ satisfying the 'Fibonacci' fusion rule

$$\tau \otimes \tau \cong 1 \oplus \tau. \tag{3.4.1}$$

Observe that the object τ *cannot* be thought of as a vector space, i.e., there is no monoidal functor from this category to Vec. Indeed, by faithfulness (MT9), a vector space V_τ satisfying $V_\tau \otimes V_\tau \cong \mathbb{C} \oplus V_\tau$ must satisfy $\dim(V_\tau)^2 = 1 + \dim(V_\tau)$, so $\dim(V_\tau) = \frac{1+\sqrt{5}}{2}$, which is not an integer.

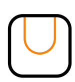
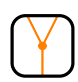
Before we begin, we observe that the pentagon associativity constraint (⬠) can be expressed solely in terms of the F-matrices as follows.



$$\sum_{h,\rho} \left[F_a^{hde}\right]_{(g,\rho,\tau)}^{(k,\mu,\nu)} \left[F_a^{bcg}\right]_{(f,\phi,\varphi)}^{(h,\rho,\sigma)} = \sum_{k,\lambda} \left[F_k^{bcd}\right]_{(j,\lambda,\delta)}^{(h,\nu,\sigma)} \sum_{j,\gamma} \left[F_a^{bje}\right]_{(f,\phi,\gamma)}^{(k,\mu,\lambda)} \left[F_f^{cde}\right]_{(g,\varphi,\psi)}^{(j,\gamma,\delta)} \tag{3.4.2}$$

Remark 3.4.3. It is worth noting that in the F-matrix formalism, the order of two hom spaces swaps in the associativity axiom by applying the symmetric swap in the fusion category Vec. When we study higher categories, namely Gray-categories, this swap operation will no longer be symmetric and can no longer be ignored.

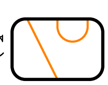
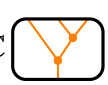
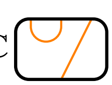
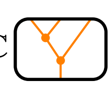
We now begin our computation of the F-matrices for Fib. First, clearly $1_{\mathcal{C}}$ and τ are both self-dual. Second, we choose bases for the 1-dimensional morphism spaces $\mathcal{C}(1 \to \tau \otimes \tau)$ and $\mathcal{C}(\tau \to \tau \otimes \tau)$; call the first basis element υ and the second basis element γ, and we denote these graphically by a cup and a trivalent vertex:

$\upsilon =$  $\qquad \gamma =$ .

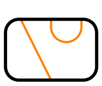
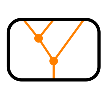
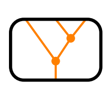
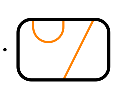
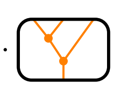
The associator is actually determined by the F-matrix

$$F_\tau^{\tau\tau\tau} : \mathcal{C}(\tau \to \tau \otimes (\tau \otimes \tau)) \longrightarrow \mathcal{C}(\tau \to (\tau \otimes \tau) \otimes \tau).$$

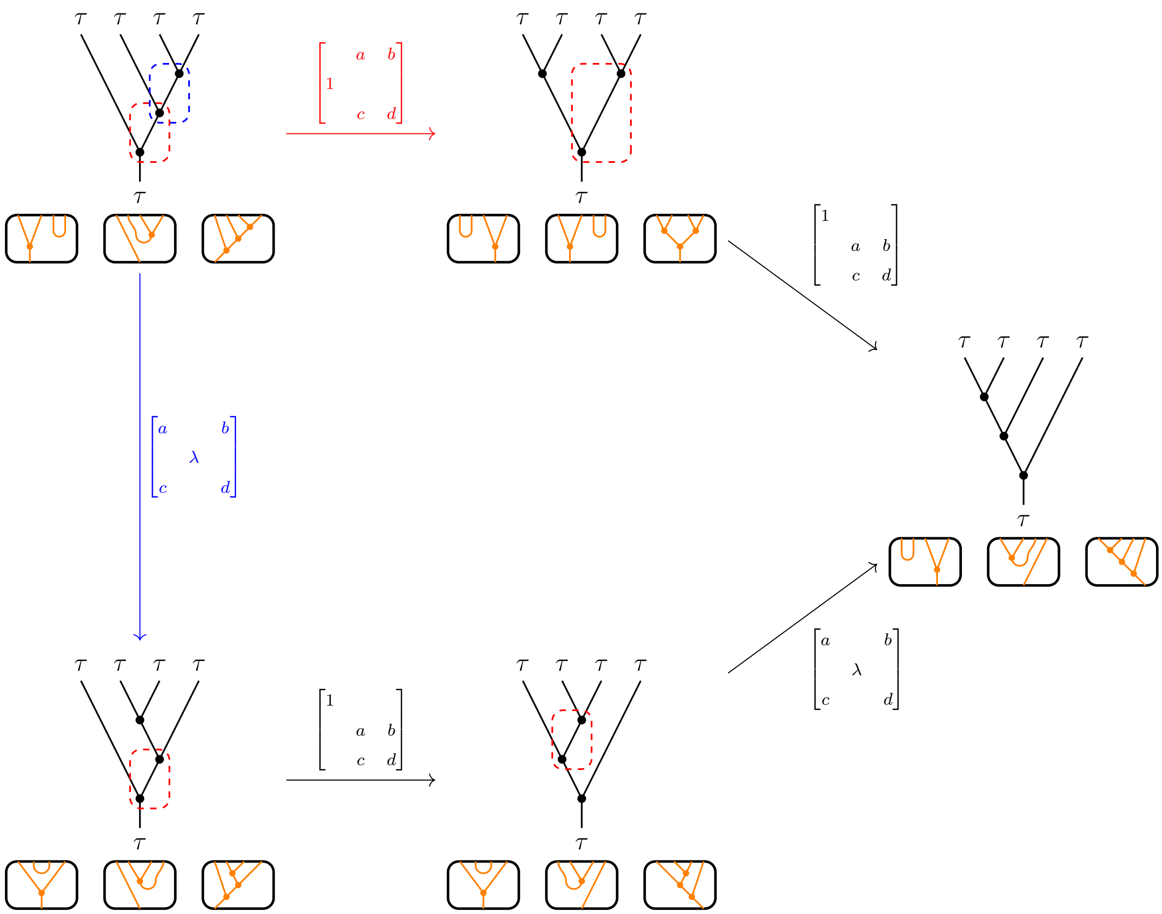
In coordinates with respect to the right and left associated tree bases, $F_\tau^{\tau\tau\tau}$ is a 2×2 matrix:

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} : \mathbb{C}$  $\oplus \mathbb{C}$  $\longrightarrow \mathbb{C}$  $\oplus \mathbb{C}$ .

This means we can apply the following *skein relations* locally in our morphism diagrams:

 $= a \cdot$  $+ c \cdot$  and  $= b \cdot$  $+ d \cdot$  (3.4.4)

We now use the constraint (3.4.2) to find all solutions for $a, b, c, d \in \mathbb{C}$. To do so, we fix tree bases for the spaces which appear in (3.4.2). We then apply the F-matrix $F^{\tau\tau\tau}_{\tau}$ locally, giving the matrices in the diagram below.



Multiplying the above matrices leads to the following system of cubic equations:

$$\begin{bmatrix} 0 & a & b \\ a & bc & bd \\ c & cd & d^2 \end{bmatrix} = \begin{bmatrix} a^2 + bcd & \lambda bc & ab + bd^2 \\ \lambda bc & \lambda^2 a & \lambda bd \\ ac + cd^2 & \lambda cd & bc + d^3 \end{bmatrix}.$$

Under the assumption that a, b, c, d are all nonzero (which is a straight forward calculation), the above system simplifies to

$$\begin{bmatrix} 0 & a & 1 \\ a & bc & 1 \\ 1 & 1 & d^2 \end{bmatrix} = \begin{bmatrix} a^2 + bcd & \lambda bc & a + d^2 \\ \lambda bc & \lambda^2 a & \lambda \\ a + d^2 & \lambda & bc + d^3 \end{bmatrix}.$$

This implies $\lambda = 1$ and $a = bc$, which further simplifies the system to

$$\begin{bmatrix} 0 & 1 \\ 1 & d^2 \end{bmatrix} = \begin{bmatrix} a + d & a + d^2 \\ a + d^2 & a + d^3 \end{bmatrix}.$$

This implies $d = -a$ and $a^2 + a - 1 = 0$, so $a = \frac{-1\pm\sqrt{5}}{2}$. Setting

$$\phi := \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad \Phi := \frac{1-\sqrt{5}}{2},$$

a straightforward calculation shows $\frac{1}{\phi} = -\Phi$ and $\frac{1}{\Phi} = -\phi$ are the two possibilities for a.

It now appears that we have a free parameter corresponding to the equation $a = bc$; the product bc is determined, but we can scale b by μ and scale c by μ^{-1} to get another solution. However, this scalar can be absorbed into the definitions of υ and γ, so without loss of generality, we may assume $b = c = \sqrt{a}$ (since both possibilities for a are strictly positive). We then see that

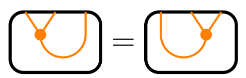
$$F_\tau^{\tau\tau\tau} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{bmatrix} \qquad \text{or} \qquad \begin{bmatrix} \Phi^{-1} & \Phi^{-1/2} \\ \Phi^{-1/2} & -\Phi^{-1} \end{bmatrix}$$

up to a gauge parameter.

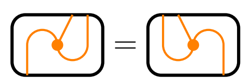
We summarize the results of this section in the following theorem.

Theorem 3.4.5. *There are exactly 2 fusion categories (up to equivalence) with the Fibonacci fusion rule* (3.4.1)*: the solution with unitary F-matrices called the* Fibonacci category *and its Galois conjugate called the* Yang-Lee *category.*

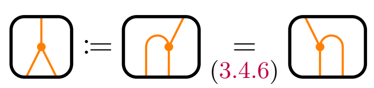
The fact that $\lambda = 1$ in the above calculation also gives us a new local skein relation:


(3.4.6)

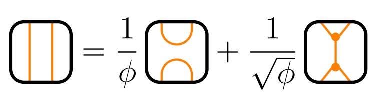
This immediately implies that our trivalent vertex γ is invariant under a *1-click rotation*:



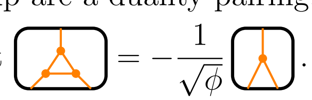
where the cap is the corresponding evaluation for $\mathrm{coev}_\tau = \upsilon$. (Since $\mathrm{Hom}(1 \to \tau \otimes \tau) = 1$ is spanned by υ, it must be part of a duality pairing.) We thus define the following morphism $\tau \otimes \tau \to \tau$:



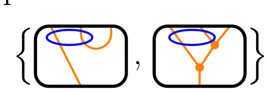
Applying a rotation to (3.4.4) combined with (3.4.6), we get the following semisimplicity fusion skein relation (assuming the first solution for $F_\tau^{\tau\tau\tau}$):


(3.4.7)

Exercise 3.4.8. One can also use *skein theory* to compute $F_\tau^{\tau\tau\tau}$ if we assume relations (3.4.6) and (3.4.7), and that the cap and cup are a duality pairing.

(1) Use (3.4.6) and (3.4.7) to prove that 

(2) Apply the fusion relation (3.4.7) to the top two left τ strands in the right associated tree basis and simplify to express them in the left associated tree basis.



The matrix of coefficients exactly give $F_\tau^{\tau\tau\tau}$.

Exercise 3.4.9. Compute the F-matrices for $\mathsf{Vec}(G,\omega)$ where $G \in Z^3(G,\mathbb{C}^\times)$.
Hint: The F-matrices are all 1×1 and equal to values of the cocycle ω.

Exercise 3.4.10. Compute the F-matrices for a fusion category with three simple objects $1,\sigma,\psi$ with fusion rules determined by

$$\sigma\otimes\sigma \cong 1\oplus\psi \qquad \text{and} \qquad \psi\otimes\psi\cong 1. \tag{3.4.11}$$

In particular, compute $F_\psi^{\psi\psi\psi}$ and $F_\sigma^{\sigma\sigma\sigma}$. How many such fusion categories are there? For each fusion category, deduce skein relations from which you can directly compute $F_\psi^{\psi\psi\psi}$ and $F_\sigma^{\sigma\sigma\sigma}$.

3.5. **Abstract fusion algebras.**

Definition 3.5.1. A *fusion rule* on a non-empty finite set $S=\{1,\dots\}$ is a collection of non-negative integers N_{ab}^c for $a,b,c\in S$ satisfying the relations

(FR1) (associativity) For all $a,b,c,d\in \operatorname{Irr}(\mathcal{C})$, $\sum_{e\in\operatorname{Irr}(\mathcal{C})} N_{ab}^e N_{ec}^d = \sum_{f\in\operatorname{Irr}(\mathcal{C})} N_{af}^d N_{bc}^f$.

(FR2) (dualizability) For each $a\in\operatorname{Irr}(\mathcal{C})$ there is a unique $a^*\in\operatorname{Irr}(\mathcal{C})$ satisfying
- $N_{aa^*}^{1c} = 1 = N_{a^*a}^{1c}$,
- for all other $b\in\operatorname{Irr}(\mathcal{C})$, $N_{ab}^{1c}=0=N_{ba}^{1c}$,
- $1^*=1$,
- $c^{**}=c$ for all $c\in\operatorname{Irr}(\mathcal{C})$, and
- For each $a,b,c\in\operatorname{Irr}(\mathcal{C})$, $N_{ab}^c = N_{bc^*}^{a^*} = N_{c^*a}^{b^*} = N_{cb^*}^{a} = N_{a^*c}^{b} = N_{b^*,a^*}^{c^*}$.

The *rank* of a fusion rule is $r:=|S|$.

Given a fusion rule $(S,N_{\bullet\bullet}^{\bullet})$, the associated *fusion algebra* is the unital complex $*$-algebra $\mathcal{FA}(S,N) := \bigoplus_{a\in S}\mathbb{C}a$ with component-wise addition, multiplication given by the bilinear extension of $a\cdot b := N_{ab}^c c$, and $*$-structure given by the anti-linear extension of $*$ from (FR2).

Remark 3.5.2. A big area of research is to determine when a given fusion rule is *categorifiable*, i.e., when does there exist a fusion category whose fusion rules reproduce the given fusion rule. This questions is surprisingly hard, whose answer is only known for $r=1,2$ [Ost03]. For unitary fusion categories, the answer is also known for $n=3$ [Ost13]. There is some progress on rank 4 [Lar14], but for now, it remains out of reach. Categorifiable multiplicity free fusion rules have been classified up to rank 6 [LPR20].

Example 3.5.3. There is a 1-parameter family of fusion rules of rank 2 on $\{1,x\}$:

$$N_{11}^1=1, \qquad N_{1x}^1=N_{x1}^1=0, \qquad N_{xx}^1=1, \qquad N_{xx}^x \text{ is arbitrary.}$$

When $N_{xx}^x>1$, this fusion rule is not categorifiable [Ost03].

When $N_{xx}^x=0$, this fusion rule is that of $\mathbb{Z}/2$, which is categorifiable in exactly two ways corresponding to $H^3(\mathbb{Z}/2,\mathbb{C}^\times)$. When $N_{xx}^x=1$, we have $x\otimes x\cong 1\oplus x$, which is the *Fibonacci* fusion rule from Theorem 3.4.5.

Exercise 3.5.4. Suppose G is a finite group.

(1) Show that $N_{gh}^k=\delta_{gh=k}$ defines a fusion rule on the set G where fusion is multiplication in G.
(2) Classify all fusion categories with these fusion rules.
Hint: As in Exercise 3.4.9, the $F_g^{hk\ell}$ for $g=hk\ell$ are all 1×1 invertible matrices. Observe they are really indexed by the three indices h,k,ℓ. Denote $F_g^{hk\ell}=\omega(h,k,\ell)$, and compute (3.4.2).

Lemma 3.5.5. *Suppose* $(S, N_{\bullet\bullet}^{\bullet})$ *is a fusion rule. For every* $a, b \in S$*, there is a* $c \in S$ *such that* $N_{ab}^c \neq 0$*.*

Proof. By associativity, we see that the coefficient of 1 in $b^*a * ab$ is non-zero. Hence $ab \neq 0$, so the result follows. □

Definition 3.5.6. Given a fusion rule $(S, N_{\bullet\bullet}^{\bullet})$, for each $a \in S$, we define $N_a \in M_S(\mathbb{N}_{\geq 0})$ to be the matrix whose (c, b)-th entry is given by N_{ab}^c. Observe N_a is exactly the adjacency matrix for the fusion graph Γ_a.

