Higher linear algebra concerns itself with higher vector/Hilbert spaces and their applications to mathematics and physics. In order to generalize ordinary linear algebra to the higher categorical context, we begin with a solid foundation of finite dimensional linear algebra, in particular, finite dimensional Hilbert spaces and operator algebras. It is assumed the reader has a solid understanding of finite dimensional linear algebra. All vector spaces will be finite dimensional unless stated otherwise, and we always work over the complex numbers.

We begin by studying finite dimensional Hilbert spaces. We do so abstractly rather than just the concrete Hilbert space $\mathbb{C}^n$, as many Hilbert spaces will arise in higher linear algebra which do not exactly look like $\mathbb{C}^n$. We study the abstract complex $*$-algebra $B(H)$ of linear operators on a Hilbert space H rather than $M_n(\mathbb{C})$ for the same reason. For this reason, some proofs may certainly be simplified by replacing $B(H)$ with $M_n(\mathbb{C})$ and a 'unitary algebra' (finite dimensional C*-algebra) with a multimatrix algebra. However, the proofs here are really operator algebraic in nature and many can be adapted to the infinite dimensional setting (with more work).

1.1. **Hilbert spaces.**

Definition 1.1.1. A *Hilbert space* is a vector space H equipped with a positive definite inner product, i.e.,

- (linear in second variable) $\langle \eta | \lambda \xi_1 + \xi_2 \rangle = \lambda \langle \eta | \xi_1 \rangle + \langle \eta | \xi_2 \rangle$ for all $\eta, \xi_1, \xi_2 \in H$,
- (anti-symmetric) $\overline{\langle \eta | \xi \rangle} = \langle \xi | \eta \rangle$ for all $\eta, \xi \in H$ and $\lambda \in \mathbb{C}$,
- (positive definite) $\langle \eta | \eta \rangle \geq 0$ for all $\eta \in H$ with equality if and only if $\eta = 0$.

(Since H was assumed to be finite dimensional, there is no completeness condition!) The *length* of $\eta \in H$ is $\|\eta\| := \sqrt{\langle \eta | \eta \rangle}$.

Example 1.1.2. The space $\mathbb{C}^n$ is a Hilbert space with $\langle \eta | \xi \rangle = \sum_{j=1}^n \overline{\eta_j} \xi_j$. Under the identification of $\mathbb{C}^n = M_{n\times 1}(\mathbb{C})$, $\langle \eta | \xi \rangle = \eta^\dagger \xi$.

Exercise 1.1.3. A *sesquilinear form* on a complex vector space V is a function $(\cdot|\cdot) : V^2 \to \mathbb{C}$ which is linear in the second variable and anti-linear in the first variable, i.e.,

$$(\lambda \eta_1 + \eta_2 | \xi) = \overline{\lambda}(\eta_1|\xi) + (\eta_2|\xi) \qquad \forall\, \eta_1, \eta_2, \xi \in H \text{ and } \lambda \in \mathbb{C}.$$

Prove that every sesquilinear form satisfies the *polarization identity*

$$4(u|v) = \sum_{k=0}^{3} i^k (v + i^k u | v + i^k u). \tag{1.1.4}$$

Remark 1.1.5. In a Hilbert space H, $\eta = 0$ if and only if $\langle \eta | \xi \rangle = 0$ for all $\xi \in H$.

Theorem 1.1.6 (Cauchy-Schwarz)**.** *For all $\eta, \xi \in H$, $|\langle \eta | \xi \rangle \leq \|\eta\| \cdot \|\xi\|$ with equality if and only if η, ξ are proportional.*

Proof. We may assume $\|\xi\| \neq 0$ and that $\langle \eta | \xi \rangle \in \mathbb{R}$ by multiplying η by a phase. The real non-negative polynomial

$$p(t) := \langle \eta - t\xi | \eta - t\xi \rangle = \|\eta\|^2 - 2t\langle \eta | \xi \rangle + t^2 \|\xi\|^2$$

achieves its minimum at $t_0 = \langle \eta|\xi\rangle/\|\xi\|^2$, at which

$$0 \le p(t_0) = \|\eta\|^2 - \langle \eta|\xi\rangle^2/\|\xi\|^2 \qquad \Longleftrightarrow \qquad \langle \eta|\xi\rangle \le \|\eta\| \cdot \|\xi\|.$$

Equality holds if and only if $p(t_0) = 0$, so that $\eta = t_0\xi$. □

Definition 1.1.7. An *orthonormal basis* (ONB) for H is a finite set $\{e_j\}_{j=1}^n \subset H$ such that

- (linearly independent) $\sum \lambda_j e_j = 0$ implies $\lambda_j = 0$ for all j,
- (spans) every $\eta \in H$ can be written as a linear combination $\eta = \sum \lambda_j e_j$ for some scalars $\lambda_1, \dots, \lambda_n \in H$, and
- (orthonormal) $\langle e_i|e_j\rangle = \delta_{i=j} := \begin{cases} 1 & \text{if } i = j \\ 0 & \text{else.} \end{cases}$

Each element e_i of an ONB is a *unit vector*, meaning it has unit length.

Example 1.1.8. The *computational basis* for $\mathbb{C}^n$ is $\{|0\rangle, \dots, |n-1\rangle\}$, where $|i\rangle$ is the vector which is one in the $(i+1)$-st coordinate and zero in every other coordinate.

Exercise 1.1.9. If $\{e_j\}_{j=1}^n \subset H$ is an ONB, then $\eta = \sum_{j=1}^n \langle e_j|\eta\rangle e_j$ for all $\eta \in H$.

Proposition 1.1.10. *An orthonormal basis exists for every Hilbert space.*

Proof. We use the Gram-Schmidt algorithm. We assume we have a basis $\{v_1, \dots, v_n\}$ of H. Set $e_1 := v_1/\|v_1\|$, and then inductively set $w_k := v_k - \sum_{j=1}^{k-1} \langle e_j|v_k\rangle e_j$ and $e_k := w_k/\|w_k\|$. □

Corollary 1.1.11. *Each Hilbert space H is isomorphic as a Hilbert space to $\mathbb{C}^n$ where $n = \dim(H)$.*

Proof. Let $\{e_1, \dots, e_n\}$ be an ONB for H. The coordinate map $[\cdot] : \eta \mapsto (\lambda_j)_{j=1}^n$ where $\eta = \sum_{j=1}^n \lambda_j e_j$ is the desired isomorphism. One checks that this map satisfies $\langle \eta|\xi\rangle_H = \langle [\eta]|[\xi]\rangle_{\mathbb{C}^n}$ for all $\eta, \xi \in H$. □

Definition 1.1.12. Given a Hilbert space H, the *conjugate space* $\overline{H}$ is the set of symbols $\{\overline{\eta} | \eta \in H\}$ with vector space structure given by

$$\overline{\eta} + \overline{\xi} := \overline{\eta + \xi} \qquad \text{and} \qquad \lambda \cdot \overline{\eta} := \overline{\overline{\lambda} \cdot \eta}$$

and inner product given by $\langle \overline{\eta}|\overline{\xi}\rangle := \langle \xi|\eta\rangle$.

The *dual space* is the space of linear functionals $H \to \mathbb{C}$.

Theorem 1.1.13 (Riesz Representation)**.** *The dual space $H^\vee$ is canonically isomorphic as a vector space to $\overline{H}$.*

Proof. Every $\eta \in H$ gives a linear functional $\langle \eta| : H \to \mathbb{C}$ by $\langle \eta|\xi := \langle \eta|\xi\rangle$. We claim the map $\overline{\eta} \mapsto \langle \eta|$ is the desired isomorphism. First, $\langle \lambda\eta + \xi| = \overline{\lambda}\langle \eta| + \langle \xi|$, so this map is linear and thus well-defined. If $\langle \eta| = \langle 0|$, then $\langle \eta|\xi\rangle = 0$ for all $\xi \in H$, so $\eta = 0$. Finally, if $f : H \to \mathbb{C}$ is a non-zero linear functional, pick an ONB $\{e_j\}_{j=1}^n$ of H, and observe that f is completely determined by $f(e_1), \dots, f(e_n)$. It is readily checked that $f = \sum_{j=1}^n f(e_j)\langle e_j|$. □

Using the above proposition, we endow $H^\vee$ canonically with an inner product by

$$\langle \langle \eta| \,|\langle \xi| \,\rangle := \langle \xi|\eta\rangle.$$

1.2. **Operators.** Given Hilbert spaces H, K, we denote the linear operators $H \to K$ by $B(H \to K)$, and we write $B(H) = B(H \to H)$.

Definition 1.2.1. Given a linear map $x : H \to K$, observe that the map $H \ni \eta \mapsto \langle \xi | x\eta \rangle$ is a linear functional in $H^\vee$ for every $\xi \in K$. Hence there is some vector $x^\dagger \xi \in H$ such that the above map is equal to $\langle x^\dagger \xi |$. It can be verified that the *adjoint map* $x \mapsto x^\dagger$ is a conjugate-linear map $B(H \to K) \to B(K \to H)$ such that:

- When x, y are composable operators, $(xy)^\dagger = y^\dagger x^\dagger$, and
- $x^{\dagger\dagger} = x$ for all operators x.

We call $x^\dagger$ the *adjoint* of x.

Exercise 1.2.2. Show the polarization identity for operators:

$$4x^\dagger y = \sum_{k=0}^{3} i^k (x + i^k y)^\dagger (x + i^k y).$$

Given ONBs $\{e_j\}_{j=1}^n$ for H and $\{f_k\}_{k=1}^m$ for K, we have a canonical isomorphism $B(H \to K) \cong M_{m \times n}(\mathbb{C})$ by

$$x \longmapsto [x] := (\langle f_i | x e_j \rangle)_{i,j}.$$

Here, the columns correspond to the ONB of the source, and the rows correspond to the ONB of the target. Under this isomorphism, composition of linear operators corresponds to matrix multiplication, and the adjoint corresponds to the conjugate transpose, also denoted $\dagger$. That is, the following diagrams commute:

$$\begin{tikzcd}
(H, \{e_j\}_{j=1}^n) \arrow[r, "x"] \arrow[d, "{[\cdot]}"] \arrow[rr, bend left, "xy"] & (K, \{f_k\}_{k=1}^m) \arrow[r, "y"] \arrow[d, "{[\cdot]}"] & (L, \{g_\ell\}_{\ell=1}^p) \arrow[d, "{[\cdot]}"] \\
\mathbb{C}^n \arrow[r, "{[x]}"] \arrow[rr, bend right, "{[x][y]}"'] & \mathbb{C}^m \arrow[r, "{[y]}"] & \mathbb{C}^p
\end{tikzcd}
\qquad
\begin{tikzcd}
(H, \{e_j\}_{j=1}^n) \arrow[r, "x^\dagger"] \arrow[d, "{[\cdot]}"] & (K, \{f_k\}_{k=1}^m) \arrow[d, "{[\cdot]}"] \\
\mathbb{C}^n \arrow[r, "{[x]^\dagger}"] & \mathbb{C}^m.
\end{tikzcd}$$

Thus studying operators between Hilbert spaces and operations between them is studying matrices and their operations. Below, we use the abstract language of Hilbert spaces, but the reader may safely replace $B(H)$ with $M_n(\mathbb{C})$ if they choose.

Definition 1.2.3. Gien $\eta \in H$ and $\xi \in K$, the *rank one* operator $|\xi\rangle\langle\eta| : H \to K$ is given by $\zeta \mapsto \langle \eta | \zeta \rangle \xi$.

Remark 1.2.4. If $x \in M_n(\mathbb{C})$ commutes with all $y \in M_n(\mathbb{C})$, then $x = \lambda 1$ for some $\lambda \in \mathbb{C}$. This can be easily seen by looking at rank one operators of the form $y = |e_i\rangle\langle e_j|$ for some ONB.

Definition 1.2.5. An operator $x \in B(H)$ is called:

- *normal* if $xx^\dagger = x^\dagger x$
- *self-adjoint* if $x^\dagger = x$
- *positive* if $\langle \xi | x\xi \rangle \geq 0$ for all $\xi \in H$ (denoted $x \geq 0$)
- a *projection* if $x^2 = x = x^\dagger$

An operator $u \in B(H \to K)$ is called:

- a *partial isometry* if $u^\dagger u$ is a projection.
- a *unitary* if u is invertible with $u^{-1} = u^\dagger$.

Example 1.2.6. Given an orthonormal set $S = \{e_1, \dots, e_k\} \subset H$, we get an orthogonal projection onto $\operatorname{span}(S)$ by $\sum_{j=1}^k |e_j\rangle\langle e_j|$.

Exercise 1.2.7. Prove that the projection $\sum_{j=1}^k |e_j\rangle\langle e_j|$ is independent of the choice of ONB of $\operatorname{span}(S)$.

Example 1.2.8. Given an orthonormal set $\{e_1, \dots, e_k\} \subset H$ and another orthonormal set $\{f_1, \dots, f_k\} \subset K$ of the same size, we get a partial isometry by $\sum_{j=1}^k |e_j\rangle\langle f_j|$.

Example 1.2.9. Suppose x is an invertible operator on H. Define a second inner product on H by $\langle \eta|\xi\rangle_x := \langle x\eta|x\xi\rangle$, and let H_x denote H with this second inner product. The operator $H_x \to H$ given by $\eta \mapsto x\eta$ is unitary.

Example 1.2.10. A *system of matrix units* in $B(H)$ is a collection of operators $\{e_{ij}\}$ satisfying

(SMU1) $e_{ij}e_{k\ell} = \delta_{j=k}e_{i\ell}$,
(SMU2) $\sum_j e_{jj} = 1$, and
(SMU3) $e_{ij}^\dagger = e_{ji}$

for all i, j, k, ℓ. Observe that each e_{ij} is a partial isometry and each e_{ii} is an orthogonal projection. What sizes of systems of matrix units can occur in $M_n(\mathbb{C})$?

Lemma 1.2.11. *For all $x \in M_{m\times n}(\mathbb{C})$, $\ker(x) = \ker(x^\dagger x)$. In particular, $x^\dagger x = 0$ implies $x = 0$.*

Proof. Clearly $x\eta = 0$ implies $x^\dagger x\eta = 0$. Conversely, if $x^\dagger x\eta = 0$, then $\|x\eta\|^2 = \langle \eta|x^\dagger x\eta\rangle = 0$, so $x\eta = 0$. For the final statement, observe $x^\dagger x = 0$ if and only if $\ker(x) = \ker(x^\dagger x) = \mathbb{C}^n$. □

Lemma 1.2.12 (Vector states separate points)**.** *An operator $x : H \to H$ is zero if and only if $\langle \eta|x\eta\rangle = 0$ for all $\eta \in H$.*

Proof. Suppose $\langle \eta|x\eta\rangle = 0$ for all $\eta \in H$. Consider the sesquilinear form $(\eta|\xi) := \langle \eta|x\xi\rangle$. By (1.1.4),

$$4\langle \eta|x\xi\rangle = 4(\eta|\xi) = \sum_{k=0}^{3} i^k(\xi + i^k\eta|\xi + i^k\eta) = \sum_{k=0}^{3} i^k\langle \xi + i^k\eta|x(\xi + i^k\eta)\rangle = 0 \qquad \forall\, \eta, \xi \in H.$$

Thus $\langle \eta|x\xi\rangle = 0$ for all $\eta, \xi \in H$, so $x\xi = 0$ for all $\xi \in H$, and $x = 0$. The other direction is trivial. □

Corollary 1.2.13. *Positive operators are self-adjoint.*

Proof. Observe that

$$\langle \xi|x^\dagger\xi\rangle = \overline{\langle x^\dagger\xi|\xi\rangle} = \overline{\langle \xi|x\xi\rangle} = \langle \xi|x\xi\rangle \geq 0 \qquad \forall\, \xi \in H$$

whenever $x \geq 0$. Hence $\langle \xi|(x - x^\dagger)\xi\rangle = 0$ for all $\xi \in H$, and thus $x = x^\dagger$ by Lemma 1.2.12. □

Exercise 1.2.14. We say two projections $p, q \in B(H)$ are *(Murray-von Neumann) equivalent*, denoted $p \approx q$, if there is a partial isometry $u \in B(H)$ such that $uu^* = p$ and $u^*u = q$. Prove that $\approx$ is an equivalence relation on $P(B(H))$, the set of projections of $B(H)$. Then describe the set of equivalence classes $P(M_n(\mathbb{C}))/\approx$.

Proposition 1.2.15. *There is a bijective correspondence between orthogonal projections in $B(H)$ and subspaces of H given by $p \mapsto pH$ and $H \supset K \mapsto \sum_{j=1}^k |e_j\rangle\langle e_j|$ where $\{e_j\}_{j=1}^k$ is an ONB of K.*

Proof. Note that the second map is well-defined by Exercise 1.2.7. Given an ONB $\{e_j\}_{j=1}^k$ of K, $\operatorname{im}\left(\sum_{j=1}^k |e_j\rangle\langle e_j|\right) = K$. Given a projection p and an ONB $\{e_j\}_{j=1}^k$ of pH, $p\eta = \sum_{j=1}^k \langle e_j|p\eta\rangle e_j = \sum_{j=1}^k |e_j\rangle\langle e_j|\eta$ by Exercise 1.1.9. □

Corollary 1.2.16. *Suppose $p \in B(H)$ is a projection and $x \in B(H)$ is an operator.*

(1) pH is invariant for x if and only if $xp = pxp$.
(2) pH is invariant for x and $x^\dagger$ if and only if $xp = px$.

Proof. To prove (1), observe that pH invariant for x means that $pxp\eta = xp\eta$ for all $\eta \in H$, and thus $pxp = xp$. Conversely, if $pxp = xp$, then $xpH \subseteq pH$.

To prove (2), observe that if pH invariant for x and $x^\dagger$ is equivalent to $xp = pxp$ and $x^\dagger p = px^\dagger p$. Hence $px = (x^\dagger p)^\dagger = (px^\dagger p)^\dagger = pxp = xp$. Conversely, $xp = px$ implies both $xp = pxp$ and $x^\dagger p = px^\dagger p$. □

Remark 1.2.17. Given an operator $x \in B(H)$ and a projection $p \in B(H)$, we can view x as matrix with operator entries

$$x = \begin{bmatrix} pxp & px(1-p) \\ (1-p)xp & (1-p)x(1-p) \end{bmatrix}$$

acting on $pH \oplus (1-p)H = H$. Under this identification,

$$p = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad 1-p = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Thus (1) of Corollary 1.2.16 above is equivalent to x being upper triangular, and (2) of Corollary 1.2.16 above is equivalent to x being diagonal.