Proposition 3.5.7. *The map* $\mathcal{FA}(S, N) \to M_S(\mathbb{C})$ *given by* $a \mapsto N_a$ *is an injective unital* $*$*-homomorphism. Hence* $\mathcal{FA}(S, N)$ *is unitary and thus semisimple, i.e., a multimatrix algebra.*

Proof. For $c \in S$, let $\delta_c : S \to \mathbb{C}$ be the function which is 1 at $c \in S$ and zero everywhere else. We consider the action of $M_s(\mathbb{C})$ on $\mathbb{C}^S$. Clearly $N_1 = 1$. Multiuplicativity follows from

$$N_a N_b \delta_c = \sum_{f \in S} N_{bc}^f N_a \delta_f = \sum_{d, f \in S} N_{bc}^f N_{af}^d \delta_d = \sum_{d, e \in S} N_{ab}^e N_{ec}^d \delta_d = \sum_{e \in S} N_{ab}^e N_e \delta_c$$

which implies $N_a N_b = \sum_{e \in S} N_{ab}^e N_e$, and $*$-preserving follows from

$$\langle \delta_c | N_a \delta_b \rangle = N_{ab}^c = N_{a^*c}^b = \langle N_{a^*} \delta_c | \delta_b \rangle.$$

Finally, if $\sum_{a \in \operatorname{Irr}(\mathcal{C})} \lambda_a N_a = 0$, then

$$0 = \left\langle \delta_b \middle| \sum_{a \in \operatorname{Irr}(\mathcal{C})} \lambda_a N_a \delta_1 \right\rangle = \sum_{a \in \operatorname{Irr}(\mathcal{C})} \lambda_a \langle \delta_b | N_a \delta_1 \rangle = \sum_{a \in \operatorname{Irr}(\mathcal{C})} \lambda_a \langle \delta_b | \delta_a \rangle = \lambda_b \qquad \forall\, b \in \operatorname{Irr}(\mathcal{C}).$$

Hence $\mathcal{FA}(S, N)$ is $*$-isomorphic to the unital complex $*$-subalgebra of $M_S(\mathbb{C})$ generated by the matrices N_a for $a \in S$, so $\mathcal{FA}(S, N)$ is a unitary algebra. □

Lemma 3.5.8. *Suppose* tr *is a positive faithful trace on a unitary algebra* A *and* $a \in A$ *such that* $\operatorname{tr}(a^k) = 0$ *for all* $k \in \mathbb{N}$*. Then* a *is nilpotent, i.e.,* $a^m = 0$ *for some* $m \in \mathbb{N}$*.*

Proof. First, we prove that if $d, \rho \in M_N(\mathbb{C})$ are diagonal matrices such that ρ has strictly positive entries and $\operatorname{Tr}_N(d^k \rho) = 0$ for all k, then $d = 0$. We induct on N, the number of diagonal entries of d. If $N = 1$, the result is obvious. Consider now the characteristic polynomial $\chi_d(\lambda) = \det(\lambda I - d) = \sum_{j=0}^m c_j \lambda^j$ of d. Since $\chi_d(d) = 0$,

$$0 = \operatorname{Tr}_N(\chi_d(d)\rho) = \sum_{j=0}^m c_j \operatorname{Tr}_N(d^j \rho) = N \operatorname{Tr}_N(\rho) c_0,$$

so $c_0 = \det(d) = 0$. Thus one of the diagonal entries of d is zero, so we may replace d with d', the diagonal matrix with that zero entry deleted. We still have that d' satisfies that the traces of all its powers is zero. By the induction hypothesis, $d' = 0$, so $d = 0$.

Now to prove the general case, we may view our unitary algebra A as embedded into $M_N(\mathbb{C})$ such that there is a positive element $\rho \in Z(A)$ with $\operatorname{tr}(a) = \operatorname{Tr}_N(a\rho)$ for all $a \in A$. (Just take the GNS representation of A on $L^2(A, \operatorname{tr})$, and write the vector state $\langle \Omega | - \Omega \rangle$ in terms of a density matrix.) Now considering the diagonal blocks of A independently in $M_N(\mathbb{C})$, for each $a \in A$, there is an invertible $s \in A$ such that $s^{-1}as = d + n_1$, a diagonal plus a nilpotent operator. Then for all $k \in \mathbb{N}$, $s^{-1}a^k s = d^k + n_k$ where n_k is some nilpotent operator, and

$$0 = \mathrm{Tr}_N(a^k\rho) = \mathrm{Tr}_N(sd^k s^{-1}\rho) + \mathrm{Tr}_N(sn_k s^{-1}\rho) = \mathrm{Tr}_N(d^k\rho) + \mathrm{Tr}_N(n_k\rho) = \mathrm{Tr}_N(d^k\rho).$$

By the first paragraph $d = 0$, so a is nilpotent. □

Exercise 3.5.9. Suppose $(S, N_{\bullet\bullet}^{\bullet})$ is a fusion rule.

(1) The map $c \mapsto \delta_{c=1}$ extends to a positive faithful trace on $\mathcal{FA}(S, N)$.
(2) If $(S, N_{\bullet\bullet}^{\bullet})$ comes from a fusion category $\mathcal{C}$, then for every $c \in \mathrm{Irr}(\mathcal{C})$, there is some $n \in \mathbb{N}$ such that $1_{\mathcal{C}}$ is a subobject of $c^{\otimes n}$.

Proof. To prove (1), we first prove that $N_{ab}^1 = N_{ba}^1$ for all $a, b \in S$. We use the final part of the dualizability axiom:

$$N_{ab}^1 = N_{b1^*}^{a^*} = N_{b1}^{a^*} = N_{1b}^{a^*} = N_{1^*b}^{a^*} = N_{ba}^1.$$

This implies traciality. To see positivity, we compute for $x = \sum_{a\in S} \lambda_a a$,

$$x^* \cdot x = \sum_{a,b\in S} \overline{\lambda_a}\lambda_b a^* \cdot b = \sum_{a,b,c\in S} \overline{\lambda_a}\lambda_b N_{a^*b}^c c \mapsto \sum_{ab\in S} \overline{\lambda_a}\lambda_b N_{a^*b}^1 = \sum_{a,b\in S} \overline{\lambda_a}\lambda_b \delta_{a=b} = \sum_{a,b\in S} |\lambda_a|^2$$

which is clearly positive and faithful.

Part (2) now follows immediately from Lemma 3.5.8; indeed, if $\mathcal{C}(1_{\mathcal{C}} \to c^{\otimes n}) = 0$ for all n, then $c \in \mathcal{FA}(S, N)$ is nilpotent, which is in contradiction with $(c^\vee)^{\otimes n} \otimes c^{\otimes n}$ containing $1_{\mathcal{C}}$ as a subobject. □

We now state facts which follow from the Frobenius-Perron Theorem. There are many references for these facts; one such reference is [EGNO15, Thm. 3.2.1].

Facts 3.5.10 (Frobenius-Perron)**.** Suppose $x \in M_n(\mathbb{C})$ is a matrix with non-negative entries.

- There is a positive eigenvalue λ of x such that $|\mu| \leq \lambda$ for all other eigenvalues $\mu \in \mathrm{spec}(x)$. Moreover, there is an eigenvector $\xi \in \mathbb{C}^n$ with strictly positive entries such that $x\xi = \lambda\xi$. If $\eta \in \mathbb{C}^n$ is any other eigenvector for x with strictly positive entries, then λ is the corresponding eigenvalue as well.

  We call λ the *Frobenius-Perron* eigenvalue of x. We call any such ξ a *Frobenius-Perron* eigenvector for x.
- If x has strictly positive entries, then $\lambda \in \mathrm{spec}(x)$ has multiplicity one, i.e., there is a unique eigenvector of x with positive entries up to positive scaling.

Definition 3.5.11. Suppose $(S, N_{\bullet\bullet}^{\bullet})$ is a fusion rule. The *Frobenius-Perron dimension* d_c of $c \in S$ is the Frobenius-Perron eigenvalue of the fusion matrix N_c from Definition 3.5.6 above. Since $N_{c^*} = N_c^T$, clearly $d_c = d_{c^*}$.

Proposition 3.5.12. *Suppose $(S, N_{\bullet\bullet}^{\bullet})$ is a fusion rule. The map $d : S \to \mathbb{C}$ extends to a $*$-algebra homomorphism $\mathcal{FA}(S) \to \mathbb{C}$. Moreover, the vector $d \in \mathbb{C}^S$ whose c-th entry is d_c is an eigenvector for every fusion matrix N_a for $a \in S$.*

Proof. The operator $X = \sum_{a\in S} N_a \in M_S(\mathbb{C})$ has strictly positive entries by Lemma 3.5.5. Indeed, for every $b, c \in S$, there is an $a \in S$ such that $N_{ab}^c \neq 0$ (using (FR2)), so $X_{bc} > 0$. Let $\xi \in \mathbb{C}^S$ be a Frobenius-Perron eigenvector with strictly positive entries and λ the corresponding eigenvalue. Translating back into $\mathcal{FA}(S, N)$, the operators $x = \sum_{a\in S} a$ and $v = \sum_{b\in S} \xi_b b$ satisfy $xv = \lambda v$. Now for every $c \in S$, $xvc = \lambda vc$, so $vc = \mu_c c$ for some

$\mu_c > 0$ by uniqueness of the Frobenius-Perron eigenvector up to positive scaling. Since v has positive entries, taking $*$ gives $c^* v^* = \mu_c v^*$ for all $c \in S$. Since v^* has positive entries, we can conclude that $\mu_c = d_{c^*} = d_c$. Since $xv^* = \left(\sum_{c\in S} \mu_c\right) v^*$, again by uniqueness of the Frobenius-Perron eigenvalue and eigenvector up to positive scaling, we can deduce that $v = v^*$ and $\lambda = \sum_{c\in S} d_c$.

We immediately see that d extends to a $*$-algebra homomorphism:

$$d_a d_b v = abv = \sum_{c\in S} N_{ab}^c cv = \sum_{c\in S} N_{ab}^c d_c v \qquad \Longrightarrow \qquad d_a d_b = \sum_{c\in S} N_{ab}^c d_c$$

as v has strictly positive entries. We can also now compute that

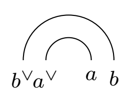
$$a\left(\sum_{b\in S} d_b b\right) = \sum_{b\in S} d_b ab = \sum_{b,c\in S} N_{ab}^c d_b c \underset{\text{(FR2)}}{=} \sum_{b,c\in S} N_{c^*a}^b d_b c = \sum_{c\in S} d_a d_{c^*} c = d_a \sum_{c\in S} d_c c,$$

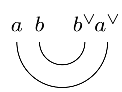
which shows the vector $d \in \mathbb{C}^S$ is a Frobenius-Perron eigenvector for N_a for all $a \in S$. $\square$

Exercise 3.5.13. A fusion rule $(S, N_{\bullet\bullet}^{\bullet})$ is *commutative* if $N_{ab}^c = N_{ba}^c$ for all $a, b, c \in S$. Prove that all fusion rules of rank at most 4 are commutative.
Hint: By Proposition 3.5.7, $\mathcal{FA}(S,N)$ is semisimple. By Proposition 3.5.12, $\mathcal{FA}(S,N)$ admits a non-trivial $*$-algebra homomorphism to $\mathbb{C}$.

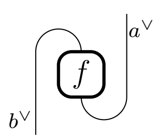
3.6. **Dual functors and pivotal structures.** For this section, $\mathcal{C}$ will always denote a multitensor category.

Observe that whenever $a, b \in \mathcal{C}$ have duals, so does $a \otimes b$; just take $b^\vee \otimes a^\vee$ with

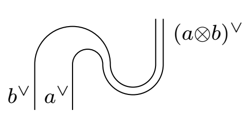
$\mathrm{ev}_{a\otimes b} :=$  $= \mathrm{ev}_b \cdot (\mathrm{id}_{b^\vee} \otimes \mathrm{ev}_a \otimes \mathrm{id}_b)$

$\mathrm{coev}_{a\otimes b} :=$  $= (\mathrm{id}_a \otimes \mathrm{coev}_b \otimes \mathrm{id}_{a^\vee}) \cdot \mathrm{coev}_a\,.$

When $\mathcal{C}$ is rigid, a choice of dual $(c^\vee, \mathrm{ev}_c, \mathrm{coev}_c)$ for every $c \in \mathcal{C}$ assembles into a *dual functor*, which is a monoidal functor $\vee : \mathcal{C} \to \mathcal{C}^{\mathrm{mop}}$. For $f \in \mathcal{C}(a \to b)$, we define

$f^\vee :=$  $= (\mathrm{ev}_b \otimes \mathrm{id}_{a^\vee}) \cdot (\mathrm{id}_{b^\vee} \otimes f \otimes \mathrm{id}_{a^\vee}) \cdot (\mathrm{id}_{b^\vee} \otimes \mathrm{coev}_a),$

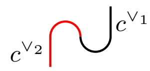
and $\vee$ has canonical tensorator $\vee^2_{a,b} : a^\vee \otimes_{\mathrm{mop}} b^\vee = b^\vee \otimes a^\vee \to (a \otimes b)^\vee$ given by

 $= (\mathrm{ev}_b \otimes \mathrm{id}_{(a\otimes b)^\vee}) \cdot (\mathrm{id}_{b^\vee} \otimes \mathrm{ev}_a \otimes \mathrm{id}_b \otimes \mathrm{id}_{(a\otimes b)^\vee}) \cdot (\mathrm{id}_{b^\vee \otimes a^\vee} \otimes \mathrm{coev}_{a\otimes b})$ (3.6.1)

and canonical unitor $\vee^0 := \mathrm{coev}_{1_\mathcal{C}} : 1_\mathcal{C} \to 1_\mathcal{C}^\vee$ (here, we suppress the unitor $\lambda_{1_\mathcal{C}^\vee} : 1_\mathcal{C} \otimes 1_\mathcal{C}^\vee \to 1_\mathcal{C}^\vee$ as usual).

Exercise 3.6.2. Prove that any other choices of duals yields a canonically naturally isomorphic monoidal equivalence $\mathcal{C} \to \mathcal{C}^{\mathrm{mop}}$. That is, while we made choices of duals to get a dual functor, there is a contractible choice of dual functors. In this sense, the existence of a dual

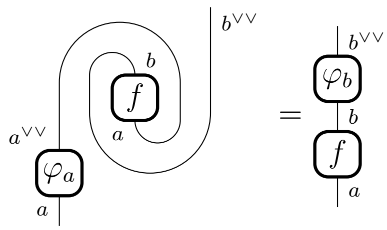
functor on a tensor category is a *property* and not extra *structure*.
Hint: For $c \in \mathcal{C}$ *with duals* $(c_i^\vee, \mathrm{coev}_i, \mathrm{ev}_i)$ *for* $i = 1, 2$ *consider the isomorphism*

$$\zeta_c :=$$


$$= (\mathrm{ev}_2 \otimes \mathrm{id}_{c^{\vee_1}}) \cdot (\mathrm{id}_{c^{\vee_2}} \otimes \mathrm{coev}_1).$$

In the category Vec of finite dimensional vector spaces, every object V is canonically isomorphic to its double dual $V^{\vee\vee}$ via the map $\varphi_V : v \mapsto \mathrm{eval}_v$ where $\mathrm{eval}_v : V^\vee \to \mathbb{C}$ is the map which evaluates every linear functional f at v. The isomorphism $\varphi_V : V \to V^{\vee\vee}$ is natural in V and monoidal, i.e., we have a monoidal natural isomorphism $\varphi : \mathrm{id}_\mathcal{C} \Rightarrow \vee \circ \vee$.

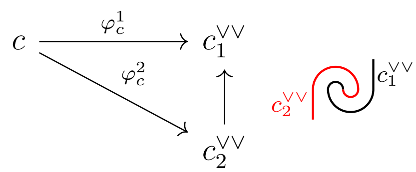
Definition 3.6.3. A *pivotal structure* is a choice of dual functor together with a monoidal natural isomorphism $\varphi : \mathrm{id}_\mathcal{C} \Rightarrow \vee \circ \vee$. Naturality means that



$$\forall f \in \mathcal{C}(a \to b).$$

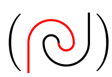
Observe that by (∨7) with $F = \mathrm{id}_\mathcal{C}$ and $G = \vee \circ \vee$, $\varphi_{c^\vee} : c^\vee \to c^{\vee\vee\vee}$ is always invertible with inverse $(\varphi_c)^\vee$. Indeed, the canonical isomorphisms $\zeta^{\mathrm{id}_\mathcal{C}} : c^\vee \to c^\vee$ and $\zeta_c^{\vee\vee} : (c^\vee)^{\vee\vee} \to (c^{\vee\vee})^\vee$ are always identity maps.

Two pivotal structures $(\vee_1, \varphi^1)$ and $(\vee_2, \varphi^2)$ on $\mathcal{C}$ are *equivalent* if for every $c \in \mathcal{C}$, the following diagram commutes:



(3.6.4)

By naturality and (3.6.4), $(\vee_1, \varphi^1)$ and $(\vee_2, \varphi^2)$ are equivalent if and only if for all $c \in \mathrm{Irr}(\mathcal{C})$,

$$(\varphi_c^1)^{-1} \cdot \Big($$

$$\Big) \cdot \varphi_c^2 = \mathrm{id}_c \,. \qquad (3.6.5)$$

Remark 3.6.6. A pivotal category $\mathcal{C}$ admits a graphical calculus where we may freely perform isotopies which rotate coupons by 2π. Again, one may suppress the pivotal isomorphisms φ with the understanding that given any complete labelling of the source and target objects, any algebraic expression for the diagram yields the same morphism in $\mathcal{C}$.

The following question is open at this time.

Question 3.6.7. *Does every fusion category admit a pivotal structure?*

Remark 3.6.8. When $\mathcal{C}$ has a pivotal structure $(\vee, \varphi)$, the set of pivotal structures on $\mathcal{C}$ with the same dual functor $\vee$ is a torsor for the group $\mathrm{Aut}_\otimes(\mathrm{id}_\mathcal{C})$ of monoidal natural isomorphisms of the identity functor. Indeed, consider the groupoid with whose objects are $\mathrm{id}_\mathcal{C}$ and $\vee \circ \vee$ and whose morphisms are monoidal natural isomorphisms. The existence of a pivotal structure means we have an isomorphism $\varphi : \mathrm{id}_\mathcal{C} \Rightarrow \vee \circ \vee$, so this groupoid is connected. This means the map $\delta \mapsto \varphi \circ \delta$ is a bijection from $\mathrm{Aut}_\otimes(\mathrm{id}_\mathcal{C})$ to the set of pivotal structures with fixed dual functor $\vee$.