Proposition 1.2.18. *Show that the following are equivalent for $u \in B(H)$:*

(1) u is a partial isometry.
(2) $u = uu^\dagger u$.
(3) $u^\dagger = u^\dagger uu^\dagger$.
(4) $u^\dagger$ is a partial isometry.

Proof.
(1) $\Rightarrow$ (2): Observe that

$$(u - uu^\dagger u)^\dagger (u - uu^\dagger u) = (u^\dagger - u^\dagger uu^\dagger)(u - uu^\dagger u) = u^\dagger u - 2u^\dagger uu^\dagger u + u^\dagger uu^\dagger uu^\dagger u = 0,$$

so $u - uu^\dagger u = 0$ by Lemma 1.2.11.
(2) $\Rightarrow$ (1): Multiply both sides on the left by $u^\dagger$ to see $u^\dagger u = u^\dagger uu^\dagger u$, which is self-adjoint.
(1) $\Leftrightarrow$ (2): Just take adjoints.

(3) $\Leftrightarrow$ (4): Apply (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (1) to $u^\dagger$. $\square$

Remark 1.2.19. Proposition 1.2.18 above gives a geometric interpretation about what it means to be a partial isometry. Indeed, $u : H \to K$ restricts to a unitary isomorphism from $u^\dagger uH$ onto $uu^\dagger K$ with inverse $u^\dagger$. Every partial isometry is of this form; that is, given projections $p \in B(H)$ and $q \in B(K)$ and a unitary isomorphism $u : pH \to qK$, we can extend u to an operator in $B(H \to K)$ satisfying $u^\dagger u = p$ and $uu^\dagger = q$ by defining u on $(1-p)H$ to be zero.

Definition 1.2.20. Define a partial order on $B(H)$ by $x \leq y$ if $y - x \geq 0$.

Exercise 1.2.21. Show that if $x \leq y$ and $z \in B(H)$, then $z^\dagger xz \leq z^\dagger yz$.

Exercise 1.2.22. Prove that for projections, $p \leq q$ if and only if $pH \subseteq qH$.

Exercise 1.2.23. Show that if $p_1, \dots, p_n$ are projections such that $\sum p_j = 1$, then $p_ip_j = 0$ when $i \neq j$. Deduce that (SMU1) can be replaced with $e_{ij}e_{jk} = e_{ik}$ for all i, j, k.

Definition 1.2.24. A non-zero projection $p \in B(H)$ is called *minimal* if $pB(H)p = \mathbb{C}p$.

Exercise 1.2.25. Show that the following are equivalent for a non-zero projection p.

(1) p is minimal.
(2) $0 \leq q \leq p$ implies $q =$ or $q = p$.
(3) pH is 1-dimensional ($\operatorname{rank}(p) = 1$).
(4) $p = |\xi\rangle\langle\xi|$ for some unit vector $\xi \in H$.

Proposition 1.2.26. *The complex $*$-algebra $B(H)$ has no non-trivial 2-sided ideals. Hence any $*$-algebra map from $M_n(\mathbb{C})$ into another complex $*$-algebra is either injective or the zero map.*

Proof. Suppose I is a 2-sided ideal, and let $x \in I$ be non-zero. Pick a unit vector $\xi \in H$ such that $x\xi \neq 0$, and set $\eta := x\xi/\|x\xi\|$. Then $|\eta\rangle\langle\eta| \cdot x \cdot |\xi\rangle\langle\xi| \in I$ is non-zero, so I contains the rank one operator $|\eta\rangle\langle\xi|$ and the minimal projection $|\xi\rangle\langle\xi|$.

Extend ξ to an ONB $\{e_1, \dots, e_n\}$ of H with $e_1 = \xi$. Observe that $|e_j\rangle\langle e_j| = |e_j\rangle\langle e_1| \cdot |e_1\rangle\langle e_1| \cdot |e_1\rangle\langle e_j| \in I$ for all j, so $1 = \sum_{j=1}^n |e_j\rangle\langle e_j| \in I$.

The last statement follows by analyzing the kernel of such a map together with the identification $M_n(\mathbb{C}) = B(\mathbb{C}^n)$. $\square$

1.3. **Direct sum and tensor product.**

Definition 1.3.1. Given two Hilbert spaces H, K, their *direct sum* is defined as a Hilbert space $H \oplus K$ together with isometries $i_H : H \to H \oplus K$ and $i_K : K \to H \oplus K$ which satisfy $i_Hi_H^\dagger + i_Ki_K^\dagger = 1$. By Proposition 1.2.18 and Exercise 1.2.23, it follows that $i_H^\dagger i_K = 0$ and $i_K^\dagger i_H = 0$. By Remark 1.2.17, operators on $H \oplus K$ can be viewed as matrices of operators:

$$x = \begin{bmatrix} i_H^\dagger x i_H & i_H^\dagger x i_K \\ i_K^\dagger x i_H & i_K^\dagger x i_H \end{bmatrix} \in \begin{bmatrix} B(H \to H) & B(K \to H) \\ B(H \to K) & B(K \to K) \end{bmatrix}.$$

Given a second direct sum $H \oplus' K$ with isometries $j_H : H \to H \oplus' K$ and $j_K : K \to H \oplus' K$ satisfying $j_Hj_H^\dagger + j_Kj_K^\dagger = 1$, there is a canonical unitary isomorphism $u := j_Hi_H^\dagger + j_Ki_K^\dagger :$

$H \oplus K \to H \oplus' K$ which is compatible with i_H, i_K and j_H, j_K in the sense that the following diagram commutes.

$$\begin{tikzcd} H \arrow[r, "i_H"] \arrow[dr, "j_H"'] & H \oplus K \arrow[d, "u"] & K \arrow[l, "i_K"'] \arrow[dl, "j_K"] \\ & H \oplus' K & \end{tikzcd}$$

This map is canonical in the sense that the canonical isomorphisms between $H \oplus K, H \oplus' K, H \oplus'' K$ for a second and third choice of direct sum fit in the following commutative diagram.

$$\begin{tikzcd} H \oplus K \arrow[rr, "\cong"] \arrow[dr, "\cong"] & & H \oplus'' K \\ & H \oplus' K \arrow[ur, "\cong"] & \end{tikzcd}$$

Thus models for $H \oplus K$ form a *contractible space*.

Exercise 1.3.2. Prove that the direct sum of Hilbert spaces is simultaneously a product and coproduct in the category of Hilbert spaces.

Definition 1.3.3. Given two Hilbert spaces H, K, their *tensor product* is the Hilbert space $H \otimes K$, which can be defined in a number of ways. The easiest is in terms of choosing ONBs $\{e_j\}$ of H and $\{f_k\}$ of K. The tensor product $H \otimes K$ then has ONB the formal symbols $\{e_j \otimes f_k\}$. Thus $\dim(H \otimes K) = \dim(H) \otimes \dim(K)$.

It can be readily checked that if we chose different ONBs $\{e'_j\}$ of H and $\{f'_k\}$ of K, there is a canonical isomorphism of Hilbert spaces from the Hilbert space $H \otimes K$ with ONB $\{e_j \otimes f_k\}$ to the Hilbert space $H \otimes' K$with ONB $\{e'_j \otimes f'_k\}$. This map is canonical in the sense that given a third choice $\{e''_j\}$ of H and $\{f''_k\}$ of K, the canonical isomorphisms between $H \otimes K, H \otimes' K, H \otimes'' K$ fit in the following commutative diagram.

$$\begin{tikzcd} H \otimes K \arrow[rr, "\cong"] \arrow[dr, "\cong"] & & H \otimes'' K \\ & H \otimes' K \arrow[ur, "\cong"] & \end{tikzcd}$$

Thus models for $H \otimes K$ form a *contractible space*.

Example 1.3.4. The computational basis for $\mathbb{C}^m \otimes \mathbb{C}^n$ is usually denoted by

$$\{|ij\rangle | i = 0, \ldots, m-1 \text{ and } j = 0, \ldots, n-1\}.$$

Exercise 1.3.5. Show that the choices of ONBs for H, K give a canonical isomorphism $B(H \otimes K) \cong B(H) \otimes B(K)$.

One can also define the tensor product via *universal property*. Any object which satisfies the universal property is unique up to unique isomorphism.

Definition 1.3.6. The tensor product Hilbert space of H, K is a Hilbert space $H \otimes K$ together with a bilinear map $\otimes : H \times K \to H \otimes K$ which satisfy the universal property that for every Hilbert space L and every *bilinear* map $T : H \times K \to L$, there is a unique *linear* map $\widetilde{T} : H \otimes K \to L$ such that the following diagram commutes.

$$\begin{tikzcd}
H \otimes K \arrow[rr, "T"] \arrow[dr, "\otimes"'] & & L \\
& H \otimes K \arrow[ur, "\widetilde{T}"'] &
\end{tikzcd}$$

Exercise 1.3.7. Use the universal property above to prove that the tensor product Hilbert space $(H \otimes K, \otimes : H \times K \to H \otimes K)$ is unique up to unique isomorphism.

1.4. **Spectral theory.**

Definition 1.4.1. The *spectrum* of an operator $x \in B(H)$ is

$$\operatorname{spec}(x) := \{\lambda \in \mathbb{C} | \lambda - x \text{ is not invertible}\}.$$

This set is the same as the set of eigenvalues of x after identifying $B(H) \cong M_n(\mathbb{C})$, which is also the set of roots of the characteristic polynomial $\chi_x(\lambda) = \det(\lambda - x)$. Recall that $\operatorname{spec}(x) \neq \emptyset$ by the Fundamental Theorem of Algebra (every complex polynomial has a root).

Exercise 1.4.2. Suppose $x \in B(H)$ is normal. Prove that $\|x\eta\| = \|x^\dagger \eta\|$ for all $\eta \in H$. Deduce that if $\lambda \in \operatorname{spec}(x)$ with corresponding eigenvector $\eta \in H$, then $\overline{\lambda} \in \operatorname{spec}(x^\dagger)$ with corresponding eigenvector η.

Theorem 1.4.3 (Spectral)**.** *The following are equivalent $x \in M_n(\mathbb{C})$.*

(1) *There is an ONB of $\mathbb{C}^n$ consisting of eigenvectors for x.*
(2) *There is a unitary $u \in M_n(\mathbb{C})$ such that $u^\dagger x u$ is diagonal.*
(3) *x is normal.*

Proof.
(1) $\Rightarrow$ (2) : Let $\{e_j\}$ be such an ONB of eigenvectors for x, and set

$$u := \begin{bmatrix} e_1 & \cdots & e_n \end{bmatrix}.$$

The eigenvalue equation implies $xu = ud$ where

$$d := \operatorname{diag}(\lambda_1, \dots, \lambda_n)$$

is the diagonal matrix whose entries are the corresponding eigenvalues of x. Then u is unitary as its columns are orthonormal, so $u^\dagger x u = d$.
(2) $\Rightarrow$ (3) : When $d = u^\dagger x u$ is diagonal,

$$x^\dagger x = ud^\dagger u^\dagger u d u^\dagger = ud^\dagger d u^\dagger = udd^\dagger u^\dagger = udu^\dagger u d^\dagger u^\dagger = xx^\dagger.$$

(3) $\Rightarrow$ (1) : Suppose x is normal and let $\lambda \in \operatorname{spec}(x)$ with eigenvector $\eta \in \mathbb{C}^n$. By Exercise 1.4.2, η is also an eigenvector of $x^\dagger$ with eigenvalue $\overline{\lambda} \in \operatorname{spec}(x^\dagger)$. Hence $\mathbb{C}\eta$ is invariant for x and $x^\dagger$. By Corollary 1.2.16, $xp = px$ where $p = |\eta\rangle\langle\eta|$. By Remark 1.2.17, x is diagonal with respect to the direct sum decomposition $\mathbb{C}^n = \operatorname{im}(p) \oplus \operatorname{im}(1-p)$. We now replace x by $(1-p)x = x(1-p)$ (which is again normal) acting on $\operatorname{im}(1-p) \subset \mathbb{C}^n$ (which has dimension $n-1$) and repeat the above procedure to obtain the desired ONB of eigenvectors. $\square$

Definition 1.4.4 (Functional calculus)**.** Suppose $x \in M_n(\mathbb{C})$ is normal. For $\lambda \in \operatorname{spec}(x)$ let $E_\lambda \subset \mathbb{C}^n$ denote the corresponding eigenspace, and let $p_\lambda \in M_n(\mathbb{C})$ be the orthogonal projection onto E_λ. We call the p_λ the *spectral projections* of x, and we note that they are mutually orthogonal ($p_\lambda p_\mu = 0$ for $\lambda \neq \mu$ in $\operatorname{spec}(x)$) and sum to 1.

Note that

$$x = \sum_{\lambda \in \operatorname{spec}(x)} \lambda p_\lambda \qquad \text{and} \qquad x^\dagger = \sum_{\lambda \in \operatorname{spec}(x)} \overline{\lambda} p_\lambda$$

as both operators agree on an orthonormal basis of $\mathbb{C}^n$, namely the orthonormal basis consisting of eigenvectors for x from the Spectral Theorem 1.4.3. For $f : \operatorname{spec}(x) \to \mathbb{C}$, we define

$$f(x) := \sum_{\lambda \in \operatorname{spec}(x)} f(\lambda) p_\lambda \in M_n(\mathbb{C}).$$

Observe that $\operatorname{spec}(f(x)) = f(\operatorname{spec}(x))$, as $f(x)$ is a diagonal operator with respect to the projections p_λ.

Theorem 1.4.5 (Gelfand)**.** *Suppose $x \in M_n(\mathbb{C})$ is normal, and let $C(\operatorname{spec}(x))$ denote the unital $*$-algebra of $\mathbb{C}$-valued functions on $\operatorname{spec}(x)$. The map $C(\operatorname{spec}(a)) \ni f \mapsto f(x) \in M_n(\mathbb{C})$ is an injective unital $*$-algebra homomorphism onto the unital $*$-algebra generated by x.*

Proof. It is straightforward to verify that $f \mapsto f(x)$ is a unital $*$-algebra map by checking the action of $f(x)$ on the ONB of eigenvectors of x from the Spectral Theorem 1.4.3. Injectivity follows as $f \neq g$ on $\operatorname{spec}(x)$ implies that $f(\lambda)p_\lambda \neq g(\lambda)p_\lambda$ for some $\lambda \in \operatorname{spec}(x)$. Since the image contains 1, x, and $x^\dagger$ by construction, it is onto the unital $*$-algebra generated by x. □

Exercise 1.4.6. Use the functional calculus to prove that every positive $x \in M_n(\mathbb{C})$ has a unique positive square root. That is, if $x \geq 0$, there is a unique positive operator $\sqrt{x} \in M_n(\mathbb{C})$ such that $\sqrt{x}^2 = x$.

Proposition 1.4.7. *Suppose $x, y \in M_n(\mathbb{C})$ with x normal and $xy = yx$. Then $f(x)y = yf(x)$ for every $f \in C(\operatorname{spec}(x))$.*

Proof. Since $\operatorname{spec}(x)$ is a finite set, there is a polynomial p such that $p = f$ on $\operatorname{spec}(x)$. Since $x^n y = yx^n$ for every n, $p(x)y = yp(x)$, and the result follows. □

Proposition 1.4.8. *The following are equivalent for $x \in M_n(\mathbb{C})$.*

(1) $x \geq 0$.
(2) *x is normal and all eigenvalues of x are non-negative.*
(3) *There is a $y \in M_n(\mathbb{C})$ such that $y^\dagger y = x$.*
(4) *There is a $y \in M_{n\times k}(\mathbb{C})$ for some $k \in \mathbb{N}$ such that $y^\dagger y = x$.*

Proof.
$\underline{(1) \Rightarrow (2):}$ Positive implies self-adjoint by Corollary 1.2.13, and self-adjoint clearly implies normal. If η is an eigenvector of x with eigenvalue λ, $0 \leq \langle \eta | x\eta = \lambda \langle \eta | \eta \rangle$, so $\lambda \geq 0$.
$\underline{(2) \Rightarrow (3):}$ Use the functional calculus to define $\sqrt{x} \in M_n(\mathbb{C})$ as in Exercise 1.4.6 above. Observe $\sqrt{x}$ is self-adjoint and satisfies $\sqrt{x}^2 = x$.
$\underline{(3) \Rightarrow (1):}$ Trivial.
$\underline{(4) \Rightarrow (1):}$ Observe that for all $\eta \in \mathbb{C}^n$, $\langle \eta | x\eta \rangle_{\mathbb{C}^n} = \langle \eta | y^\dagger y \eta \rangle_{\mathbb{C}^n} = \langle y\eta | y\eta \rangle_{\mathbb{C}^k} \geq 0$. □

Definition 1.4.9. For an operator $x \in M_n(\mathbb{C})$ its *support projection* is $\operatorname{supp}(x) := 1 - p_{\ker(x)}$ where $p_{\ker(x)}$ is the orthogonal projection onto $\ker(x)$. Observe that $x = x \operatorname{supp}(x)$.

Remark 1.4.10. When x is normal, $x = x \operatorname{supp}(x) = \operatorname{supp}(x)x$, and $\operatorname{supp}(x)$ is the sum of all spectral projections of x except for p_0 if $0 \in \operatorname{spec}(x)$. Thus $\operatorname{supp}(x)$ is well-defined independent of the action of $M_n(\mathbb{C})$ on $\mathbb{C}^n$.

Definition 1.4.11 (Polar decomposition)**.** Suppose $x \in M_{m\times n}(\mathbb{C})$. Using functional calculus, we define $|x| := \sqrt{x^\dagger x}$. The map $u : |x|\xi \mapsto x\xi$ on $\operatorname{supp}(x)\mathbb{C}^n$ and and $u = 0$ on $(1 - \operatorname{supp}(x))\mathbb{C}^n$ is an isometric linear operator and thus well-defined:

$$\||x|\xi\|^2 = \langle\, |x|\xi \,|\, |x|\xi \,\rangle = \langle\, \xi \,|\, |x|^2\xi \,\rangle = \langle x^\dagger x\xi, \xi\rangle = \langle x\xi|x\xi\rangle = \|x\xi\|^2.$$

Hence we may write $x = u|x|$ where u is a partial isometry and $|x| \geq 0$; this is called the *polar decomposition* of x.