Let us now compute $\mathrm{Aut}_\otimes(\mathrm{id}_\mathcal{C})$ for a multitensor category $\mathcal{C}$. Using naturality, the unital condition (3.1.13), and the monoidal condition (3.1.14), a monoidal natural isomorphism $\delta : \mathrm{id}_\mathcal{C} \Rightarrow \mathrm{id}_\mathcal{C}$ is exactly a choice of non-zero scalar $\delta_c \in \mathbb{C}^\times$ for each simple $c \in \mathrm{Irr}(\mathcal{C})$ satisfying

- $\delta_{1_i} = \mathrm{id}_{1_i}$ for all $i = 1, \ldots, n$ and
- $\delta_a \delta_b = \delta_c$ for $a, b, c \in \mathrm{Irr}(\mathcal{C})$ whenever $N_{ab}^c \neq 0$.

In light of this calculation, we make the following definition.

Definition 3.6.9. The *universal grading groupoid* $\mathcal{U}_\mathcal{C}$ of a multitensor category $\mathcal{C}$ has n objects corresponding to the simple summands of $1_\mathcal{C}$. The morphisms $1_i \to 1_j$ are the simples $c \in \mathrm{Irr}(\mathcal{C})$ with $s(c) = i$ and $t(c) = j$, subject to the relations $ab = c$ whenever $N_{ab}^c \neq 0$.

We postpone a discussion of gradings on multitensor categories to Definition 3.8.4 below.

Observe that we can identify the n objects of $\mathcal{U}_\mathcal{C}$ in bijective correspondence with the n idempotents 1_i in $\mathcal{U}_\mathcal{C}$. We thus see that we can identify a monoidal natural isomorphism $\delta : \mathrm{id}_\mathcal{C} \Rightarrow \mathrm{id}_\mathcal{C}$ with a function δ from the set of arrows of $\mathcal{U}_\mathcal{C}$ to $\mathbb{C}^\times$ satisfying $\delta_{1_i} = 1$ for all $i = 1, \ldots, n$ and $\delta(ab) = \delta(a)\delta(b)$ for composable $a, b \in \mathrm{Irr}(\mathcal{C})$. This is exactly a *groupoid homomorphism* $\delta : \mathcal{U}_\mathcal{C} \to \mathbb{C}^\times$. (Here, we use a slight abuse of notation. Since we may recover the objects of $\mathcal{G}$ from the idempotent morphisms, we simply consider $\mathcal{G}$ as its set of morphisms. More precisely, δ is a functor $\mathcal{U}_\mathcal{C} \to \mathrm{B}\mathbb{C}^\times$.) Moreover, the $\circ$ composition of monoidal natural isomorphisms exactly corresponds to the pointwise multiplication of such homomorphisms, turning $\mathrm{Hom}(\mathcal{U}_\mathcal{C} \to \mathbb{C}^\times)$ into a group. We record this discussion as the following proposition.

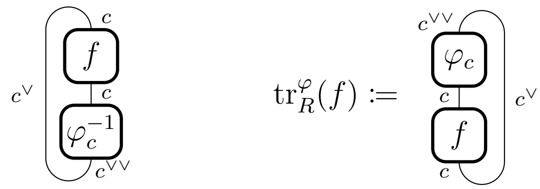
Proposition 3.6.10. *When a multitensor category $\mathcal{C}$ admits a pivotal structure, then the set of pivotal structures is a torsor for the group* $\mathrm{Aut}_\otimes(\mathrm{id}_\mathcal{C}) \cong \mathrm{Hom}(\mathcal{U}_\mathcal{C} \to \mathbb{C}^\times)$.

Example 3.6.11. Consider the multitensor category $\mathrm{End}(\mathcal{M})$ for $\mathcal{M}$ a finite semisimple category. Identifying $\mathcal{M} = \mathsf{Vec}^{\boxplus n}$ for some n, the simple objects of $\mathrm{End}(\mathcal{M})$ have canonical representatives E_{ij} taking $\mathbb{C}$ in the j-th summand of $\mathcal{M}$ to $\mathbb{C}$ in the i-th summand of $\mathcal{M}$. Observe that these simples compose strictly by the rule $E_{ij} \circ E_{k\ell} = \delta_{j=k} E_{i\ell}$. The dual of E_{ij} is E_{ji} with evaluation and coevaluation given by

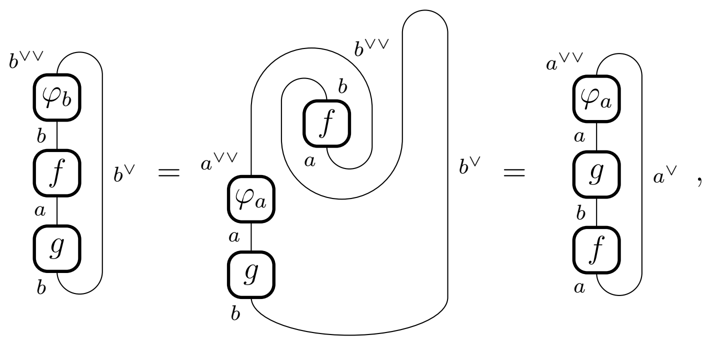
$$\mathrm{ev}_{ij} : E_{ij} \circ E_{ji} = E_{ii} \xrightarrow{\mathrm{id}_{E_{ii}}} E_{ii} \hookrightarrow 1_{\mathrm{End}(\mathcal{M})}$$
$$\mathrm{coev}_{ij} : 1_\mathcal{C} \twoheadrightarrow E_{jj} \xrightarrow{\mathrm{id}_{E_{jj}}} E_{jj} = E_{ji} \circ E_{jj}.$$

Hence setting $\varphi_{ij} := \mathrm{id}_{E_{ii}}$ defines a pivotal structure on $\mathrm{End}(\mathcal{M})$. By Proposition 3.6.10, all other pivotal structures on $\mathrm{End}(\mathcal{M})$ are obtained as $\varphi \circ \delta$ with $\delta \in \mathrm{Hom}(\mathcal{U} \to \mathbb{C}^\times)$. Observe that the universal grading groupoid $\mathcal{U}$ of $\mathrm{End}(\mathcal{M})$ is given by the *matrix groupoid* $\mathcal{M}_n$ which has n objects (say $1, \ldots n$) and a unique isomorphism between any two (say $e_{ij} : i \to j$). A groupoid map $\delta : \mathcal{M}_n \to \mathbb{C}^\times$ is completely determined by where it sends the $n-1$ elements $e_{i,i+1}$, which can be sent to arbitrary numbers. Hence $\mathrm{Hom}(\mathcal{U} \to \mathbb{C}^\times) \cong (\mathbb{C}^\times)^{n-1}$.

Definition 3.6.12. A pivotal structure on $\mathcal{C}$ yields two $\operatorname{End}_{\mathcal{C}}(1_{\mathcal{C}})$-valued *traces* on every endomorphism algebra:

$\operatorname{tr}_L^{\varphi}(f) :=$  $\operatorname{tr}_R^{\varphi}(f) :=$ $\quad \forall\, f \in \mathcal{C}(c \to c). \qquad (3.6.13)$

Observe that for all $f : a \to b$ and $g : b \to a$,



and similarly for the left trace.

Lemma 3.6.14. *When $\mathcal{C}$ is pivotal multitensor, the traces $\operatorname{tr}_L^{\varphi}$ and $\operatorname{tr}_R^{\varphi}$ are nondegenerate, i.e., for every nonzero $f \in \mathcal{C}(a \to b)$, there is a $g \in \mathcal{C}(b \to a)$ such that $\operatorname{tr}_L^{\varphi}(g \cdot f) \neq 0$, and similarly for $\operatorname{tr}_R^{\varphi}$. In particular, for each $c \in \operatorname{Irr}(\mathcal{C})$, $\operatorname{tr}_L^{\varphi}(\operatorname{id}_c) \neq 0 \neq \operatorname{tr}_R^{\varphi}(\operatorname{id}_{\mathcal{C}})$.*

Proof. Suppose $f \in \mathcal{C}(a \to b)$ is nonzero. Then there is a simple $s \in \operatorname{Irr}(\mathcal{C})$, and nonzero maps $g \in \mathcal{C}(s \to a)$ and $h \in \mathcal{C}(b \to s)$ such that $h \cdot f \cdot g \neq 0$. Then

$$0 \neq \epsilon := \operatorname{ev}_c \cdot [\operatorname{id}_{c^\vee} \otimes (h \cdot f \cdot g \cdot \varphi_c^{-1})] \in \mathcal{C}(c^\vee \otimes c^{\vee\vee} \to 1_{\mathcal{C}}).$$

Since we also know $0 \neq \operatorname{coev}_{c^\vee} \in \mathcal{C}(1_{\mathcal{C})} \to c^\vee \otimes c^{\vee\vee})$, by (MT8).

$$\operatorname{tr}_L^{\varphi}((g \cdot h) \cdot f) = \operatorname{tr}_L^{\varphi}(h \cdot f \cdot g) = e \cdot \operatorname{coev}_{c^\vee} \neq 0.$$

Hence $\operatorname{tr}_L^{\varphi}$ is nondegenerate.

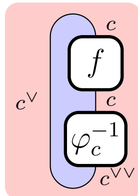
To prove the final statement, observe that $\operatorname{coev}_{c^\vee} \neq 0$ and $\operatorname{ev}_{\mathcal{C}} \cdot (\operatorname{id}_{c^\vee} \otimes \varphi_c^{-1}) \neq 0$, so again by (MT8), $\operatorname{tr}_L^{\varphi}(\operatorname{id}_c) = \operatorname{ev}_{\mathcal{C}} \cdot (\operatorname{id}_{c^\vee} \otimes \varphi_c^{-1}) \cdot \operatorname{coev}_{c^\vee} \neq 0$.

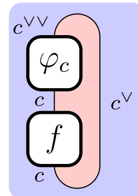
The proofs for $\operatorname{tr}_R^{\varphi}$ are similar. □

Recall we have a decomposition $1_{\mathcal{C}} = \bigoplus_{i=1}^n 1_i$ into simples, and $p_i \in \mathcal{C}(1_{\mathcal{C}} \to 1_{\mathcal{C}})$ is the minimal idempotent onto 1_i. We get $M_n(\mathbb{C})$-valued traces $\operatorname{Tr}_{L/R}^{\varphi}$ on each endomorphism algebra determined by the formulas

$$(\operatorname{Tr}_{L/R}^{\varphi}(f))_{ij} \operatorname{id}_{1_j} = \operatorname{tr}_{L/R}^{\varphi}(p_i \otimes f \otimes p_j) \qquad \forall f \in \mathcal{C}(c \to c). \qquad (3.6.15)$$

If we shade regions in our diagrams to denote simple summands of $1_{\mathcal{C}}$, $\mathrm{Tr}^{\varphi}_{L/R}$ can be represented graphically by

$\mathrm{Tr}^{\varphi}_{L}(f)_{ij} =$ 

$\mathrm{Tr}^{\varphi}_{R}(f)_{ij} =$ 

 $= p_i,$ $= p_j.$

Definition 3.6.16. Given a pivotal multitensor category $\mathcal{C}$ and $c \in \mathrm{Irr}(\mathcal{C})$, by Lemma 3.6.14, $\mathrm{Tr}^{\varphi}_{L/R}(\mathrm{id}_c)$ has exactly one non-zero entry which we call the *left/right quantum dimension* of c respectively, denoted $\dim^{\varphi}_{L/R}(c)$.

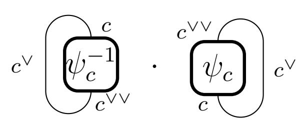
Exercise 3.6.17. Suppose $\mathcal{C}$ is a pivotal multitensor category. Prove that $\dim^{\varphi}_{L}(c) = \dim^{\varphi}_{R}(c^{\vee})$ for all $c \in \mathrm{Irr}(\mathcal{C})$.
Hint: Use $\varphi_{c^{\vee}}^{-1} = \varphi_c^{\vee}$.

Exercise 3.6.18. Suppose $\mathcal{C}$ is a fusion category with a pivotal structure $(\vee, \varphi)$. Prove that the quantum dimensions $\dim^{\varphi}_{L/R}$ give algebra homomorphisms $\mathcal{FA}(\mathrm{Irr}(\mathcal{C})) \to \mathbb{C}$. Deduce that

- $(\dim^{\varphi}_{L/R}(c))_{c \in \mathrm{Irr}(\mathcal{C})}$ are both eigenvectors for N_a with eigenvalue $\dim^{\varphi}_{L/R}(a)$ respectively.
- $|\dim^{\varphi}_{L/R}(c)| \leq d_c$ for all $c \in \mathrm{Irr}(\mathcal{C})$.
  Hint: Use that d_c dominates $|\mu_c|$ for all other eigenvalues of N_c.
- If $(\vee, \varphi)$ is pseudounitary, then
$$\dim^{\varphi}_{L}(c) = \dim^{\varphi}_{R}(c) = d_c \qquad \forall\, c \in \mathrm{Irr}(\mathcal{C}). \tag{3.6.19}$$
  Hint: Use the uniqueness of the Frobenius-Perron eigenvector (up to positive scaling).

Remark 3.6.20. Our definition of psedounitarity differs from [EGNO15, §9.4] for a fusion category, as we wish to use this adjective for (infinite) multitensor categories. However, in the case of fusion categories, our definitions can be shown to be equivalent.

Without a pivotal structure, one can define the quantum dimension of a fusion category as follows. For each $c \in \mathrm{Irr}(\mathcal{C})$, pick a non-canonical isomorphism $\psi_c \in \mathcal{C}(c \to c^{\vee\vee})$ for each $c \in \mathrm{Irr}(\mathcal{C})$ using (MT6). Note, however, that the ψ_c may not assemble into a pivotal structure. We get a categorical dimension

$$\dim(\mathcal{C}) := \sum_{c \in \mathrm{Irr}(\mathcal{C})} \tag{3.6.21}$$


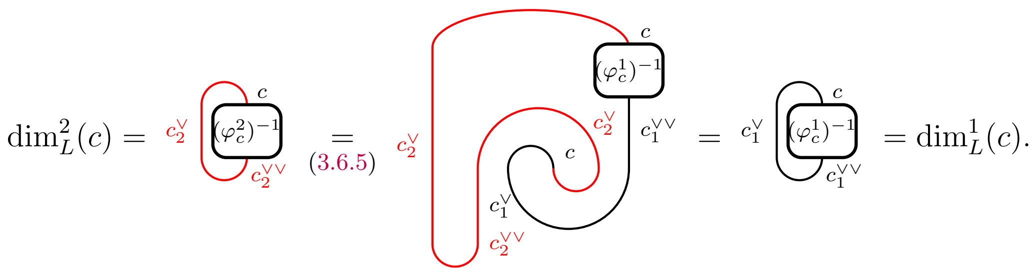
which is *independent* of the choices of ψ_c, as we are summing over ψ_c and their inverses. (One should view this as summing over a basis of $\mathcal{C}(c \to c^{\vee\vee})$ and a dual basis.)

Following [EGNO15, Def. 9.4.4] a fusion category is pseudounitary if
$$\dim(\mathcal{C}) = \mathrm{FPdim}(\mathcal{C}) := \sum_{c \in \mathrm{Irr}(\mathcal{C})} d_c^2.$$
It can be shown that in this case, $\mathcal{C}$ has a canonical pivotal structure satisfying (3.6.19). Conversely, if $\mathcal{C}$ is pseudounitary in our sense, then (3.6.19) holds, and thus choosing $\psi_c = \varphi_c$ for all $c \in \mathrm{Irr}(\mathcal{C})$, $\dim(\mathcal{C}) = \mathrm{FPdim}(\mathcal{C})$.

Proposition 3.6.22. *Suppose $\mathcal{C}$ is multitensor and $(\vee_1, \varphi_1)$ and $(\vee_2, \varphi_2)$ are two pivotal structures. The following are equivalent.*

(1) $(\vee_1, \varphi^1)$ and $(\vee_2, \varphi^2)$ are equivalent.
(2) For all $c \in \operatorname{Irr}(\mathcal{C})$, $\dim_L^1(c) = \dim_L^2(c)$.
(3) For all $c \in \operatorname{Irr}(\mathcal{C})$, $\dim_R^1(c) = \dim_R^2(c)$.

Proof.
(1) $\Rightarrow$ (2): Assuming (1), (3.6.5) holds. Hence



(2) $\Rightarrow$ (1): The left hand side of (3.6.5) is a scalar multiple of id_c. By Lemma 3.6.14, we may determine this scalar by applying tr_L^1 to both sides as $\dim_L^i(c) \neq 0$ for $i = 1, 2$. It is straightforward to check that tr_L^1 applied to the left hand side is equal to $\dim_L^2(c)$, which is equal to $\dim_L^1(c)$ by assumption. Hence (3.6.5) holds.
(2) $\Leftrightarrow$ (3): Immediate from Exercise 3.6.17. $\square$

Remark 3.6.23. Two equivalent/non-equivalent pivotal structures can look very different/similar given the choice of dual functor. If we choose our duals appropriately by incorporating various cube roots of unity, we may arrange so that each equivalence class of pivotal structure has a representative where $\vee \circ \vee = \mathrm{id}_{\mathcal{C}}$ and $\varphi = \mathrm{id}$.

First, there is a spherical structure on $\mathsf{Vec}(\mathbb{Z}/3)$ where we declare the dual of $\mathbb{C}_g$ to be $\mathbb{C}_{g^{-1}}$ for $g \in \mathbb{Z}/3$. We can then take each $\varphi_g : \mathbb{C}_g \to \mathbb{C}_g$ to be the identity map, and all quantum dimensions are equal to 1.

Since the universal grading group of $\mathsf{Vec}(\mathbb{Z}/3)$ is obviously $\mathbb{Z}/3$, we can distort φ by a group homomorphism $\delta : \mathbb{Z}/3 \to \mathbb{C}^\times$. There are exactly 2 non-trivial such homomorphisms, mapping the generator g to either ζ or ζ^{-1} where $\zeta = \exp(2\pi i/3)$. Thus there is another pivotal structure with the same duality pairings as above where $\varphi_g^\zeta = \zeta \cdot \mathrm{id}_g$ and $\varphi_{g^{-1}}^\zeta = \zeta^{-1} \cdot \mathrm{id}_{g^{-1}}$.