Remark 1.4.12. When $x \in M_n(\mathbb{C})$, the partial isometry u constructed above commutes with all unitaries $v \in M_n(\mathbb{C})$ which commute with x and $x^\dagger$. Indeed, such a v commutes with $x^\dagger x$, and thus with $|x|$ and $\operatorname{supp}(|x|)$ by Proposition 1.4.7. This means $v = 0$ on $(1 - \operatorname{supp}(x))\mathbb{C}^n$ and on $\operatorname{supp}(x)\mathbb{C}^n$,

$$vuv^*|x|\xi = vu|x|v^*\xi = vxv^*\xi = x\xi.$$

Thus $vuv^* = u$, so $vu = uv$.

Exercise 1.4.13. In this exercise, we will prove the uniqueness of the polar decomposition.

(1) Show that $|x|$ is the unique positive operator that squares to $x^\dagger x$.
(2) Prove that u is the unique partial isometry such that $x = u|x|$ and $\ker(u) = \ker(x)$.
(3) Deduce that u is the unique partial isometry such that with $x = u|x|$ and $u^\dagger u = \operatorname{supp}(|x|)$. In this sense, the polar decomposition is independent of the action of $M_{m\times n}(\mathbb{C})$ on $\mathbb{C}^n$.

The following lemma was worked out with David Reutter and Jan Steinebrunner.

Lemma 1.4.14. *Suppose $x \in M_{m\times n}(\mathbb{C})$, and let $x = u|x|$ be its polar decomposition.*

(1) $u^\dagger u = \operatorname{supp}(x)$ and $uu^\dagger = \operatorname{supp}(x^\dagger)$, and
(2) $u^\dagger x = |x|$ and $x = |x^\dagger|u$, and
(3) the polar decomposition of $x^\dagger$ is given by $u^\dagger|x^\dagger|$.

Proof.
(1): First, since $\ker(u) = \ker(x)$, $u^\dagger u = 1 - p_{\ker(x)} = \operatorname{supp}(x)$.

Second, since $x^\dagger = |x|u^\dagger$, $\ker(u^\dagger) \subseteq \ker(x^\dagger)$. If $\eta \in \ker(x^\dagger)$, then $0 = x^\dagger\eta = |x|u^\dagger\eta$, so $u^\dagger\eta \in \ker(|x|) = \ker(u)$. Hence $uu^\dagger\eta = 0$, so $\eta \in \ker(uu^\dagger) = \ker(u^\dagger)$ by Lemma 1.2.11. Thus $\ker(x^\dagger) = \ker(u^\dagger)$, so $uu^\dagger = 1 - p_{\ker(x^\dagger)} = \operatorname{supp}(x^\dagger)$.
(2): Since $\ker(x) = \ker(|x|)$, $\operatorname{supp}(|x|) = u^\dagger u$ by (1). Thus $u^\dagger x = u^\dagger u|x| = \operatorname{supp}(|x|)|x| = |x|$ by Remark 1.4.10.

Since $uu^\dagger|x^\dagger| = |x^\dagger|$ and $|x|u^\dagger u = |x|$,

$$(u^\dagger|x^\dagger|u)^2 = u^\dagger|x^\dagger|uu^\dagger|x^\dagger|u = u^\dagger|x^\dagger|^2u = u^\dagger xx^\dagger u = |x|^2 = x^\dagger x.$$

Hence $u^\dagger|x^\dagger|u = |x|$ by uniqueness of the positive square root (Exercise 1.4.6). Hence

$$x = u|x| = uu^\dagger|x^\dagger|u = \operatorname{supp}(|x^\dagger|)|x^\dagger|u = |x^\dagger|u.$$

(3): Taking $\dagger$ in the second equation in (2) gives $x^\dagger = u^\dagger|x^\dagger|$. Since we showed $\ker(u^\dagger) = \ker(x^\dagger)$ in (1), it is indeed the polar decomposition. $\square$

Corollary 1.4.15. *For $x \in M_{m\times n}(\mathbb{C})$, the following are equivalent.*

(1) x has a left inverse.
(2) $x^\dagger x$ is invertible.
(3) In the polar decomposition $x = u|x|$, u is an isometry.

Dually, x has a right inverse if and only if $xx^\dagger$ is invertible if and only if u is a coisometry.

Proof. Since $\ker(x) = \ker(x^\dagger x)$, x has a left inverse if and only if $\ker(x^\dagger x) = \ker(x) = 0$ if and only if $x^\dagger x$ is invertible (by the Rank-Nullity Theorem). Moreover, $\ker(x) = 0$ if and only if $u^\dagger u = 1 - p_{\ker(x)} = 1$.

The dual statement for (1) $\Leftrightarrow$ (2) follows formally by considering $f^\dagger$. The dual statement for (2) $\Leftrightarrow$ (3) follows as $uu^\dagger = \operatorname{supp}(f^\dagger) = 1 - \ker(f^\dagger) = 1 - \ker(ff^\dagger)$. $\square$

1.5. **Complex $*$-algebras and states.**

Definition 1.5.1. A *complex algebra* is a complex vector space equipped with a compatible associative multiplication satisfying

- (distributive) $(a+b)\cdot c = a\cdot c + b\cdot c$ and $a\cdot(b+c) = a\cdot b + a\cdot c$ for all $a,b,c \in A$, and
- (compatibility with scalars) $(\lambda a)\cdot(\mu b) = (\lambda\mu)(a\cdot b)$ for all $a,b \in A$ and $\lambda,\mu \in \mathbb{C}$.

These conditions just say that $\cdot : A^2 \to A$ is bilinear. We assume that complex algebras are finite dimensional unless stated otherwise.

A *complex $*$-algebra* is a complex algebra A equipped with an anti-linear involution $* : A \to A$ satisfying $(ab)^* = b^*a^*$ and $a^{**} = a$ for all $a,b \in A$.

Lemma 1.5.2. *Every algebra automorphism of $M_n(\mathbb{C})$ is inner, i.e., if $\theta : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is a complex algebra isomorphism, then there is an invertible $h \in M_n(\mathbb{C})$ such that $\theta(x) = h^{-1}xh$ for all $x \in M_n(\mathbb{C})$.*

Proof. Let $\{e_j\}$ be an ONB for $\mathbb{C}^n$. Then $\{|e_i\rangle\langle e_j|\}$ is a system of matrix units (see Example 1.2.10) for $M_n(\mathbb{C})$. Since θ is an algebra map, $\{p_{ij} := \theta(|e_i\rangle\langle e_j|)\}$ is a collection of rank one operators satisfying (SMU1) and (SMU2), i.e., $p_{ij}p_{k\ell} = \delta_{j=k}p_{i\ell}$ and $\sum_j p_j = 1$. Pick $f_1 \in \operatorname{im}(p_{11})$ and set $f_j := p_{j1}f_1$ for all $j > 1$. Note that

$$p_{ij}f_k = p_{ij}p_{k1}f_1 = \delta_{j=k}p_{i1}f_1 = \delta_{j=k}f_i. \tag{1.5.3}$$

Now define $h \in M_n(\mathbb{C})$ by $hf_j := e_j$. Then since $f_k = p_{kk}f_k$ for all k,

$$h^{-1}|e_i\rangle\langle e_j|hf_k = h^{-1}|e_i\rangle\langle e_j|e_k = \delta_{j=k}h^{-1}e_i = \delta_{j=k}f_i \underset{(1.5.3)}{=} p_{ij}f_k = \theta(|e_i\rangle\langle e_j|)f_k \qquad \forall\, i,j,k.$$

Since $\{|e_i\rangle\langle e_j|\}$ is a basis for $M_n(\mathbb{C})$ and $\{f_k\}$ is a basis for $\mathbb{C}^n$, the result follows. $\square$

Theorem 1.5.4.

(1) *Any involution $*$ on $M_n(\mathbb{C})$ is of the form $x^* = hx^\dagger h^{-1}$ for some invertible $h \in M_n(\mathbb{C})$ such that $h = h^\dagger$.*
(2) *We have an isomorphism $(M_n(\mathbb{C}), *) \cong (M_n(\mathbb{C}), \dagger)$ as complex $*$-algebras if and only if the corresponding h for $*$ is positive or negative definite.*

Proof. To prove (1), observe that $x \mapsto x^{*\dagger}$ is an automorphism of $M_n(\mathbb{C})$, and is thus inner by Lemma 1.5.2. Thus there is an $k \in M_n(\mathbb{C})$ such that $x^{*\dagger} = k^{-1}xk$. Taking adjoints and setting $h = k^\dagger$, we have $x^* = hx^\dagger h^{-1}$. The condition that $x^{**} = x$ for all $x \in M_n(\mathbb{C})$ is then

$$x = x^{**} = (hx^\dagger h^{-1})^* = h(h^\dagger)^{-1}xh^\dagger h^{-1} \qquad \Longleftrightarrow \qquad xh^\dagger h^{-1} = h^\dagger h^{-1}x \qquad \forall\, x \in M_n(\mathbb{C}).$$

Thus $h^\dagger h^{-1} \in Z(M_n(\mathbb{C})) = \mathbb{C}1$, so $h^\dagger = \lambda h$ for some $\lambda \in \mathbb{C}$. Taking adjoints,

$$h = \overline{\lambda}h^\dagger = |\lambda|^2 h,$$

so $\lambda \in U(1)$, the unimodular complex scalars. Replacing h by $\lambda^{1/2}h$ which does not affect conjugation by h, we may assume $h = h^\dagger$.

To prove (2), first suppose h is positive or negative definite. We may assume h is positive definite by replacing h with $-h$ if necessary. The map $x \mapsto h^{-1/2}xh^{1/2}$ is the desired $*$-algebra isomorphism $(M_n(\mathbb{C}), *) \to (M_n(\mathbb{C}), \dagger)$. Conversely, if $\theta : (M_n(\mathbb{C}), *) \to (M_n(\mathbb{C}), \dagger)$ is a $*$-algebra isomorphism, then θ is an algebra automorphism, so there is a $k \in M_n(\mathbb{C})$ such that $\theta(x) = k^{-1}xk$. Similar to above, the $*$-algebra isomorphism condition then reduces to $h^{-1}kk^\dagger \in Z(M_n(\mathbb{C}))$, so $h = \lambda kk^\dagger$ for some $\lambda \in \mathbb{C}^\times$. Since $h = h^\dagger$, $\lambda \in \mathbb{R}^\times$, so h is positive or negative definite as claimed. □

Definition 1.5.5. Let A be a unital complex $*$-algebra. We call a linear functional $\varphi : A \to \mathbb{C}$:

- a *trace* or *tracial* if $\varphi(ab) = \varphi(ba)$ for all $a, b \in A$.
- *positive* if $\varphi(a^*a) \geq 0$ for all $a \in A$.
- a *state* if φ is positive and $\varphi(1) = 1$.
- *faithful* if φ is positive and $\varphi(a^*a) = 0$ implies $a = 0$.

Example 1.5.6. The trace on $M_n(\mathbb{C})$ given by $\operatorname{tr}(x) := \sum_{j=1}^n x_{jj}$ is a tracial state.

Lemma 1.5.7. *The complex $*$-algebra $(M_n(\mathbb{C}), \dagger)$ has a unique normalized trace.*

Proof. Suppose $\varphi : M_n(\mathbb{C}) \to \mathbb{C}$ is another trace with $\varphi(1) = 1$. Then

$$\varphi(|e_i\rangle\langle e_i|) = \varphi(|e_i\rangle\langle e_j| \cdot |e_j\rangle\langle e_i|) = \varphi(|e_j\rangle\langle e_i| \cdot |e_i\rangle\langle e_j|) = \varphi(|e_j\rangle\langle e_j|) \qquad \forall\, i, j,$$

and

$$\varphi(|e_i\rangle\langle e_j|) = \varphi(|e_i\rangle\langle e_j| \cdot |e_j\rangle\langle e_j|) = \varphi(|e_j\rangle\langle e_j| \cdot |e_i\rangle\langle e_j|) = \langle e_j|e_i\rangle\varphi(|e_j\rangle\langle e_j|) = 0 \qquad \forall\, i \neq j.$$

The result follows. □

Exercise 1.5.8. Suppose φ is a state on $M_n(\mathbb{C})$ such that $\varphi(|e_j\rangle\langle e_j|) = \frac{1}{n}$ for all e_j in an ONB of $\mathbb{C}^n$. Show that $\varphi = \operatorname{tr}$.

Exercise 1.5.9. For $\eta, \xi \in \mathbb{C}^n$, show that $\operatorname{tr}(|\eta\rangle\langle\xi|) = \langle\xi|\eta\rangle$.

Exercise 1.5.10. Suppose $\varphi : A \to \mathbb{C}$ is a linear functional on a unital complex $*$-algebra. Use Exercise 1.2.2 to prove that φ is a trace if and only if $\varphi(a^*a) = \varphi(aa^*)$ for all $a \in A$.

Exercise 1.5.11. Let $A = \mathbb{C}^2$ with coordinate-wise multiplication and $(a, b)^* := (\overline{b}, \overline{a})$. Prove that A has no states.

Definition 1.5.12. Suppose φ is a faithful state on a (finite dimensional) complex $*$-algebra A. Then $\langle a|b\rangle_\varphi := \varphi(a^*b)$ defines a positive definite inner product on A (thought of as a $\mathbb{C}$-vector space). We denote the corresponding Hilbert space by $L^2(A, \varphi)$; this is called the *GNS-Hilbert space.*

We denote the image of $1 \in A$ in $L^2(A, \varphi)$ by Ω, so $a\Omega$ is the image of $a \in A$.

Proposition 1.5.13. *For any state φ on $M_n(\mathbb{C})$, there exists a unique $d \in M_n(\mathbb{C})$ with $d \geq 0$ and $\operatorname{tr}(d) = 1$ (called the* density matrix *of φ) such that $\varphi(a) = \operatorname{tr}(da)$ for all $a \in M_n(\mathbb{C})$. Moreover, φ is a faithful if and only if d is invertible.*

Proof. Since tr is a state by Lemma 1.5.7, $L^2(M_n(\mathbb{C}), \operatorname{tr})$ is a Hilbert space. By the Riesz-Representation Theorem 1.1.13, every linear map $M_n(\mathbb{C}) \to \mathbb{C}$ can be uniquely expressed as $\langle d|$ for some $d \in M_n(\mathbb{C})$ for the trace inner product. Thus there is a unique $d \in M_n(\mathbb{C})$ such that $\varphi(x) = \operatorname{tr}(d^\dagger x)$ for all $x \in M_n(\mathbb{C})$. Taking $x = |\xi\rangle\langle\xi|$ for a unit vector $\xi \in H$, we have

$$0 \leq \varphi(|\xi\rangle\langle\xi|) = \operatorname{tr}(d^\dagger \cdot |\xi\rangle\langle\xi|) = \operatorname{tr}(|\xi\rangle\langle\xi| \cdot d^\dagger \cdot |\xi\rangle\langle\xi|) = \langle\xi|d^\dagger\xi\rangle \operatorname{tr}(|\xi\rangle\langle\xi|) \underset{\text{(Exer. 1.5.9)}}{=} \frac{1}{n}\langle\xi|d^\dagger\xi\rangle,$$

so $d = d^\dagger \geq 0$. Clearly $1 = \varphi(1) = \operatorname{tr}(d)$. □

Proposition 1.5.14. *Suppose φ is a faithful state on A. For $a \in A$, the map given by $b\Omega \mapsto ab\Omega$ defines a left multiplication operator $\lambda_a \in B(L^2(A, \varphi))$. The adjoint of this operator is λ_{a^*} given by $b\Omega \mapsto a^*b\Omega$.*

Proof. We compute that

$$\langle b\Omega|\lambda_a c\Omega\rangle_\varphi = \langle b\Omega|ac\Omega\rangle_\varphi = \varphi(b^*ac) = \varphi((a^*b)^*c) = \langle a^*b\Omega|c\Omega\rangle_\varphi.$$

It follows that $\lambda_a^\dagger = \lambda_{a^*}$. □

Exercise 1.5.15. Prove that if $a \in A$, the map given by $b\Omega \mapsto ba\Omega$ defines a right multiplication operator $\rho_a \in B(L^2(A, \varphi))$. Calculate the adjoint of ρ_a. When does $\rho_a^\dagger = \rho_{a^*}$?

Remark 1.5.16. If $f : A \to A$ commutes with right multiplication in A, then f is left multiplication by an element of A. That is, $\operatorname{End}(A_A) = A$. Thus $\lambda A = \{\lambda_a | a \in A\} \subset B(L^2(A, \varphi))$ is the set of all operators which commute with $\rho A = \{\rho_a | a \in A\}$.

1.6. **Operator algebras.** We now have all the background material necessary to study finite dimensional operator algebras. For this section, A is a unital complex $*$-algebra (always assumed to be finite dimensional).

Definition 1.6.1. We call A a C*-*algebra* if there exists a norm $\|\cdot\|$ on A which is submultiplicative ($\|ab\| \leq \|a\| \cdot \|b\|$) such that

$$\|a^*a\| = \|a\|^2 \qquad \forall\, a \in A \tag{1.6.2}$$

Example 1.6.3. On $M_n(\mathbb{C})$, define

$$\|x\| := \sup_{\|\eta\|=1} \|x\eta\|,$$

and observe that $\|x\xi\| \leq \|x\| \cdot \|\xi\|$ for all $\xi \in H$ (divide both sides by $\|\xi\|$ assuming $\xi \neq 0$). One verifies this defines a norm. Submultiplicativity follows from the fact that

$$\|xy\eta\| \leq \|x\| \cdot \|y\eta\| \leq \|x\| \cdot \|y\| \cdot \|\eta\| \qquad \forall \eta \in H.$$

To prove the C*-axiom 1.6.2, First note that

$$\|x\eta\|^2 = \langle x\eta | x\eta\rangle = \langle \eta | x^\dagger x\eta\rangle \underset{\text{(Cauchy-Schwarz)}}{\leq} \|\eta\|\cdot\|x^\dagger x\eta\| \leq \|x^\dagger x\|\cdot\|\eta\|^2 \qquad \forall\, \eta\in H.$$

Thus $\|x\|^2 \leq \|x^\dagger x\| \leq \|x\|\cdot\|x^\dagger\|$. Similarly, $\|x^\dagger\|^2 \leq \|xx^\dagger\| \leq \|x\|\cdot\|x^\dagger\|$. These two sets of inequalities together imply $\|x\| = \|x^\dagger\|$, and thus these inequalities are all equalities.