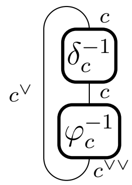
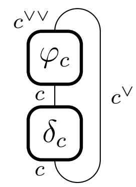
Now instead of altering the pivotal structure, we can instead change our duality pairing on $\mathsf{Vec}(\mathbb{Z}/3)$. For $g \in \mathbb{Z}/3 = \{0, 1, 2\}$ (modulo 3), we define $\mathrm{ev}_g^\zeta : \mathbb{C}_{g^{-1}} \otimes \mathbb{C}_g \cong \mathbb{C} \to \mathbb{C}$ to simply be multiplication by $\zeta^{g/2}$, where $\zeta^{-1/2} = \exp(2\pi i/6)$. The map $\mathrm{coev}_g^\zeta : \mathbb{C} \to \mathbb{C}_g \otimes \mathbb{C}_{g^{-1}} \cong \mathbb{C}$ is then multiplication by $\zeta^{-g/2}$. Then taking the pivotal structure $\varphi_g = \mathrm{id}_g$ for all g yields the quantum dimensions

$$\dim_L^\zeta(g) = \mathrm{ev}_g^\zeta \cdot \mathrm{coev}_{g^{-1}}^\zeta = \zeta^{g/2}\zeta^{g/2} = \zeta^g$$
$$\dim_R^\zeta(g) = \mathrm{ev}_{g^{-1}}^\zeta \cdot \mathrm{coev}_g^\zeta = \zeta^{-g/2}\zeta^{-g/2} = \zeta^{-g}.$$

Hence by picking a different dual functor, we can still arrange for $\varphi = \mathrm{id}$, but since the quantum dimensions are different, this pivotal structure is inequivalent to the spherical structure.

We end this section by unpacking how the group $\mathrm{Hom}(\mathcal{U}_{\mathcal{C}} \to \mathbb{C}^\times)$ acts on the set of pivotal structures using the quantum dimensions.

Remark 3.6.24. Given $\delta : \mathcal{U}_{\mathcal{C}} \to \mathbb{C}^\times$, the pivotal structure $\varphi \circ \delta$ has left and right quantum dimensions given by

$\dim_L^{\varphi\circ\delta} =$  $= \delta_c^{-1} \dim_L^{\varphi}(c)$ and $\dim_R^{\varphi\circ\delta}(c) =$  $= \delta_c \dim_R^{\varphi}(c).$ (3.6.25)

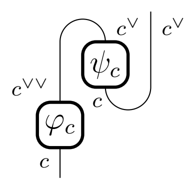
This means that for all pivotal structures, the product of the left and right dimensions of a simple object is always the same!

Corollary 3.6.26. *If $\mathcal{C}$ is a pivotal multitensor category, then for every $c \in \mathrm{Irr}(\mathcal{C})$, the product $\dim_L^{\varphi}(c) \cdot \dim_R^{\varphi}(c)$ is* independent *of the choice of pivotal structure.*

3.7. **Self-duality and Frobenius-Schur indicator.** In this section, $\mathcal{C}$ is a pivotal multitensor category.

Definition 3.7.1. A *self-duality* for an object $c \in \mathcal{C}$ is an isomorphism $\psi_c : c \to c^\vee$. (A self-duality need not exist.) An object is called *self-dual* if it has a self-duality.

Suppose now $c \in \mathrm{Irr}(\mathcal{C})$ with self-duality $\psi_c : c \to c^\vee$. Observe that $\mathcal{C}(c \to c^\vee) \cong \mathcal{C}(c \to c) = \mathbb{C}\,\mathrm{id}_c$, so $\mathcal{C}(c \to c^\vee) = \mathbb{C}\psi_c$, i.e., any other choice of self-duality differs from ψ_c by a scalar. Consider the π-rotation operator on $\mathcal{C}(c \to c^\vee)$:

$\rho(\psi_c) :=$  $\in \mathcal{C}(c \to c^\vee) = \mathbb{C}\psi_c.$

We see there is some $\lambda \in \mathbb{C}^\times$ such that $\rho(\psi_c) = \lambda\psi_c$. Since $(\mathcal{C}, \vee, \varphi)$ is pivotal,

$$\psi_c = \rho^2(\psi_c) = \lambda\rho(\psi_c) = \lambda^2\psi_c \qquad \Longrightarrow \qquad \lambda^2 = 1.$$

Observe that the scalar λ is *independent* of the choice of ψ_c. We call λ the (second) *Frobenius-Schur indicator* of the simple object c [NS07].

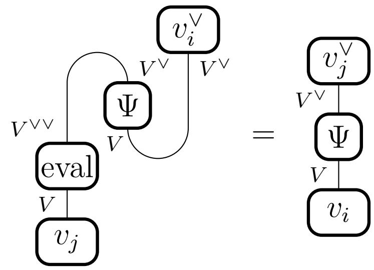
When $c \in \mathcal{C}$ is not simple, we will still have that $\rho^2 = \mathrm{id}$, but ψ_c may no longer be an eigenvector.

Definition 3.7.2. Suppose $c \in \mathcal{C}$ with self-duality $\psi_c : c \to c^\vee$. If $\rho(\psi_c) = \psi_c$, we call ψ_c a *symmetric self-duality.* If $\rho(\psi_c) = -\psi_c$, we call ψ_c an *anti-symmetric self-duality.*

When c is simple, then a self-duality is either symmetrically or anti-symmetrically self-dual, and this property is independent of the choice of self-duality. We thus call c symmetrically/anti-symmetrically self-dual accordingly.

Exercise 3.7.3. Show that any object of the form $c \oplus c^\vee$ has a canonical symmetric self-duality.

Example 3.7.4. Suppose V is a finite dimensional complex vector space, let $\{v_i\}$ be a basis for V, and let $\{v_i^\vee\}$ be the dual basis for $V^\vee$ determined by the formula $v_i^\vee(v_j) = \delta_{i=j}$. Every isomorphism $\psi : V \to V^\vee$ is of the form $v_i \mapsto \sum_j \Psi_{ij} v_j^\vee$ where $\Psi \in M_n(\mathbb{C})$ is invertible. To calculate $\rho(\psi)$, we compute

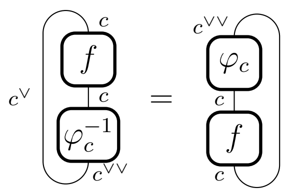
$\rho(\Psi)_{ij} =$  $= \Psi_{ji}.$

Thus ψ is symmetric if and only if $\Psi = \Psi^T$, and ψ is anti-symmetric if and only if $\Psi = -\Psi^T$.

Exercise 3.7.5. There is a non-trivial 3-cocycle ω on $\mathbb{Z}/2 = \{1, g\}$ such that $\omega(g, g, g) = -1$ and all other values are $+1$. Prove that the non-trivial simple object $g \in \mathsf{Vec}(\mathbb{Z}/2, \omega)$ is anti-symmetrically self-dual, i.e., g has Frobenius-Schur indicator -1.

3.8. **Sphericality.** In the beginning of this section, we assume $\mathcal{C}$ is a pivotal tensor category; very soon after, we assume $\mathcal{C}$ is pivotal multitensor.

Definition 3.8.1. A pivotal tensor category is called *spherical* if

$\operatorname{tr}_L^\varphi(f) =$  $= \operatorname{tr}_R^\varphi(f) \qquad \forall\, f : c \to c, \forall\, c \in \mathcal{C};$

equivalently, $\dim_L^\varphi(c) = \dim_R^\varphi(c)$ for all simple $c \in \operatorname{Irr}(\mathcal{C})$.

Proposition 3.8.2. *If the tensor category $\mathcal{C}$ admits a spherical structure $(\vee, \varphi)$, then the set of pivotal structures (with the same $\vee$) is a torsor for the group $\operatorname{Hom}(\mathcal{U}_\mathcal{C} \to \mathbb{Z}/2)$ where $\mathbb{Z}/2 = \{\pm 1\} \subset \mathbb{C}^\times$.*

Proof. Since φ is a pivotal structure, all other pivotal structures are of the form $\varphi \circ \delta$ for some $\delta \in \operatorname{Aut}_\otimes(\operatorname{id}_\mathcal{C}) \cong \operatorname{Hom}(\mathcal{U}_\mathcal{C} \to \mathbb{C}^\times)$ by Remark 3.6.8 and Proposition 3.6.10. Now looking at (3.6.25), $\varphi \circ \delta$ is spherical if and only if $\delta_c = \delta_c^{-1}$ for all $c \in \operatorname{Irr}(\mathcal{C})$, i.e., $\delta_c = \pm 1$. $\square$

Remark 3.8.3. A tensor category admits at most one pseudounitary spherical structure. Indeed, by Proposition 3.8.2, only the constant function $\delta : \mathcal{U}_\mathcal{C} \to 1_\mathbb{C}$ preserves sphericality and positivity of dimensions.

We now discuss sphericality for multitensor categories. Since $1_\mathcal{C}$ is no longer simple, the left and right traces are no longer comparable, as we may have $s(c) \neq t(c)$ for $c \in \operatorname{Irr}(\mathcal{C})$. We now return to our discussion of gradings.

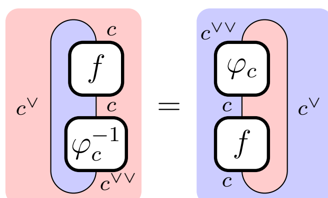
Definition 3.8.4. A *grading* on a multitensor category $\mathcal{C}$ is a groupoid $\mathcal{G}$ (which we identify with its morphisms as above) and a decomposition $\mathcal{C} = \bigoplus_{g \in \mathcal{G}} \mathcal{C}_g$ into semisimple subcategories such that $\mathcal{C}_g \otimes \mathcal{C}_h \subseteq \mathcal{C}_{gh}$ for all composable $g, h \in \mathcal{G}$. When g, h are not composable, $\mathcal{C}_g \otimes \mathcal{C}_h = 0$. A grading is *faithful* if $\mathcal{C}_g \neq 0$ for all $g \in \mathcal{G}$.

Exercise 3.8.5. First prove that $\mathcal{C}$ is faithfully graded by $\mathcal{U}_{\mathcal{C}}$. Then prove that every faithful $\mathcal{G}$-grading on $\mathcal{C}$ induces a canonical groupoid surjection $\mathcal{U}_{\mathcal{C}} \twoheadrightarrow \mathcal{G}$. Deduce that a groupoid map $\delta : \mathcal{G} \to \mathbb{C}^\times$ can be extended to a groupoid map $\mathcal{U}_{\mathcal{C}} \to \mathbb{C}^\times$.

Example 3.8.6. The $n \times n$ *matrix groupoid* $\mathcal{M}_n$ has n simple objects and a unique isomorphism between any two objects. Observe that $\mathcal{M}_n$ is highly reminiscent of a system of matrix units for $M_n(\mathbb{C})$.

First, given a multitensor category $\mathcal{C}$ and a decomposition $1_{\mathcal{C}} = \bigoplus_{i=1}^n 1_i$ into simples, we get a $\mathcal{M}_n$ grading on $\mathcal{C}$ by $\mathcal{C}_{ij} \coloneqq 1_i \otimes \mathcal{C} \otimes 1_j$. If this grading is faithful, we say $\mathcal{C}$ is an $n \times n$ multitensor category. Observe that a non-zero multitensor category $\mathcal{C}$ is *indecomposable* (not a direct sum of non-zero multitensor categories) if and only if $\mathcal{C}$ is $n \times n$ for some $n \in \mathbb{N}$.

Definition 3.8.7. A pivotal structure on a multitensor category $\mathcal{C}$ is called *spherical* if for all simples $1_i, 1_j \subseteq 1_{\mathcal{C}}$, $c \in \mathcal{C}_{ij}$, and $f : c \to c$, $\operatorname{Tr}_L^\varphi(f)_{ij} = \operatorname{Tr}_R^\varphi(f)_{ij}$, where $\operatorname{Tr}_{L/R}^\varphi$ are the $M_n(\mathbb{C})$-valued traces from (3.6.15). By an abuse of notation, this is the condition that



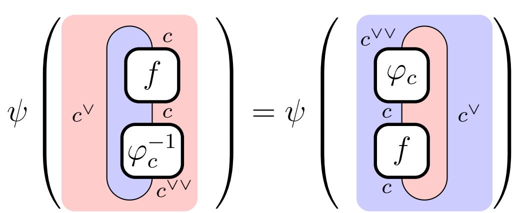
$= p_i$, $= p_j$. (3.8.8)

Strictly speaking, this does not make sense, as the regions are shaded differently; we mean only the scalar multiple of the corresponding identities of $1_i, 1_j$. To be more precise, sphericality is the condition that for every simple object $c \in \operatorname{Irr}(\mathcal{C})$, $\dim_L^\varphi(c) = \dim_R^\varphi(c)$.

In the multitensor setting, we can sometimes introduce an extra structure which allows one to 'correct' for a non-spherical pivotal structure.

Definition 3.8.9. Suppose $(\mathcal{C}, \vee, \varphi)$ is a pivotal category. A *spherical weight* is a linear functional $\psi : \operatorname{End}_{\mathcal{C}}(1_{\mathcal{C}}) \to \mathbb{C}$ such that

- (faithful) $\psi(p_i) \neq 0$ for all minimal idempotents $p_i \in \operatorname{End}_{\mathcal{C}}(1_{\mathcal{C}})$, and
- (spherical) $\psi \circ \operatorname{tr}_L^\varphi = \psi \circ \operatorname{tr}_R^\varphi$, i.e.,



$\forall f : c_{ij} \to c_{ij}$, $= p_i$, $= p_j$.

Equivalently, for all simple $c_{ij} \in \operatorname{Irr}(\mathcal{C}_{ij})$, $\psi(p_j) \dim_L^\varphi(c_{ij}) = \psi(p_i) \dim_R^\varphi(c_{ij})$.

Example 3.8.10. If $\mathcal{C}$ is a spherical multitensor category in the sense of (3.8.8) above, then every constant weight is spherical, e.g., the weight $\psi(p_i) = 1$ for all minimal idempotents $p_i \in \operatorname{End}(1_{\mathcal{C}})$ is spherical.

Suppose in addition $\mathcal{C}$ is $n \times n$ and $\delta : \mathcal{M}_n \to \mathbb{C}^\times$ is a groupoid map from the matrix groupoid as in Example 3.6.11, where we write $\delta_{ij} \in \mathbb{C}^\times$ for the image of the unique isomorphism $i \to j$. Using the matrix grading, we get a new pivotal structure on $\mathcal{C}$ by $\varphi \circ \delta$. Moreover, by (3.6.25), this new pivotal structure satisfies

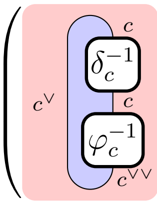
$$\dim_R^{\varphi\circ\delta}(c_{ij}) = \delta_{ij} \dim_R^\varphi(c_{ij}) = \delta_{ij} \dim_L^\varphi(c_{ij}) = \delta_{ij}^2 \dim_L^{\varphi\circ\delta}(c_{ij}) \qquad \forall\, c_{ij} \in \operatorname{Irr}(\mathcal{C}_{ij}).$$

In this case, this new pivotal structure also admits a spherical weight given by setting $\psi(p_1) = 1$ and

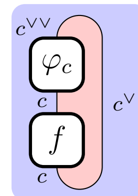
$$\psi(p_j) \coloneqq \delta_{1j}^2 \qquad\qquad \forall\, j > 1.$$

Indeed, for $c \in \operatorname{Irr}(\mathcal{C}_{ij})$, we have

$$\psi\left(\vphantom{\int}\right.$$



$$\left.\vphantom{\int}\right) = \underbrace{\psi(p_j)}_{=\delta_{1j}^2} \cdot \dim_L^{\varphi\circ\delta}(c_{ij}) = \underbrace{\frac{\delta_{1j}^2}{\delta_{ij}^2}}_{=\delta_{1i}^2 = \psi(p_i)} \underbrace{\delta_{ij}^2 \dim_L^{\varphi\circ\delta}(c_{ij})}_{=\dim_R^{\varphi\circ\delta}(c_{ij})} = \psi\left(\vphantom{\int}\right.$$



$$\left.\vphantom{\int}\right).$$

Example 3.8.11. Suppose $\mathcal{M}$ is a finite semisimple category. By the previous example, every pivotal structure on $\operatorname{End}(\mathcal{M})$ from Example 3.6.11 admits a spherical weight.

The above example is extremely informative about when one should expect the existence of a spherical weight.

Corollary 3.8.12. *Suppose $\mathcal{C}$ is an $n \times n$ multitensor category. A pivotal structure φ on $\mathcal{C}$ admits a spherical weight if and only if there is a $\delta \in \operatorname{Aut}_\otimes(\operatorname{id}_\mathcal{C}) \cong \operatorname{Hom}(\mathcal{U}_\mathcal{C} \to \mathbb{C}^\times)$ coming from a map $\mathcal{M}_n \to \mathbb{C}^\times$ such that $\varphi \circ \delta$ is spherical.*

Proof. If $\varphi\circ\delta$ is spherical, then $\varphi = \varphi\circ\delta\circ\delta^{-1}$ admits a spherical weight by Example 3.8.10. Conversely, if φ admits a spherical weight ψ, then for all $c \in \operatorname{Irr}(\mathcal{C}_{ij})$,

$$\psi(p_j)\dim_L^\varphi(c_{ij}) = \psi(p_i)\dim_R^\varphi(c_{ij}) \qquad \Longrightarrow \qquad \dim_L^\varphi(c_{ij}) = \frac{\psi(p_i)}{\psi(p_j)}\dim_R^\varphi(c_{ij}).$$

For each $j > 1$, let δ_{1j} be any square root of $\frac{\psi(p_1)}{\psi(p_j)}$, which completely determines δ as a groupoid homomorphism $\mathcal{M}_n \to \mathbb{C}^\times$. Observe that $\delta_{ij}^2 = \delta_{1i}^{-2}\delta_{1j}^2 = \frac{\psi(p_i)}{\psi(p_j)}$ for all i, j. By (3.6.25), we calculate for all $c_{ij} \in \operatorname{Irr}(\mathcal{C}_{ij})$,

$$\dim_L^{\varphi\circ\delta}(c_{ij}) \underset{(3.6.25)}{=} \delta_{ij}^{-1}\dim_L^\varphi(c_{ij}) = \delta_{ij}^{-1}\underbrace{\frac{\psi(p_i)}{\psi(p_j)}}_{=\delta_{ij}^2}\dim_L^\varphi(c_{ij}) = \delta_{ij}\dim_R^{\varphi\circ\delta}(c_{ij}) \underset{(3.6.25)}{=} \dim_R^{\varphi\circ\delta}(c_{ij})$$

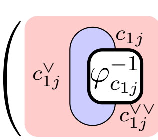
as desired. ☐

Lemma 3.8.13. *Suppose $\mathcal{C}$ is a multitensor category which admits a pivotal structure.*

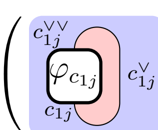
(1) Each pivotal structure $(\varphi, \vee)$ admits at most one spherical weight up to scaling by $(\mathbb{C}^\times)^{\#\text{components of } \mathcal{C}}$.
(2) Each weight $\psi : \operatorname{End}_\mathcal{C}(1_\mathcal{C}) \to \mathbb{C}$ admits at most one pivotal structure under which it is spherical.

Proof. To prove (1), it suffices to consider the case $\mathcal{C}$ is $n \times n$. Given $(\varphi, \vee)$, for each $j \neq 1$, pick $c_{1j} \in \operatorname{Irr}(\mathcal{C}_{1j})$. If ψ is a spherical weight, then

$$\dim_L^\varphi(c_{1j}) \cdot \psi(1_j) = \psi\left(\vphantom{\int}\right.$$



$$\left.\vphantom{\int}\right) = \psi\left(\vphantom{\int}\right.$$



$$\left.\vphantom{\int}\right) = \dim_R^\varphi(c_{1j}) \cdot \psi(1_1).$$

Since $\dim^{\varphi}_{L/R}(c_{1j}) \neq 0$, we see that $\psi(1_j)$ is completely determined by $\psi(1_1)$, so there is a unique choice up to scaling.

To prove (2), we show that if ψ is a spherical weight for the pivotal structure φ on $\mathcal{C}$, then for every non-trivial $\delta \in \operatorname{Aut}_\otimes(\operatorname{id}_\mathcal{C}) \cong \operatorname{Hom}(\mathcal{U}_\mathcal{C} \to \mathbb{C}^\times)$, ψ is not spherical for $\varphi \circ \delta$. Indeed, for a non-trivial δ, choose such that $\delta_c \neq 1$. Then by (3.6.25),

$$\psi(p_i)\dim_R^{\varphi\circ\delta}(c_{ij}) = \delta_c\psi(p_i)\dim_R^{\varphi}(c_{ij}) = \delta_c\psi(p_j)\dim_L^{\varphi}(c_{ij}) = \delta_c^2\psi(p_j)\dim_L^{\varphi\circ\delta}(c_{ij}).$$

We conclude ψ is not spherical for $\varphi \circ \delta$ for any non-trivial δ. □

Exercise 3.8.14. Suppose $\mathcal{C}$ is a pivotal multitensor category and $1_\mathcal{C} = \bigoplus_{i=1}^n 1_i$ is a decomposition into simples. We say an object $c \in \mathcal{C}_{ij}$ has *constant distortion* $\Delta \in \mathbb{C}^\times$ if

$$\operatorname{tr}_L^{\varphi}(f) = \Delta \cdot \operatorname{tr}_R^{\varphi}(f) \qquad \forall\, f \in \operatorname{End}_\mathcal{C}(c)$$

under the identification $\operatorname{End}_\mathcal{C}(1_i) \cong \mathbb{C}$ and $\operatorname{End}_\mathcal{C}(1_j) \cong \mathbb{C}$ by mapping the identities to $1_\mathbb{C}$.

(1) Show all simples have constant distortion.
(2) Show that if $c \in \mathcal{C}_{ij}$ has constant distortion Δ_c, then so does every subobject $b \subseteq c$.
(3) Show that if $a \in \mathcal{C}_{ij}$ and $b \in \mathcal{C}_{jk}$ have constant distortion Δ_a and Δ_b respectively, then $a \otimes b \in \mathcal{C}_{ik}$ has constant distortion $\Delta_a\Delta_b$.
(4) Show that every corner $\mathcal{C}_{ii}$ of $\mathcal{C}$ has a $\mathbb{C}^\times$ grading $\mathcal{C}_{ii} = \bigoplus(\mathcal{C}_{ii})_z$ where $(\mathcal{C}_{ii})_z$ is the semisimple subcategory of $\mathcal{C}_{ij}$ whose objects have constant distortion z. Taking $\mathcal{G} \subset \mathbb{C}^\times$ to be the subgroup such that $(\mathcal{C}_{ii})_z \neq 0$, deduce that $\mathcal{C}$ is faithfully $\mathcal{G}$-graded.
(5) Show that $\mathcal{C}_e$ is spherical for every idempotent $e \in \mathcal{U}_\mathcal{C}$.
(6) Prove that the map $\Delta : \operatorname{Irr}(\mathcal{C}) \to \mathbb{C}^\times$ given by

$$c \longmapsto \Delta_c := \frac{\dim_L(c)}{\dim_R(c)}$$

gives a groupoid homomorphism from the universal grading groupoid $\mathcal{U}_\mathcal{C}$ to $\mathbb{C}^\times$.

4. Unitary fusion categories

4.1. Dagger monoidal categories.

Definition 4.1.1. A *dagger monoidal category* is a dagger category with a monoidal structure such that $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a linear †-functor, and all associators and unitor isomorphisms are unitary. We call a dagger monoidal category a C* *monoidal category* if the underlying dagger category is unitary. We call a C* monoidal category:

- a *unitary (multi)tensor category* if $\mathcal{C}^\natural$ is a (multi)tensor category (recall that ♮ denotes forgetting the † structure.),
- and a *unitary (mulit)fusion category* if $\mathcal{C}^\natural$ is (multi)fusion.

Just as with the adjective *tensor*, we reserve the desirable adjective *unitary* for rigid C* monoidal categories. (It would make no difference at this categorical level to use unitary monoidal category instead of C* monoidal, but when we get to 2-categories, we will want the adjective 'unitary' to include having adjoints for 1-morphisms.)

Example 4.1.2. Let G be a finite group, and let $U(1)$ denote the unitary group of unimodular complex scalars. The unitary category $\mathsf{Hilb}(G, \omega)$ for $\omega \in Z^3(G, U(1))$ has objects G-graded finite dimensional Hilbert spaces with grading preserving linear maps. The tensor structure is given similar to Example 3.1.3.

Example 4.1.3. For G a finite group, $\mathsf{Rep}^\dagger(G)$ is a C* monoidal category with tensor product similar to that in Example 3.1.4.

Similar to Examples 3.1.18, 3.1.19, and 3.1.20, when $\mathcal{C}$ is a unitary monoidal category, $\mathsf{Add}^\dagger(\mathcal{C})$, $\mathsf{Proj}(\mathcal{C})$, and $\natural^\dagger(\mathcal{C})$ are also C* monoidal categories. The canonical inclusions are strict monoidal, and the obvious universal properties hold.

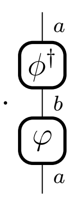
Definition 4.1.4. A *dagger monoidal functor* $F : \mathcal{C} \to \mathcal{D}$ between dagger monoidal categories is a dagger functor equipped with unitary tensorator and unitor coherence isomorphisms satisfying the same associative and unital axioms.

Two dagger monoidal categories $\mathcal{C}, \mathcal{D}$ are equivalent if there are dagger monoidal functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ together with unitary monoidal natural isomorphisms $F \circ G \cong \mathrm{id}_{\mathcal{D}}$ and $G \circ F \cong \mathrm{id}_{\mathcal{C}}$.

Exercise 4.1.5. Show that a dagger monoidal functor $F : \mathcal{C} \to \mathcal{D}$ between C* monoidal categories whose underlying functor is an equivalence of categories can be augmented to an equivalence of unitary monoidal categories.

4.2. **Unitary F-matrices and $6j$-symbols.** In this section, we give a *Yoneda-free* treatment of F-matrices and $6j$-symbols analogous to the way physicists think about unitary fusion categories, which we believe goes back to [MS89]. In particular, we show how to find the *unitary* F-matrices given a unitary fusion category $\mathcal{C}$ with fusion rule $(\mathrm{Irr}(\mathcal{C}), N_{\bullet\bullet}^{\bullet})$ using inner products.

For each $a \in \mathrm{Irr}(\mathcal{C})$ and $b \in \mathcal{C}$, we endow the hom space $\mathcal{C}(a \to b)$ with the *skein module inner product* determined by the formula

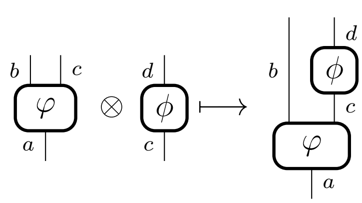
$$\langle \phi | \varphi \rangle_{\text{Skein}} \cdot \mathrm{id}_a = \frac{\sqrt{d_a}}{\sqrt{d_b}} \cdot (\phi^\dagger \cdot \varphi) = \frac{\sqrt{d_a}}{\sqrt{d_b}} \cdot$$  $$\in \mathcal{C}(a \to a) \cong \mathbb{C} \qquad \text{via} \qquad \mathrm{id}_a \mapsto 1_{\mathbb{C}}$$

where d_a, d_b denotes the Frobenius-Perron dimensions as in Definition 3.5.11. (Dimensions of open strings go on top, and dimensions of closed strings go on the bottom.)

In §4.4 below, we will show that this inner product comes from normalizing the unique unitary spherical structure on $\mathcal{C}$, so in particular, it is well-behaved under rotations. For the time being, we will content ourselves with proving that the skein module inner product behaves well under gluing.

Proposition 4.2.1. *Suppose $a \in \mathrm{Irr}(\mathcal{C})$ and $b, d \in \mathcal{C}$. The gluing map*

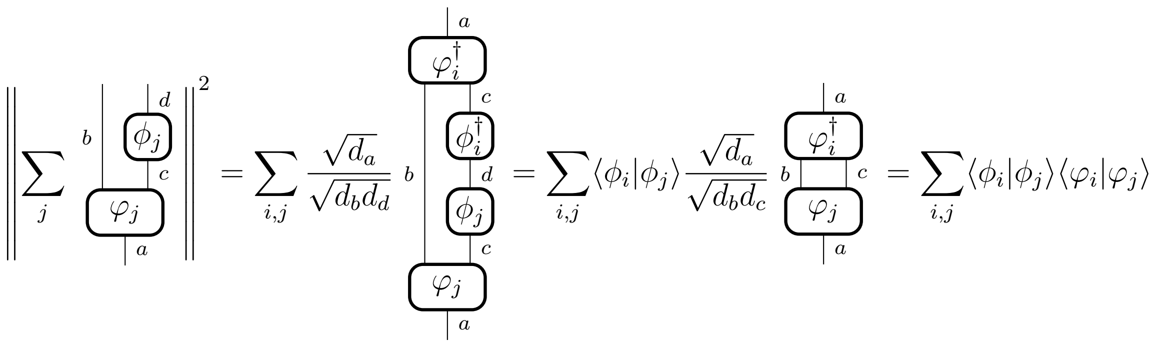
$$\bigoplus_{c \in \mathrm{Irr}(\mathcal{C})} \mathcal{C}(a \to b \otimes c) \otimes \mathcal{C}(c \to d) \longrightarrow \mathcal{C}(a \to b \otimes d)$$



is unitary using the skein module inner product.

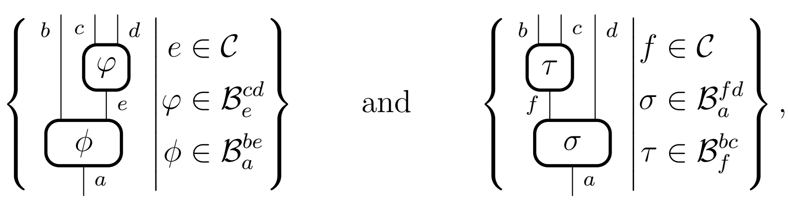
Proof. By semisimplicity, this map is an isomorphism; indeed, this can be seen by turning the b-string down to the left using dualizability and applying the semisimplicity criterion that composition is an isomorphism. We can then turn the b-string back up to get that gluing is an isomorphism.

Now since the direct sum on the left hand side is orthogonal, it suffices to show gluing is isometric on each summand. Suppose $\sum_j \varphi_j \otimes \phi_j \in \mathcal{C}(a \to b \otimes c) \otimes \mathcal{C}(c \to d)$. We calculate



which is exactly equal to $\| \sum_j \varphi_j \otimes \phi_j \|^2$ as claimed. □

For each triple $a, b, c \in \operatorname{Irr}(\mathcal{C})$, we fix an orthonormal basis $\mathcal{B}_c^{ab}$ (which has size N_{ab}^c) of the hom space $\mathcal{C}(c \to a \otimes b)$. By Proposition 4.2.1, for each $a, b, c, d \in \operatorname{Irr}(\mathcal{C})$, we get two orthonormal *tree bases* of the Hilbert space $\mathcal{C}(a \to b \otimes c \otimes d)$:

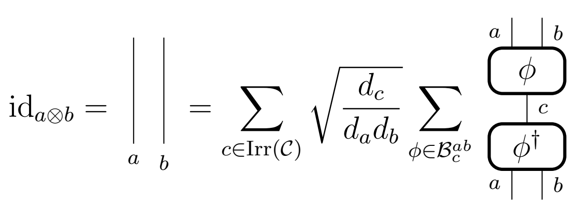


(4.2.2)

as the tensor product of two ONBs is again an ONB.

It will be convenient later on to have the following result.

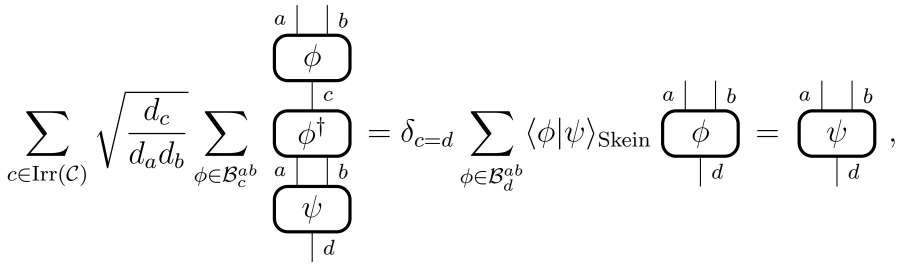
Lemma 4.2.3. *For $a, b \in \operatorname{Irr}(\mathcal{C})$, we have the following* fusion relation



(4.2.4)

Proof. To show two morphisms in $\mathcal{C}(a \otimes b \to a \otimes b)$ are equal, by the Yoneda Lemma, it suffices to prove that their actions by post-composition agree on $\mathcal{C}(d \to a \otimes b)$ for every $d \operatorname{Irr}(\mathcal{C})$. We look at the post-composition action of both sides of (4.2.4) on the ONB $\mathcal{B}_d^{ab}$ for

$d \in \mathrm{Irr}(\mathcal{C})$. For $\psi \in \mathcal{B}_d^{ab}$,

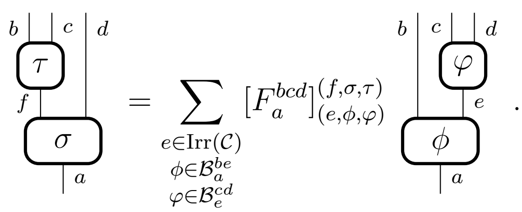


which agrees with the post-composition action by $\mathrm{id}_{a\otimes b}$. □

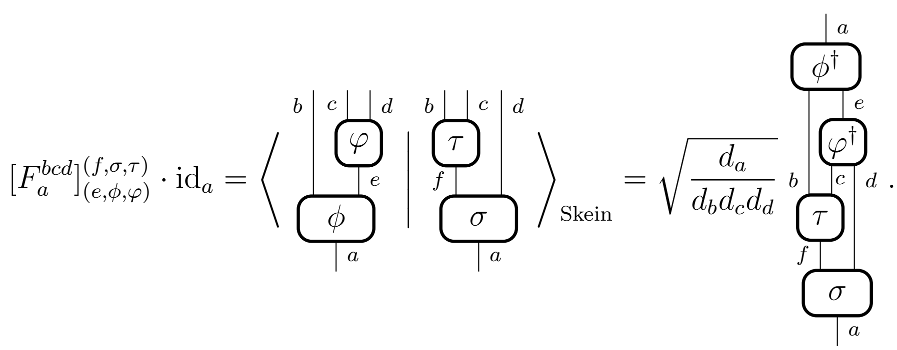
Since the two tree bases from (4.2.2) are orthonormal bases of the same Hilbert space, there is a unitary matrix F_a^{bcd} which maps between them. That is, F_a^{bcd} is a unitary transformation between the Hilbert spaces

$$F_a^{bcd} : \bigoplus_{e\in\mathrm{Irr}(\mathcal{C})} \mathcal{C}(a \to b\otimes e)\otimes\mathcal{C}(e\to c\otimes d) \longrightarrow \bigoplus_{f\in\mathrm{Irr}(\mathcal{C})} \mathcal{C}(a\to f\otimes d)\otimes \mathcal{C}(f\to b\otimes c).$$

The entries of the F-matrix are called $6j$*-symbols*, since in the multiplicity free case, they are determined by the 6 parameters a, b, c, d, e, f. In general, they are determined by 10 parameters, where we include the 4 basis elements $\phi, \varphi, \sigma, \tau$. In the graphical calculus, the $6j$-symbols are determined by the following formula:



Taking inner products with the tree basis diagram on the right hand side in the linear combination yields the following formula for the $6j$-symbols:

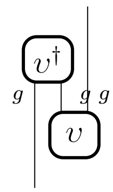

(4.2.5)

Remark 4.2.6. One might be concerned that in order to obtain the F-matrices and associated $6j$-symbols, we made choices of simples and orthonormal bases. However, different choices lead to what are called *gauge equivalent* F-matrices. We will not discuss this further.