Lemma 1.6.4. *All norms on $\mathbb{C}^n$ are equivalent. That is, if $\|\cdot\|_1, \|\cdot\|_2$ are two norms on $\mathbb{C}^n$, there is a $C>0$ such that $C^{-1}\|\cdot\|_2 \leq \|\cdot\|_1 \leq C\|\cdot\|_2$.*

Proof. Without loss of generality, we may assume that $\|\cdot\|_2$ is our favorite norm on $\mathbb{C}^n$. We fix our favorite for which we know that the unit ball is compact. (Mine is $\|\cdot\|_\infty$, for which the unit ball is $[-1,1]^n$.) Then the unit sphere (the $x\in\mathbb{C}^n$ such that $\|x\|_2 = 1$) is also compact. Pick $C>0$ such that both

$$C^{-1} \leq \min_{\|x\|_2=1} \|x\|_1 \qquad \text{and} \qquad \max_{\|x\|_2=1}\|x\|_1 \leq C.$$

Then whenever $x\in\mathbb{C}^n$ is non-zero,

$$C^{-1} \leq \left\|\frac{x}{\|x\|_2}\right\|_1 \leq C \qquad \Longleftrightarrow \qquad C^{-1}\|x\|_2 \leq \|x\|_1 \leq C\|x\|_2. \qquad \square$$

Proposition 1.6.5. *The only* C* *norm on $\mathbb{C}^n = C(\{1,\dots,n\})$ is $\|f\|_\infty := \max_{j=1}^n |f_j|$.*

Proof. We leave it to the reader to verify $\|\cdot\|_\infty$ is a C* norm.

Suppose $\|\cdot\|$ is another C* norm. By (1.6.2), $\|\cdot\|$ is completely determined by its values on elements of the form $\overline{f}f$, which only take positive values.

First, observe that for an orthogonal projection $p\in\mathbb{C}^n$, $\|p\| = \|p^*p\| = \|p\|^2$, so $\|p\|\in\{0,1\}$. Consider a positive function $f = (f_1,\dots,f_n)$. By replacing f with $f_j^{-1}f = (f_1/f_j,\dots,f_n/f_j)$ where $f_j = \max(f)$, we may assume that $f_i\leq 1$ for all i, and at least one f_j is equal to 1. The C* axiom (1.6.2) tells us that $\|f^2\| = \|f\|^2$, and iterating, $\|f^{2^n}\| = \|f\|^{2^n}$ for all n. If $r<1$, $r^n\to 0$ as $n\to\infty$, so f^{2^n} converges point-wise (and thus in some norm!) to some non-zero orthogonal projection p. Since all norms are equivalent on $\mathbb{C}^n$ by Lemma 1.6.4, $\|f\|^{2^n} = \|f^{2^n}\|\to\|p\| = 1$. This is only possible if $\|f\| = 1$. We conclude that $\|f\| = \max_{j=1}^n f_j$. $\square$

Theorem 1.6.6 (Fundamental Theorem of finite dimensional operator algebras)**.** *The following conditions are equivalent for a finite dimensional unital complex $*$-algebra A.*

- (C*1) *A is a* C*-*algebra.*
- (C*2) *(multimatrix) There exists a $*$-isomorphism $A\cong\bigoplus_{i=1}^k M_{a_i}(\mathbb{C})$ where each summand has the usual conjugate transpose $\dagger$ operation.*
- (C*3) *(matrix $\dagger$-subalgebra) There exists an injective unital $*$-homomorphism $A\to M_n(\mathbb{C})$ for some $n\in\mathbb{N}$, where $M_n(\mathbb{C})$ has the usual conjugate transpose $\dagger$ operation.*
- (C*4) *($\exists$ faithful state) There exists a faithful state $\varphi: A\to\mathbb{C}$, i.e., $\varphi(a^*a)\geq 0$ for all $a\in A$, and $\varphi(a^*a) = 0$ implies $a = 0$.*
- (C*5) *($*$-definite) For every $a\in A$, $a^*a = 0$ implies $a = 0$.*

Proof. We prove the following implications:

$$\begin{tikzcd}
(\mathrm{C}^*2) \arrow[dd, Rightarrow] & & (\mathrm{C}^*5) \arrow[ll, Rightarrow] \\
& (\mathrm{C}^*1) \arrow[ur, Rightarrow] & \\
(\mathrm{C}^*3) \arrow[ur, Rightarrow] \arrow[rr, Rightarrow] & & (\mathrm{C}^*2) \arrow[uu, Rightarrow]
\end{tikzcd}$$

The interesting part is proving (C*5) $\Rightarrow$ (C*2).

(C*2) $\Rightarrow$ (C*3): Set $n := \sum_{i=1}^{k} a_i$ and embed A as block-diagonal matrices.

(C*3) $\Rightarrow$ (C*1): By Example 1.6.3, $M_n(\mathbb{C})$ is a C*-algebra, so we may restrict its norm to the image of A.

(C*3) $\Rightarrow$ (C*4): Take $\varphi = \operatorname{tr}$ from Example 1.5.6.

(C*1) $\Rightarrow$ (C*5): If $a^*a = 0$, then by 1.6.2, $\|a\|^2 = \|a^*a\| = 0$, so $a = 0$.

(C*4) $\Rightarrow$ (C*5): If $a^*a = 0$, then $\varphi(a^*a) = 0$, so $a = 0$.

(C*5) $\Rightarrow$ (C*2): We proceed in 4 steps.

Step 1: Recall that one description of the *Jacobson radical* of A is

$$J(A) = \{b \in A | 1 + abc \text{ is invertible } \forall a, c \in A\}.$$

We first show every element of $J(A)$ is nilpotent.

Proof. Suppose $b \in J(A)$. Since A is finite dimensional, eventually a^n is a linear combination of the a^k for $k < n$. Thus there is a polynomial of the form

$$p(x) = x^n + \lambda_{n-1}x^{n-1} + \cdots + \lambda_1 x + \lambda_0$$

such that $p(b) = 0$. If j is minimal such that $\lambda_j \neq 0$, then

$$\begin{aligned}
0 &= \frac{1}{\lambda_j} p(b) \\
&= \frac{1}{\lambda_j} b^n + \frac{\lambda_{n-1}}{\lambda_j} b^{n-1} + \cdots + \frac{\lambda_{j+1}}{\lambda_j} b^{j+1} + b^j \\
&= b^j \underbrace{\left(1 + \frac{\lambda_{j+1}}{\lambda_j} b + \cdots \frac{\lambda_{n-1}}{\lambda_j} b^{n-1-j} + \frac{1}{\lambda_j} b^{n-j}\right)}_{\text{invertible as } b \in J(A)}.
\end{aligned}$$

Since $b \in J(A)$, the the term on the right hand side is invertible, and thus $b^j = 0$, so b is nilpotent. □

Step 2: A is semisimple. Thus by the Artin-Wedderburn Theorem, A is a finite direct sum of matrix algebras, i.e., a multimatrix algebra.

Proof. We must prove that $J(A) = 0$. Suppose for contradiction that $b \in J(A)$ and $b \neq 0$. Since the Jacobson radical is an ideal, if $b \in J(A)$, $b^*b \in J(A)$, and $b^*b \neq 0$ by (C*5). So we may assume our original b is self-adjoint.

By Step 1 above, b is nilpotent. Pick $k > 1$ minimal such that $b^k = 0$. If k is even, then $0 = b^k = c^*c$ where $c = b^{k/2}$. If k is odd, then $0 = b^{k+1} = c^*c$ where $c = b^{(k+1)/2}$. But both $k/2$ and $(k+1)/2$ are strictly less than k when $k > 1$, a contradiction. □

Step 3: Each full matrix algebra summand $M_n(\mathbb{C})$ of A is preserved under $*$.

Proof. We know that $A \cong \bigoplus_{i=1}^k M_{a_i}(\mathbb{C})$. Consider the k mutually orthogonal central projections $p_1, \ldots, p_k$ where p_i corresponds to the unit of $M_{a_i}(\mathbb{C})$. Then $p_1^*, \ldots, p_k^*$ are also mutually orthogonal central projections, so $p_i^* = p_j$ for some $j = 1, \ldots, n$. Since each $p_j \neq 0$, we also have $p_j^* p_j \neq 0$ by (C*5), so $p_j^* = p_j$ for all j. $\square$

Step 4: Restricting $*$ to a full matrix algebra summand $M_n(\mathbb{C})$ of A, by Theorem 1.5.4, there is a self-adjoint $h \in M_n(\mathbb{C})$ such that $x^* = hx^\dagger h^{-1}$ for all $x \in M_n(\mathbb{C})$. We show h is positive or negative definite, which proves $(A, *) \cong \left(\bigoplus_{i=1}^k M_{a_i}(\mathbb{C}), \dagger\right)$ as complex $*$-algebras.