Exercise 4.2.7. Given a set of fusion rules $(S, N_{\bullet\bullet}^{\bullet})$ and F-matrices satisfying (3.4.2), explain how to construct a corresponding *skeletal* fusion category.

4.3. **Examples of unitary fusion categories.** We give some important examples and exercises. Some of them are from the perspective of someone who knows algebraic fusion categories but does not think about them in terms of F-matrices. Others are from the F-matrix/$6j$-symbol perspective.

Exercise 4.3.1. Consider the non-trivial 3-cocycle on $\mathbb{Z}/2 = \{1, g\}$ such that $\omega(g, g, g) = -1$ and all other values are $+1$ from Exercise 3.7.5. Observe there is a unique normalized ONB element $\upsilon \in \mathcal{B}_1^{gg}$ up to phase. (Here we use υ to evoke the image of a cup/coevaluation.) Calculate the scalar λ such that

$$\lambda \operatorname{id}_g = (\upsilon^\dagger \otimes \operatorname{id}_g) \cdot (\operatorname{id}_g \otimes \upsilon) =$$  .

(Note that an associator is suppressed in the above string diagram!) Why is it dangerous to represent $\upsilon^\dagger$ by a cap?

Exercise 4.3.2. Classify all unitary fusion categories with fusion rules given by a finite group.

Example 4.3.3 (Fibonacci/Golden category)**.** One of the fusion categories with fusion rules

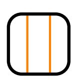
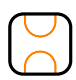
$$\tau \otimes \tau = 1 \oplus \tau$$

from Theorem 3.4.5 is unitary, namely the Fibonacci category (as opposed to the Yang-Lee category). Observe that the F-matrix

$$F_\tau^{\tau\tau\tau} = \begin{bmatrix} \phi^{-1} & \phi^{-1/2} \\ \phi^{-1/2} & -\phi^{-1} \end{bmatrix}$$

is unitary.

We can actually work out the semisimplicity skein relation

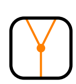
 $= \dfrac{1}{\phi}$  $+ \dfrac{1}{\sqrt{\phi}}$  , (3.4.7)

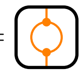
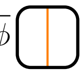
directly from the fusion rules. Suppose Fib is a unitary fusion category with the Fibonacci fusion rules. Since Frobenius-Perron dimension d is a $*$-algebra homomorphism $\mathcal{FA}(\mathsf{Fib}) \to \mathbb{C}$ by Proposition 3.5.12, we have

$$d_\tau^2 = 1 + d_\tau \qquad \Longrightarrow \qquad d_\tau = \frac{1 \pm \sqrt{5}}{2}.$$

Since Fib is unitary, we must have $d_\tau = \frac{1+\sqrt{5}}{2} = \phi > 0$, the golden ratio.

As before, we pick basis elements $\upsilon \in \mathsf{Fib}(1 \to \tau \otimes \tau)$ and $\gamma \in \mathsf{Fib}(\tau \to \tau \otimes \tau)$, but this time, we require they are the unique ONB elements up to phase in the skein module inner product. Again, we represent them graphically by

$\upsilon =$  $\qquad \gamma =$  , and we represent their adjoints by vertical reflection. Using the skein module inner product yields the relations

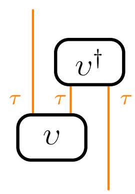
$$\upsilon^\dagger \upsilon =$$  $$= \phi \operatorname{id}_1 \qquad \gamma^\dagger\gamma =$$  $$= \sqrt{\phi}$$  .

Since 1 and τ are distinct simple objects, we have the relation

$$\upsilon^\dagger\gamma =$$  $$= 0.$$

We now see that the semisimplicity relation (3.4.7) is easily seen to be a decomposition of $\operatorname{id}_{\tau\otimes\tau}$ into a sum of minimal central projections in $\operatorname{End}(\tau\otimes\tau) \cong \mathbb{C}^2$.

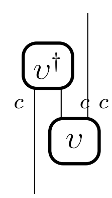
As a final comment, we observe that we can compute the zig-zag of υ and $\upsilon^\dagger$ directly in terms of $F_\tau^{\tau\tau\tau}$:



$$\underset{(4.2.5)}{=} \phi \cdot [F_\tau^{\tau\tau\tau}]_1^1 = \phi\cdot\phi^{-1} = 1.$$

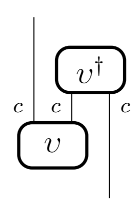
We thus see that $(\tau, \upsilon^\dagger, \upsilon)$ is a dual for τ.

Remark 4.3.4. When $\mathcal{C}$ is a unitary fusion category and $c \in \operatorname{Irr}(\mathcal{C})$ is self-dual, we will see that we can compute the Frobenius-Schur indicator λ such that $\rho(\psi_c) = \lambda\psi_c$ from §3.7 by the formula

$$\lambda \operatorname{id}_c = (\upsilon^\dagger \otimes \operatorname{id}_c)\cdot(\operatorname{id}_c \otimes \upsilon) = \tag{4.3.5}$$ 

where $\upsilon \in \mathcal{B}_1^{cc}$ is the unique normalized ONB element up to phase. Indeed, choosing this υ allows us to identify the objects $c = c^\vee = c^{\vee\vee}$, and we may choose $\psi_c = \operatorname{id}_c$. By Theorem 4.4.15 below, $\mathcal{C}$ has a unique unitary spherical structure, and by (4.4.4) below, the pivotal isomorphism $\varphi_c : c \to c$ is exactly (4.3.5).

Since $\upsilon^\dagger$ is also the unique spherical evaluation map $c\otimes c \to 1_{\mathcal{C}}$ up to phase, by (†∨3) below, we also have that λ satisfies the equation

$$\lambda \operatorname{id}_c = (\operatorname{id}_c \otimes \upsilon^\dagger)\cdot(\upsilon\otimes\operatorname{id}_c) =$$  $$= d_c \cdot [F_c^{ccc}]_1^1$$

as in the above calculation.

Example 4.3.6. At least one of the examples from Exercise 3.4.10 is unitary. That is, there is a unitary fusion category Ising with three simple objects $1, \sigma, \psi$ with fusion rules determined by

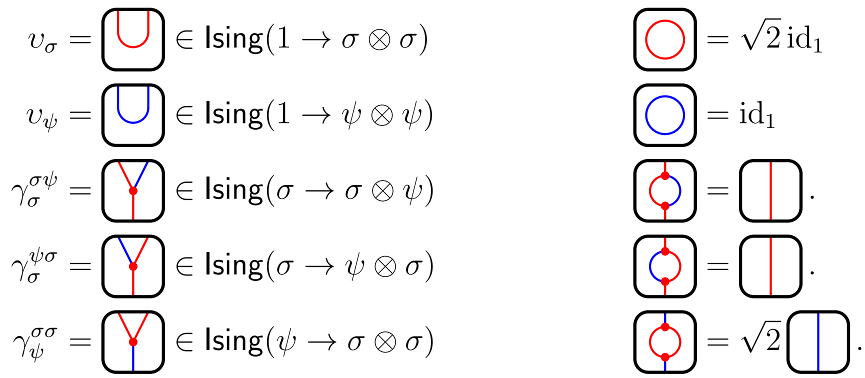
$$\sigma\otimes\sigma \cong 1\oplus\psi \qquad \text{and} \qquad \psi\otimes\psi\cong 1. \tag{3.4.11}$$

The subcategory generated by $1, \psi$ is equivalent to $\mathsf{Hilb}(\mathbb{Z}/2)$. The simple object σ is sometimes called a *Majorana fermion*. (The category Ising actually admits a modular *braiding*; more on this in [[]] below.)

A simple dimension calculation shows

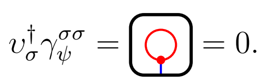
$$d_\sigma^2 = 1 + d_\psi = 2 \qquad \Longrightarrow \qquad d_\sigma = \sqrt{2}.$$

We denote σ by a red string and ψ by a blue string. We denote the unique normalized basis elements up to phase by

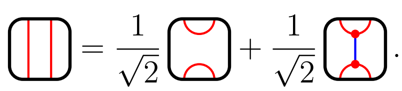


We denote their daggers by their vertical reflections, and we have the normalizations on the right hand side above. We warn the reader that while the blue cup and its adjoint satisfy the zig-zag relation (3.2.5), the red cup and its adjoint do *not*; rather, they zig-zag up to a minus sign as in Exercise 4.3.1. (Perhaps there is another such unitary fusion category where the red cup and its adjoint satisfy the zig-zag relation; see Exercise 4.3.7 below.)

Since 1 and ψ are distinct simple objects, we have the relation



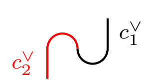
The fusion relation (4.2.4) is then given by



Exercise 4.3.7. How many of the fusion categories from Exercise 3.4.10 with fusion rules (3.4.11) are unitary?

TODO: maybe do Tambara-Yamagami diagrammatically earlier?

4.4. **Unitary dual functors.** We saw in Exercise 3.6.2 above that a dual functor on a multitensor category is unique up to canonical monoidal natural isomorphism.

$\zeta_c =$  $= (\mathrm{ev}_2 \otimes \mathrm{id}_{c^{\vee_1}}) \cdot (\mathrm{id}_{c^{\vee_2}} \otimes \mathrm{coev}_1)$

Thus a dual functor is not really additional structure on a multitensor category.

However, this is no longer the case when we consider dual functors compatible with the dagger structure on a unitary multitensor category. Indeed, the canonical monoidal natural isomorphism $\zeta : \vee_1 \Rightarrow \vee_2$ need not be unitary, so what was once a contractible space may now fracture into many disconnected components.

In this section, we will see that a choice of *unitary dual functor* (which always exists!) is a structure on a unitary multitensor category, and each unitary dual functor induces a canonical unitary pivotal structure. This is surprising in comparison with ordinary multitensor categories, where the dual functor is not interesting at all, and a pivotal structure is an important piece of additional structure (which is not known to always exist!).

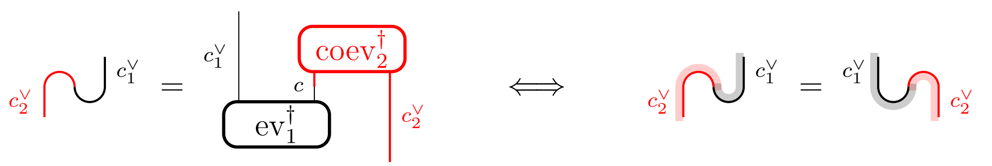
As we will often us the adjoints of evaluations and coevaluations in this section, we introduce the following graphical calculus.

Notation 4.4.1. To differentiate between strands for c and $c^\vee$, we include a *framing*, which is a lighter shaded thick line to one side of the strand for c or $c^\vee$ as follows:

We can then represent $\mathrm{ev}_c, \mathrm{coev}_c, \mathrm{ev}_c^\dagger, \mathrm{coev}_c^\dagger$ unambiguously as

Unfortunately, framing does not really help us at this point when $c^{\vee\vee}$ also appears in a diagram with c and $c^\vee$, so we will not be able to completely avoid the use of coupons labelled with $\mathrm{ev}_c^\dagger$ and $\mathrm{coev}_c^\dagger$ at this time.

Example 4.4.2. Using the framed graphical calculus, ζ_c is unitary if and only if



(4.4.3)

Given a unitary multitensor category $\mathcal{C}$ and an arbitrary dual $(c^\vee, \mathrm{ev}_c, \mathrm{coev}_c)$ for $c \in \mathcal{C}$, we can automatically write down a canonical isomorphism $c \to c^{\vee\vee}$.

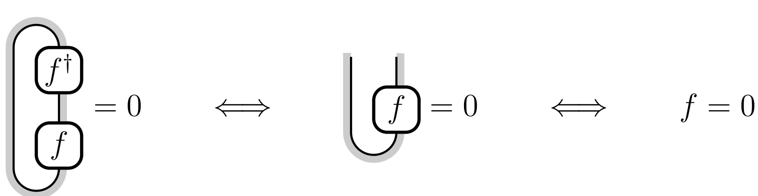
(4.4.4)

In our study of unitary dual functors, we will determine when these maps assemble into a unitary monoidal natural isomorphism $\mathrm{id}_\mathcal{C} \Rightarrow \vee \circ \vee$. We begin with some preliminary results.

Lemma 4.4.5. *Suppose $a \in \mathcal{C}$ and $(a^\vee, \mathrm{ev}_a, \mathrm{coev}_a)$ is an arbitrary dual. Then*

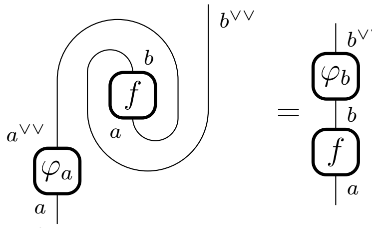
are both faithful $\mathcal{C}(1 \to 1)$-valued positive linear functionals on $\mathrm{End}_\mathcal{C}(a)$ in the sense that for all $f : a \to b$, $f^\dagger f \mapsto 0$ if and only if $f = 0$.

Proof. We prove the first map is faithful and the second is similar. By positivity,

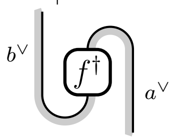


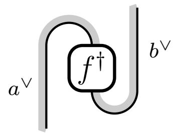
□

Proposition 4.4.6. *Suppose $a, b \in \mathcal{C}$ with arbitrary duals $(a^\vee, \mathrm{ev}_a, \mathrm{coev}_a)$ and $(b^\vee, \mathrm{ev}_b, \mathrm{coev}_b)$ respectively. The following are equivalent.*

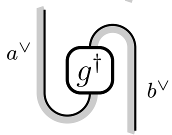
(1) 

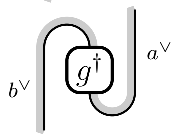
for all $f \in \mathcal{C}(a \to b)$.

(2) 

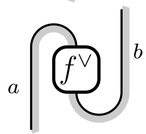
$= f^{\vee\dagger} = f^{\dagger\vee} =$ 

for all $f \in \mathcal{C}(a \to b)$.

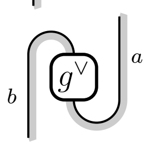
(3) 

$= g^{\vee\dagger} = g^{\dagger\vee} =$ 

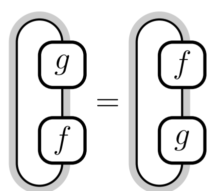
for all $g \in \mathcal{C}(b \to a)$.

(4) $f =$ 

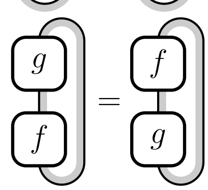
for all $f \in \mathcal{C}(a \to b)$.

(5) $g =$ 

for all $g \in \mathcal{C}(b \to a)$.

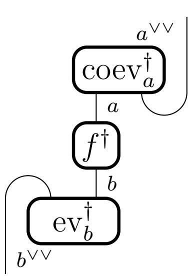
(6) For all $f \in \mathcal{C}(a \to b)$ and $g \in \mathcal{C}(b \to a)$. 

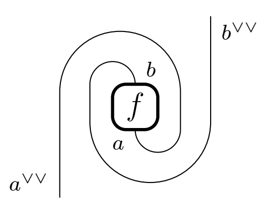
.

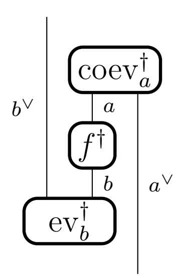
(7) For all $f \in \mathcal{C}(a \to b)$ and $g \in \mathcal{C}(b \to a)$, 

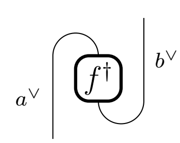
.

Proof.
(1) $\Leftrightarrow$ (2): Using the formula for φ_b^{-1} from (4.4.4), (1) is equivalent to



$=$ 

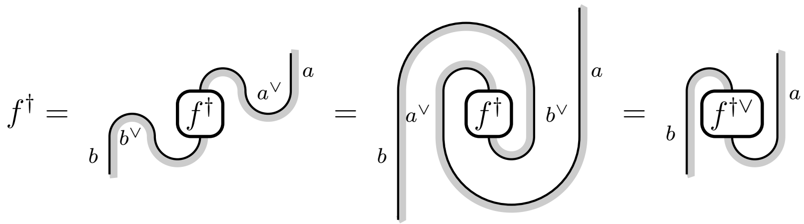
$\iff$ 

$=$ 

which is exactly (2).
(2) $\Leftrightarrow$ (3): Observe that $f^{\dagger\vee} = f^{\vee\dagger}$ for all $f \in \mathcal{C}(a \to b)$ if and only if

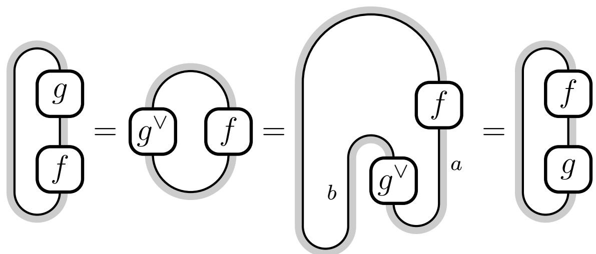
$$(f^\dagger)^{\vee\dagger} = f^{\dagger\vee\dagger} = f^{\vee\dagger\dagger} = f^{\dagger\dagger\vee} = (f^\dagger)^{\dagger\vee} \qquad \forall\, f^\dagger \in \mathcal{C}(b \to a)$$

which is exactly (3) as every $g : b \to a$ arises as the dagger of $g^\dagger : a \to b$.
(2) $\Leftrightarrow$ (5): Condition (2) is equivalent to



$\forall f^\dagger : b \to a$

which is exactly (5) as every $g : b \to a$ arises as the dagger of $g^\dagger : a \to b$.
(3) $\Leftrightarrow$ (4): Similar to (2) $\Leftrightarrow$ (5) and omitted.
(5) $\Leftrightarrow$ (6): Suppose $g : b \to a$. By non-degeneracy of the left capping map from Lemma 4.4.5, (5) is equivalent to



$\forall\, f : a \to b.$

Since $g : b \to a$ was arbitrary, (5) is equivalent to (6).
(4) $\Leftrightarrow$ (7): Similar to (5) $\Leftrightarrow$ (6) and omitted. $\square$

Definition 4.4.7. A *unitary dual functor* on $\mathcal{C}$ is a choice of dual functor $\vee : \mathcal{C} \to \mathcal{C}^{\text{mop}}$ which is dagger monoidal, i.e., $f^{\vee\dagger} = f^{\dagger\vee}$ for all $f : a \to b$, and the canonical tensorator $\vee^2_{a,b} : b^\vee \otimes a^\vee \to (a \otimes b)^\vee$ is unitary.

Example 4.4.8. Consider the pivotal structures on $\text{End}(\mathcal{M})$ for a finite semisimple category $\mathcal{M}$ as in Exercise 3.6.11. When $\mathcal{M}$ is unitary, for any groupoid map $\delta : \mathcal{M}_n \to \mathbb{R}_{>0}$, we can define evaluations and coevaluations

$$\text{ev}_{ij} : E_{ij} \circ E_{ji} = E_{ii} \xrightarrow{\delta_{ij}^{-1/2}\,\text{id}_{E_{ii}}} E_{ii} \hookrightarrow 1_{\text{End}(\mathcal{M})}$$

$$\text{coev}_{ij} : 1_{\mathcal{C}} \twoheadrightarrow E_{jj} \xrightarrow{\delta_{ij}^{1/2}\,\text{id}_{E_{jj}}} E_{jj} = E_{ji} \circ E_{ij}.$$

One can check that this defines a UDF associated to any groupoid map $\delta : \mathcal{M}_n \to \mathbb{R}_{>0}$ whose quantum dimensions are given by

$$\dim_L^\delta(E_{ij}) = \delta_{ij}^{-1} \qquad \text{and} \qquad \dim_R^\delta(E_{ij}) = \delta_{ij}.$$

We will see in §4.5 below (specifically ($\dagger\vee$6)) that these UDFs are inequivalent, as the loop parameters do not agree.

We now curate a list of facts which will be essential in our study of unitary dual functors.

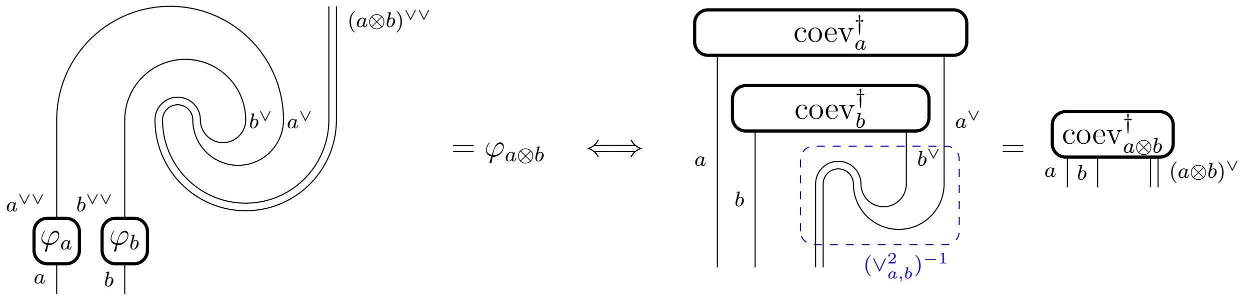
Facts 4.4.9. Let $\vee : \mathcal{C} \to \mathcal{C}^{\text{mop}}$ be a dual functor which is not necessarily unitary.

($\dagger\vee$1) By Proposition 4.4.6, $f^{\vee\dagger} = f^{\dagger\vee}$ for all $f : a \to b$ for all $a, b \in \mathcal{C}$ if and only if the isomorphisms $\varphi_c : c \to c^{\vee\vee}$ from (4.4.4) assemble into a natural transformation $\varphi : \text{id} \Rightarrow \vee \circ \vee$.

(†∨2) The canonical tensorator $\vee^2_{a,b} : b^\vee \otimes a^\vee \to (a \otimes b)^\vee$ is unitary if and only if φ from (4.4.4) satisfies the monoidality condition (3.1.14). Indeed, since

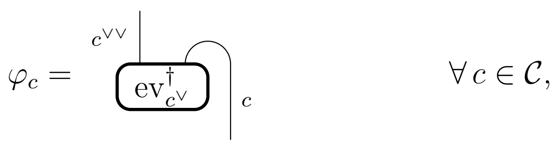
$$(\vee \circ \vee)^2_{a,b} = ((\vee^2_{a,b})^{-1})^\vee \circ \vee^2_{b^\vee, a^\vee}$$

φ is monoidal if and only if

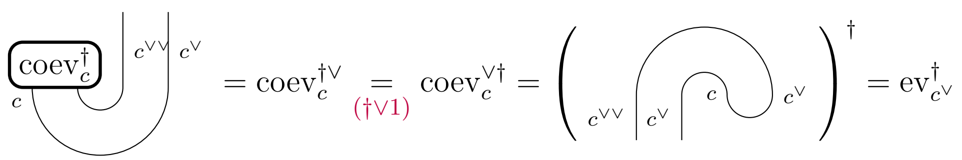


Now turning up the bottom left a, b strings using $\mathrm{ev}_a^\dagger, \mathrm{ev}_b^\dagger$ respectively in both diagrams on the right hand side, we see this equation is equivalent to unitarity of $\vee^2_{a,b}$.

(†∨3) If both (†∨1) and (†∨2) hold, then

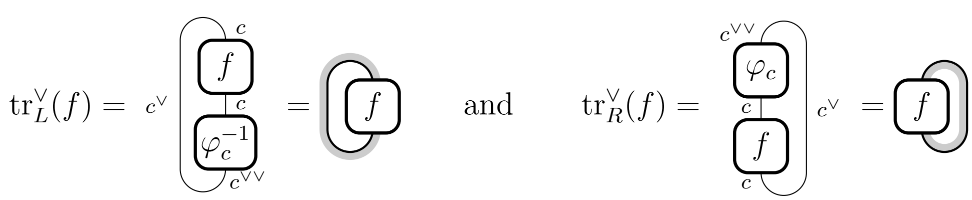


which is equivalent to unitarity of φ_c by (4.4.4). Hence a unitary dual functor induces a canonical unitary pivotal structure. Indeed,



which is equivalent to the alternate formula for φ_c above.

(†∨4) Given a unitary dual functor with its canonical unitary pivotal structure, the left and right pivotal traces can be written as

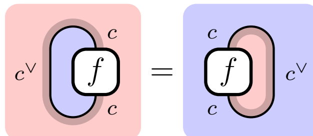
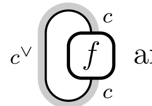
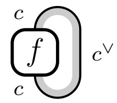


which shows they automatically take positive values on positive morphisms of the form $f^\dagger f$. An immediate corollary is that the quantum dimensions of all simple objects are always strictly positive.

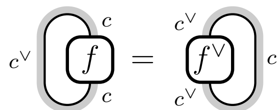
As a consequence, if $\delta \in \mathrm{Aut}_\otimes(\mathrm{id}_\mathcal{C}) \cong \mathrm{Hom}(\mathcal{U}_\mathcal{C} \to \mathbb{C}^\times)$, then by (3.6.25), it is only possible for $\varphi \circ \delta$ to come from a unitary dual functor if δ takes values in $\mathbb{R}_{>0}$.

We now turn to the question of constructing unitary dual functors.

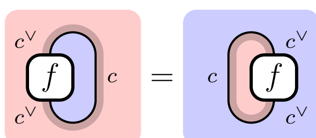
Definition 4.4.10. Suppose $\mathcal{C}$ is unitary multitensor and let $1_{\mathcal{C}} = \bigoplus 1_j$ be a decomposition into simples. As before, denote by $p_j \in \operatorname{End}_{\mathcal{C}}(1_{\mathcal{C}})$ the orthogonal projection onto 1_j. A dual $(c^\vee, \operatorname{ev}_c, \operatorname{coev}_c)$ for $c \in \mathcal{C}$ is called

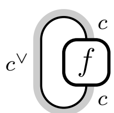
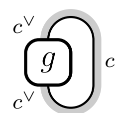
- *balanced* if  for all $f : c \to c$ and $= p_i$, $= p_j$, and
- *tracial* if the maps $f \mapsto$  and $f \mapsto$  are tracial on $\operatorname{End}_{\mathcal{C}}(c)$.

Remark 4.4.11. Suppose $(c^\vee, \operatorname{ev}_c, \operatorname{coev}_c)$ is a dual for $c \in \mathcal{C}$. Since

 $\qquad \forall f : c \to c,$

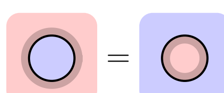
$(c^\vee, \operatorname{ev}_c, \operatorname{coev}_c)$ is balanced if and only if

 $\qquad \forall f : c^\vee \to c^\vee \qquad$ and $= p_i$, $= p_j$,

and $f \mapsto$  is tracial on $\operatorname{End}_{\mathcal{C}}(c)$ if and only if $g \mapsto$  is tracial on $\operatorname{End}_{\mathcal{C}}(c^\vee)$.

Proposition 4.4.12. *There exists a balanced tracial dual for each $c \in \mathcal{C}$.*

Proof. First, consider the simple $s \in \operatorname{Irr}(\mathcal{C})$. Choosing arbitrary duals $(s^\vee, \operatorname{ev}_s, \operatorname{coev}_s)$, we rescale $\operatorname{ev}_s, \operatorname{coev}_s$ so that

$d_s :=$  $\qquad$ $= p_i$, $= p_j$.

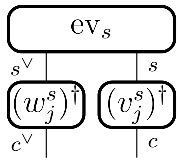
Here, 1_i is the source summand of $1_{\mathcal{C}}$ of s and 1_j is the target summand of $1_{\mathcal{C}}$ of s. As above in the definition of sphericality, these diagrams do not type-check; we mean only the scalar multiple of the corresponding identities of $1_i, 1_j$ here agree. We remark that the choice of scaling for $\operatorname{ev}_s, \operatorname{coev}_s$ to achieve sphericality is unique up to a phase in $U(1)$, so any other spherical choice of $\operatorname{ev}'_s, \operatorname{coev}'_s$ differs from $\operatorname{ev}_s, \operatorname{coev}_s$ up to a unique phase in $U(1)$. This proves both (1) and (2) for simple $s \in \mathcal{C}$.

We now extend the definition of duals on simples to duals on all objects of $\mathcal{C}$ as in (†∨7). We we pick isometries $\{v_j^s : s \to c\}_{j=1}^{m_s}$ and $\{w_j^s : s^\vee \to c^\vee\}_{j=1}^{m_s}$ decomposing $c = \bigoplus_{s \in \operatorname{Irr}(\mathcal{C})} s^{\oplus m_s}$ and $c^\vee = \bigoplus_{s \in \operatorname{Irr}(\mathcal{C})} (s^\vee)^{\oplus m_s}$ as an orthogonal direct sum of simples, i.e.,

$$\sum_{s \in \operatorname{Irr}(\mathcal{C})} \sum_{j=1}^{m_s} v_j^s (v_j^s)^\dagger = \operatorname{id}_c \qquad \text{and} \qquad \sum_{s \in \operatorname{Irr}(\mathcal{C})} \sum_{j=1}^{m_s} w_j^s (w_j^s)^\dagger = \operatorname{id}_{c^\vee}.$$

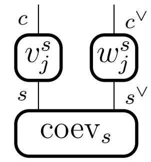
We define define $\mathrm{ev}_c, \mathrm{coev}_c$ in terms of $\mathrm{ev}_s, \mathrm{coev}_s$:

$$\mathrm{ev}_c^\delta := \sum_{s\in \mathrm{Irr}(\mathcal{C})}\sum_{j=1}^{m_s}$$



and

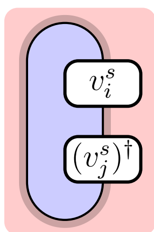
$$\mathrm{coev}_c^\delta := \sum_{s\in \mathrm{Irr}(\mathcal{C})}\sum_{j=1}^{m_s}$$



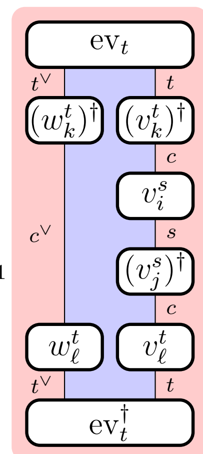
. (4.4.13)

It is clear $\mathrm{ev}_c^\delta, \mathrm{coev}_c^\delta$ satisfy the zig-zag relations.

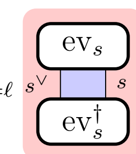
Now observe $\{v_i^s \cdot (v_j^s)^\dagger : s \to c\}_{i,j=1}^{m_s}$ is a system of matrix units for the $M_{m_s}(\mathbb{C})$ summand of $\mathrm{End}_{\mathcal{C}}(c)$. We calculate that



$$= \sum_{s\in \mathrm{Irr}(\mathcal{C})}\sum_{k,\ell=1}^{m_s}$$



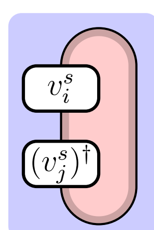
$$= \delta_{s=t}\delta_{k=\ell}\delta_{k=i}\delta_{j=\ell}$$



$$= \delta_{i=j} d_s,$$

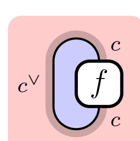
which is clearly a trace on the $M_{m_s}(\mathbb{C})$ summand of $\mathrm{End}_{\mathcal{C}}(c)$ as desired. Summing over p_i, p_j, we get the tracial property for the left capping map.

Similarly, we see that

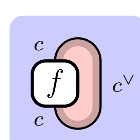


$$= \delta_{i,j} d_s,$$

so the right capping map is also a trace. Taking linear combinations of matrix units, we see that we have thus arranged so that for all $f : c \to c$ and all simple summands $1_i, 1_j$ of $1_{\mathcal{C}}$,



$=$



.

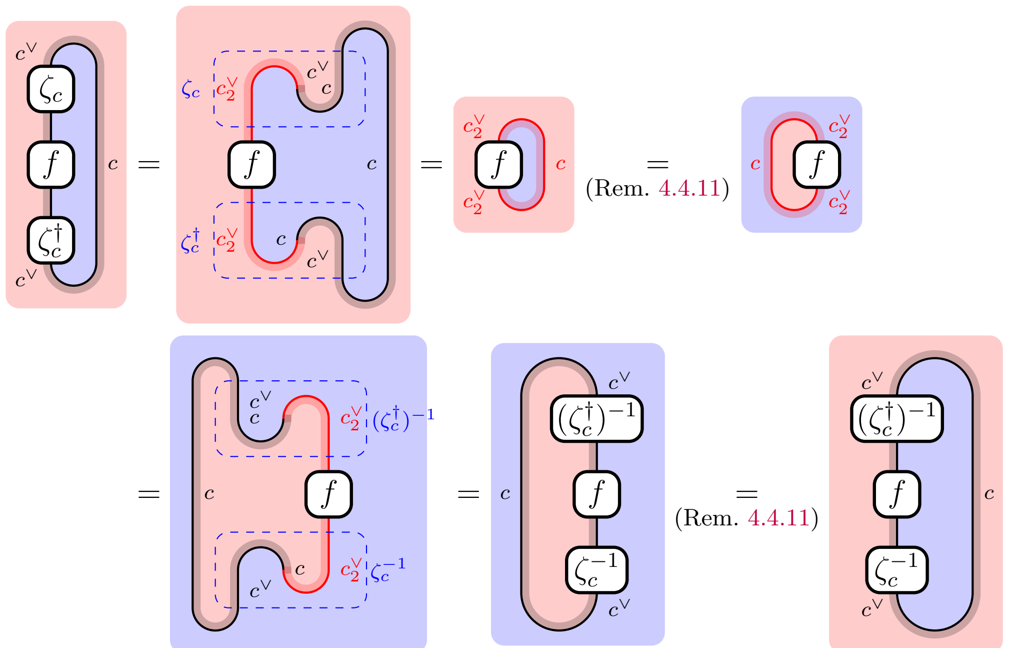
We have thus constructed a balanced tracial dual for c. □

Proposition 4.4.14. *Given two balanced duals for $c \in \mathcal{C}$, the canonical unitary isomorphism $\zeta_c : c_2^\vee \to c_1^\vee$ is unitary. In particular, every balanced dual is automatically tracial.*[2]

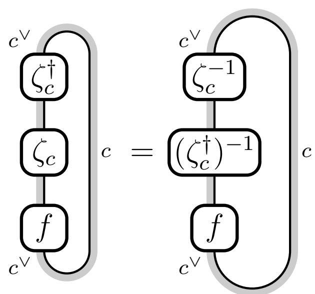
[2]There is a subtle mistake in the proof of [Pen20, Prop. 3.30]; the first paragraph on p40 implicitly assumes that a balanced dual is automatically tracial without proof.