Proof. If h is not positive or negative definite, choose $-\infty < r < 0 < s < \infty$ such that $r, s \in \operatorname{spec}(h)$, and pick unit length eigenvectors $\eta, \xi \in \mathbb{C}^n$ for h corresponding to r, s respectively. Observe that η, ξ are also eigenvectors of h^{-1} corresponding to eigenvalues $\frac{1}{r}, \frac{1}{s}$ respectively. Since η, ξ are eigenvectors corresponding to distinct eigenvalues, $\eta \perp \xi$, i.e., $\langle \eta | \xi \rangle = 0$. Setting

$$x := \begin{bmatrix} \sqrt{-r}\eta + \sqrt{s}\xi & 0 & \cdots & 0 \end{bmatrix} \in M_n(\mathbb{C}),$$

we have

$$\begin{aligned} hx^\dagger h^{-1} x &= h \begin{bmatrix} \sqrt{-r}\eta^\dagger + \sqrt{s}\xi^\dagger \\ 0 \\ \vdots \\ 0 \end{bmatrix} h^{-1} \begin{bmatrix} \sqrt{-r}\eta + \sqrt{s}\xi & 0 & \cdots & 0 \end{bmatrix} \\ &= h \begin{bmatrix} \sqrt{-r}\eta^\dagger + \sqrt{s}\xi^\dagger \\ 0 \\ \vdots \\ 0 \end{bmatrix} \begin{bmatrix} \frac{\sqrt{-r}}{r}\eta + \frac{\sqrt{s}}{s}\xi & 0 & \cdots & 0 \end{bmatrix} \\ &= h \begin{bmatrix} \frac{-r}{r} + \frac{s}{s} & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix} = 0. \end{aligned}$$

Thus $x^* x = hx^\dagger h^{-1} x = 0$, contradicting (C*5). $\square$

Definition 1.6.7. A *unitary algebra* is a finite dimensional unital complex $*$-algebra that satisfies the equivalent conditions of Theorem 1.6.6. Note that unitary algebras are more commonly called *finite dimensional* C*-*algebras*.

Corollary 1.6.8. *Every unitary algebra A has a unique* C* *norm.*

Proof. By (1.6.2), every C* norm is completely determined by its values on positive operators. Suppose $a \in A$ is positive, and consider A unitally $*$-embedded in $M_n(\mathbb{C})$ from (C*3). The Gelfand Theorem 1.4.5 says that the unital $*$-algebra generated by a is isomorphic to $C(\operatorname{spec}(a))$, so it suffices to prove the result for C*-algebras of the form $C(X)$ for $X \subset \mathbb{C}$ a finite set. Since $C(X) \cong \mathbb{C}^n$ as a unital complex $*$-algebra, the result now follows from Proposition 1.6.5. $\square$

We now prove that every finite dimensional C*-algebra A is also a von Neumann algebra, which means we can perform polar decomposition *internal* to A.

Proposition 1.6.9. *Every unitary algebra A is closed under the functional calculus and polar decomposition.*

Proof. Identify A with a $*$-closed subalgebra of $M_n(\mathbb{C})$. If $a \in A$ is normal and $f : \operatorname{spec}(a) \to \mathbb{C}$, then $f(a)$ is in the unital $*$-algebra generated by a and $a^\dagger$, which again lies in A.

Next, identifying $A \cong \bigoplus_{i=1}^k M_{n_i}(\mathbb{C})$, each $a \in A$ corresponds to a tuple $(x_i) \in \bigoplus_{i=1}^k M_{n_i}(\mathbb{C})$. Then $a_i = u_i|a_i|$ is the polar decomposition in $M_{n_i}(\mathbb{C})$, and $a = u|a|$ where $u = (u_i)$ and $|a| = (|a_i|)$. ☐

Exercise 1.6.10. Suppose A is a unitary algebra and $a \in A$.

(1) Show that a can be written uniquely as $\operatorname{Re}(a) + i\operatorname{Im}(a)$ where both $\operatorname{Re}(a), \operatorname{Im}(a) \in A$ are self-adjoint.
(2) Show that if a is self-adjoint, then a can be written uniquely as $a = a_+ - a_-$ where a_+, a_- are both positive and $a_+a_- = 0$.
(3) Show that if a is self-adjoint, then $a \leq \|a\|$, i.e., $\|a\| - a \geq 0$.
(4) Show that if a is self-adjoint, then a can be written as a linear combination of two unitaries in A.
Hint: if $\|a\| \leq 1$, consider $u := a + i\sqrt{1-a^2}$.

Definition 1.6.11. For a subset $S \subset B(H)$, the *commutant* of S is

$$S' := \{x \in B(H) | xs = sx \text{ for all } s \in S\}.$$

Exercise 1.6.12. Show that if $S \subset T \subset B(H)$, then $T' \subset S'$, $S \subset S''$, and $S' = S'''$.

Definition 1.6.13. Let $A \subset B(H)$ be a $*$-closed subalgebra. For $k \in \mathbb{N}$, we define the *k-amplification* of H is the Hilbert space $\bigoplus_{j=1}^k H$. The algebra A acts on the amplified Hilbert space $\bigoplus_{j=1}^k H$ by diagonal operators. That is, as in Remark 1.2.17, we may think of $B\left(\bigoplus_{j=1}^k H\right)$ as $k \times k$ matrices over $B(H)$. The A-action is given by

$$a \cdot \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_k \end{bmatrix} := \begin{bmatrix} a\eta_1 \\ \vdots \\ a\eta_k \end{bmatrix} = \begin{bmatrix} a & & \\ & \ddots & \\ & & a \end{bmatrix} \begin{bmatrix} \eta_1 \\ \vdots \\ \eta_k \end{bmatrix}.$$

Exercise 1.6.14. Suppose $S \subseteq B(H)$ is a subset, and let $\alpha : B(H) \to M_n(B(H))$ be the amplification

$$x \longmapsto \begin{pmatrix} x & & \\ & \ddots & \\ & & x \end{pmatrix}.$$

Prove that:

(1) $\alpha(S)' = M_n(S')$, and
(2) If $0, 1 \in S$, then $M_n(S)' = \alpha(S')$.
(3) Deduce that when $0, 1 \in S$, $\alpha(S)'' = \alpha(S'')$.

Theorem 1.6.15 (von Neumann Bicommutant)**.** *If $A \subset B(H)$ is a unital $*$-subalgebra, then $A = A''$.*

Proof. Our proof follows [Jon15, Thm. 3.2.1]. Consider the n-amplification $\bigoplus_{j=1}^n H$ where $n = \dim(H)$ which carries the diagonal A-action $\alpha : A \to M_n(B(H))$. Let $\{e_i\}$ be an ONB of H, and consider the vector

$$\eta := \begin{bmatrix} e_1 \\ \vdots \\ e_n \end{bmatrix} \in \bigoplus_{j=1}^n H,$$

i.e., η is the j-th standard basis vector in the j-th summand of the amplified Hilbert space. Consider the subspace $K := \alpha(A)\eta \subset \bigoplus_{j=1}^n H$, and let $p_K \in B\left(\bigoplus_{j=1}^n H\right)$ be the projection onto K. Since A is $*$-closed, $p_K \in \alpha(A)' = M_n(A')$ by Exercise 1.6.14.

If $x \in A''$, then $\alpha(x) \in M_n(A')'$ and thus commutes with p_K. Thus $\alpha(x)K \subseteq K$. Since A is unital, there is an $a \in A$ such that $\alpha(x)\alpha(1)\eta = \alpha(a)\eta$. In particular, $xe_j = ae_j$ for all j, so $x = a \in A$. Hence $A'' \subseteq A$, so $A = A''$. □

Unital $*$-subalgebras $A \subset B(H)$ such that $A = A''$ are called *von Neumann algebras*. By Exercise 1.6.12, A' is also a von Neumann algebra, von Neumann algebras always come in pairs: A and A'. Combining Theorems 1.6.6 and 1.6.15, we immediately have the following corollary

Corollary 1.6.16. *Unitary algebras are the same thing as finite dimesional von Neumann algebras.*

Although the following corollary was already proven in Proposition 1.6.9 above, we provide a second von Neumann algebraic proof.

Corollary 1.6.17. *Suppose $A \subset B(H)$ is a unitary algebra. For $a \in A$, let $a = u|a|$ be the polar decomposition from Definition 1.4.11. Then $|a|$ and u are again in A.*

Proof. We know $|a| = \sqrt{a^*a} \in A$ by Proposition 1.6.9. Recall that the u constructed in Definition 1.4.11 commutes will all unitaries v which commute with a. This means that u commutes with all unitaries $v \in A'$. Since A' is a unitary algebra, it is spanned by its unitaries by Exercise 1.6.10. This means u commutes with all of A', so $u \in A'' = A$. □

References

[Jon15] Vaughan F. R. Jones. Von Neumann algebras, 2015. https://math.vanderbilt.edu/jonesvf/VONNEUMANNALGEBRAS2015/VonNeumann2015.pdf.