Proof. Suppose $(c_2^\vee, \mathrm{ev}_2, \mathrm{coev}_2)$ is another balanced dual for c. Without loss of generality, we may assume that $c_2^\vee = c^\vee$. Indeed, there is some unitary isomorphism $c^\vee \to c_2^\vee$, and the dual $(c^\vee, (u \otimes \mathrm{id}_c) \cdot \mathrm{ev}_2, \mathrm{coev}_2 \cdot (\mathrm{id}_c \otimes u^\dagger)$ is balanced if and only if $(c_2^\vee, \mathrm{ev}_2, \mathrm{coev}_2)$ is balanced.

We now compute that for all $f : c^\vee \to c^\vee$,



Since $(c^\vee, \mathrm{ev}_c, \mathrm{coev}_c)$ is tracial and balanced, by Remark 4.4.11, this means that

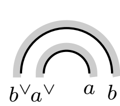
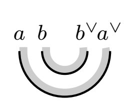


$\forall f : c^\vee \to c^\vee.$

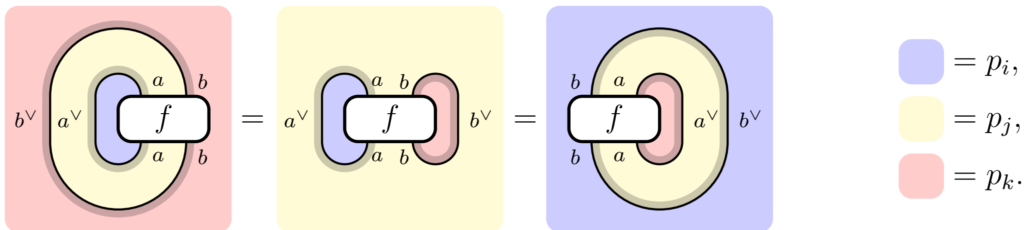
By non-degeneracy, we conclude that $\zeta_c^\dagger \zeta_c = \zeta_c^{-1}(\zeta_c^\dagger)^{-1} = (\zeta_c^\dagger \zeta_c)^{-1}$. But since $\zeta_c^\dagger \zeta_c$ is positive, the only way it can equal its own inverse is if it is equal to $\mathrm{id}_{c^\vee}$ by the Spectral Theorem [[]]. We conclude that ζ_c is unitary. This immediately implies that $(c_2^\vee, \mathrm{ev}_2, \mathrm{coev}_2)$ is tracial. □

Theorem 4.4.15. *Let $\mathcal{C}$ be a unitary multitensor category. There is a unique spherical UDF up to canonical unitary monoidal natural isomorphism.*

Proof. For each $c \in \mathcal{C}$, we set $(c^\vee, \mathrm{ev}_c, \mathrm{coev}_c)$ to be a balanced tracial dual from Proposition 4.4.12. By Proposition 4.4.6, $f^{\vee\dagger} = f^{\dagger\vee}$ for all $f : a \to b$. It remains to show that the canonical tensorator $\vee^2_{a,b}$ is unitary. Given balanced tracial duals $(a^\vee, \mathrm{ev}_a, \mathrm{coev}_a)$ and $(b^\vee, \mathrm{ev}_b, \mathrm{coev}_b)$ for $a, b \in \mathcal{C}$, the dual

$\Big(b^\vee \otimes a^\vee, \mathrm{ev} :=$  $, \mathrm{coev} :=$  $\Big)$

is also balanced as



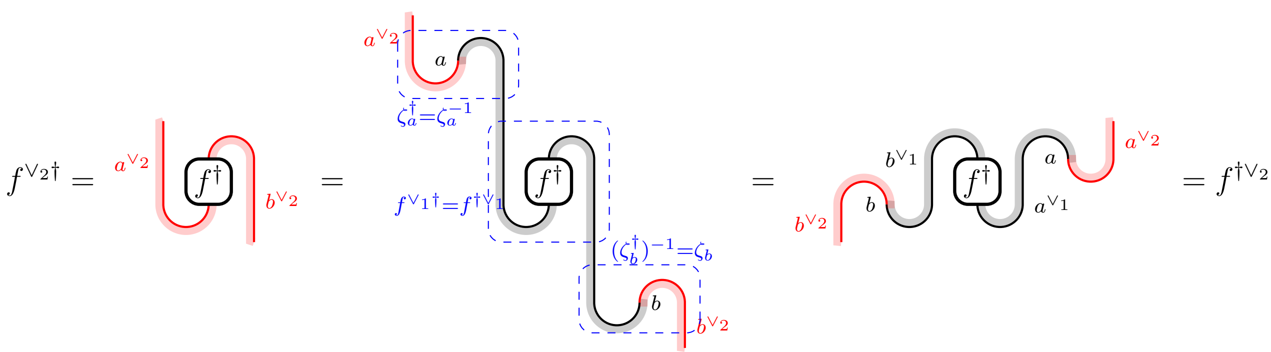
Now we have two balanced duals; the above one, and the balanced tracial dual $((a \otimes b)^\vee, \mathrm{ev}_{a\otimes b}, \mathrm{coev}_{a\otimes b})$ from Proposition 4.4.12. By Proposition 4.4.14, the canonical isomorphism $\zeta_{a\otimes b} : b^\vee \otimes a^\vee \to (a \otimes b)^\vee$ is unitary. But $\zeta_{a\otimes b}$ is exactly thecanonical tensorator $\vee^2_{a,b}$.

Uniqueness now follows directly from Proposition 4.4.14. □

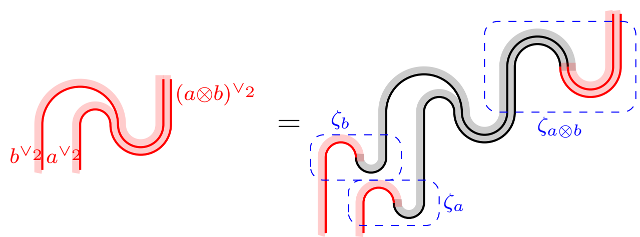
4.5. **Classifying unitary dual functors and bi-involutivity.** We now classify the unitary dual functors on a unitary multitensor category $\mathcal{C}$.

Facts 4.5.1. We proceed with additional facts about unitary dual functors.

(†∨5) If $\vee_1$ is a unitary dual functor and $\vee_2$ is another dual functor such that the canonical isomorphism $\zeta_c : c^{\vee_2} \Rightarrow c^{\vee_1}$ is unitary as in (4.4.3), then $\vee_2$ is unitary. First, for all $f : a \to b$,



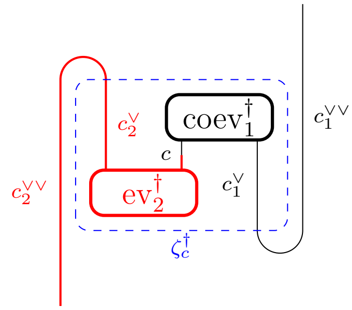
so (†∨1) holds. Second,



is visibly a composite of unitaries, so (†∨2) holds.

(†∨6) Two UDFs $\vee_1, \vee_2$ are unitarily equivalent if and only if the corresponding pivotal structures φ^1, φ^2 are equivalent, in which case the equivalence is necessarily unitary.

Indeed, unitarity of ζ_c is the condition (4.4.3), which is equivalent to the equation

$\varphi_c^1 \cdot (\varphi_c^2)^{-1} =$ 

$\underset{(4.4.3)}{=}$  .

(†∨7) Let $\mathrm{Aut}_\otimes^+(\mathrm{id}_\mathcal{C})$ be the subgroup of $\mathrm{Aut}_\otimes(\mathrm{id}_\mathcal{C})$ of $\delta : \mathrm{id}_\mathcal{C} \Rightarrow \mathrm{id}_\mathcal{C}$ such that $\delta_c > 0$ for all $c \in \mathrm{Irr}(\mathcal{C})$. Observe that the isomorphism $\mathrm{Aut}_\otimes(\mathrm{id}_\mathcal{C}) \cong \mathrm{Hom}(\mathcal{U}_\mathcal{C} \to \mathbb{C}^\times)$ restricts to an isomorphism $\mathrm{Aut}_\otimes^+(\mathrm{id}_\mathcal{C}) \cong \mathrm{Hom}(\mathcal{U}_\mathcal{C} \to \mathbb{R}_{>0})$.

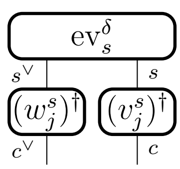
If $\vee$ is a unitary dual functor and $\delta \in \mathrm{Aut}_\otimes^+(\mathrm{id}_\mathcal{C}) \cong \mathrm{Hom}(\mathcal{U}_\mathcal{C} \to \mathbb{R}_{>0})$, we can define another unitary dual functor by

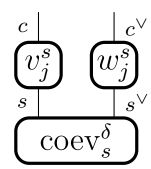
$$\mathrm{ev}_s^\delta := \delta_s^{-1/2}\,\mathrm{ev}_s \qquad \text{and} \qquad \mathrm{coev}_s^\delta := \delta_s^{1/2}\,\mathrm{coev}_s \qquad\qquad \forall\, s \in \mathrm{Irr}(\mathcal{C}),$$

and for arbitrary $c \in \mathcal{C}$, we define ev_c^δ and coev_c^δ by decomposing $c = \bigoplus_{s\in\mathrm{Irr}(\mathcal{C})} s^{\oplus m_s}$ and $c^\vee = \bigoplus_{s\in\mathrm{Irr}(\mathcal{C})} (s^\vee)^{\oplus m_s}$ as an orthogonal direct sum of simples, i.e., we pick isometries $\{v_j^s : s \to c\}_{j=1}^{m_s}$ and $\{w_j^s : s^\vee \to c^\vee\}_{j=1}^{m_s}$ such that

$$\sum_{s\in\mathrm{Irr}(\mathcal{C})}\sum_{j=1}^{m_s} v_j^s (v_j^s)^\dagger = \mathrm{id}_c \qquad \text{and} \qquad \sum_{s\in\mathrm{Irr}(\mathcal{C})}\sum_{j=1}^{m_s} w_j^s (w_j^s)^\dagger = \mathrm{id}_{c^\vee},$$

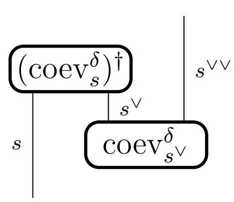
and we define $\mathrm{ev}_c^\delta, \mathrm{coev}_c^\delta$ in terms of $\mathrm{ev}_s^\delta, \mathrm{coev}_\delta^s$ as in (4.4.13):

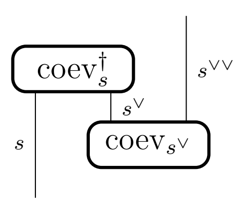
$\mathrm{ev}_c^\delta := \sum_{s\in\mathrm{Irr}(\mathcal{C})}\sum_{j=1}^{m_s}$ 

and $\mathrm{coev}_c^\delta := \sum_{s\in\mathrm{Irr}(\mathcal{C})}\sum_{j=1}^{m_s}$  .

It is clear $\mathrm{ev}_c^\delta, \mathrm{coev}_c^\delta$ satisfy the zig-zag relations. We also see that if we chose different isometries, the resulting $\mathrm{ev}_c^\delta, \mathrm{coev}_c^\delta$ would change by conjugating by unitaries, resulting in a unitarily equivalent dual functor.

To see $\vee^\delta$ is unitary, we see that we can arrange so that we *have not changed* φ_s *nor* $\vee_{s,t}^2$ for $s, t \in \mathrm{Irr}(\mathcal{C})$. Indeed,

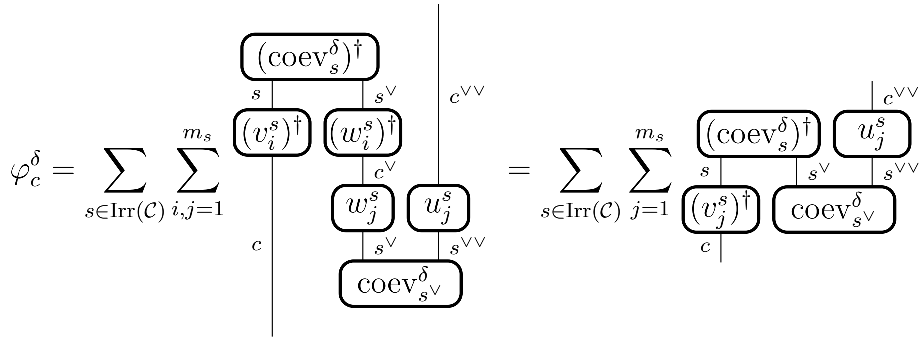
$\varphi_s^\delta =$ 

$= \delta_s^{1/2} \cdot \delta_s^{-1/2} \cdot$ 

$= \varphi_s$

for all simples $s \in \mathrm{Irr}(\mathcal{C})$. Picking isometries $\{w_j^s : s^\vee \to c^\vee\}_{j=1}^{m_s}$ as above and using the unitary isomorphisms $\varphi_s : s \to s^{\vee\vee}$ and $\varphi_c : c \to c^{\vee\vee}$ to get isometries

$\{u_j^s := \varphi_c \cdot v_j^s \cdot \varphi_s^\dagger : s^{\vee\vee} \to c^{\vee\vee}\}_{j=1}^{m_s}$, we have



$$= \sum_{s\in\operatorname{Irr}(\mathcal{C})} \sum_{j=1}^{m_s} u_j^s \cdot \varphi_s \cdot (v_j^s)^\dagger = \sum_{s\in\operatorname{Irr}(\mathcal{C})} \sum_{j=1}^{m_s} \varphi_c v_j^s \cdot \varphi_s^\dagger \cdot \varphi_s \cdot (v_j^s)^\dagger = \varphi_c.$$

Similarly, one shows $(\vee^\delta)^2_{a,b} = \vee^2_{a,b}$ for all $a, b \in \mathcal{C}$.

We thus get a new unitary dual functor $\vee^\delta$ whose quantum dimensions have been *distorted* by δ, just as in (3.6.25):

$$\dim_L^{\vee^\delta}(c) = \delta_c^{-1} \dim_L^\vee(c) \qquad \text{and} \qquad \dim_R^{\vee^\delta}(c) = \delta_c \dim_R^\vee(c) \qquad \forall\, c \in \operatorname{Irr}(\mathcal{C}).$$

Moreover, this δ-distorted UDF is unique up to unique unitary monoidal natural isomorphism.

(†∨8) Combining (†∨4), (†∨6), and (†∨7) with Proposition 3.6.22 and Theorem 4.4.15, we see that UDFs are classified by $\operatorname{Aut}_\otimes^+(\operatorname{id}_\mathcal{C}) \cong \operatorname{Hom}(\mathcal{U}_\mathcal{C} \to \mathbb{R}_{>0})$ up to unique unitary monoidal natural isomorphism. Observe that we may really view this classifying object as a group and not a torsor, as there is a canonical basepoint: the unitary spherical UDF.

(†∨9) When $\mathcal{C}$ is unitary fusion, $\mathcal{U}_\mathcal{C}$ is a finite group, and every group homomorphism $\mathcal{U}_\mathcal{C} \to \mathbb{C}^\times$ takes values in $U(1)$ (more precisely, in the roots of unity). Since $U(1) \cap \mathbb{R}_{>0} = \{1\}$, we see $\operatorname{Hom}(\mathcal{U}_\mathcal{C} \to \mathbb{R}_{>0}) = \{1\}$. Thus there is a unique UDF, which is the canonical spherical one constructed in Theorem 4.4.15. In particular, the left and right quantum dimensions must be equal to the Frobenius-Perron dimension from Definition 3.5.11.

TODO: bi-involutivity

References

[EGNO15] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor categories*, volume 205 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015. MR3242743 DOI:10.1090/surv/205.

[HV19] Chris Heunen and Jamie Vicary. *Categories for quantum theory*, volume 28 of *Oxford Graduate Texts in Mathematics*. Oxford University Press, Oxford, 2019. An introduction, MR3971584 DOI:10.1093/oso/9780198739623.001.0001.

[Lar14] Hannah K. Larson. Pseudo-unitary non-self-dual fusion categories of rank 4. *J. Algebra*, 415:184–213, 2014. MR3229513, DOI:10.1016/j.jalgebra.2014.05.032.

[LPR20] Zhengwei Liu, Sebastien Palcoux, and Yunxiang Ren. Classification of Grothendieck rings of complex fusion categories of multiplicity one up to rank six, 2020. arXiv:2010.10264.

[MS89] Gregory Moore and Nathan Seiberg. Classical and quantum conformal field theory. *Comm. Math. Phys.*, 123(2):177–254, 1989. MR1002038.

[NS07] Siu-Hung Ng and Peter Schauenburg. Higher Frobenius-Schur indicators for pivotal categories. In *Hopf algebras and generalizations*, volume 441 of *Contemp. Math.*, pages 63–90. Amer. Math. Soc., Providence, RI, 2007. MR2381536, arXiv:math/0503167.

[Ost03] Viktor Ostrik. Fusion categories of rank 2. *Math. Res. Lett.*, 10(2-3):177–183, 2003. MR1981895 arXiv:math.QA/0203255.

[Ost13] Victor Ostrik. Pivotal fusion categories of rank 3, 2013. (with an Appendix written jointly with Dmitri Nikshych), arXiv:1309.4822.

[Pen20] David Penneys. Unitary dual functors for unitary multitensor categories. *High. Struct.*, 4(2):22–56, 2020. MR4133163 arXiv:1808.00323.