The fundamental philosophy of calculus is to a) approximate, b) refine the approximation, and c) apply a limit process. In order to deal with c), we’ll need the concept of a limit, specifically, for functions of several variables.

**Definition 1.** The limit as \((x, y)\) approaches \((a, b)\) of \(f(x, y)\) equals \(L\), denoted \(\lim_{(x, y)\to(a, b)} f(x, y) = L\), means that for all \(\epsilon > 0\), there exists \(\delta > 0\) such that if \(0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta\), then \(|f(x, y) - L| < \epsilon\). \(^1\)

In laymen’s terms, this means that \(f(x, y)\) can be made arbitrarily close to \(L\), say within \(\epsilon > 0\), provided that \((x, y)\) is sufficiently close to \((a, b)\), say within \(\delta > 0\). Using the definition of the limit to prove that the value of a limit is \(L\) amounts to being explicit about the relationship between \(\epsilon\) and \(\delta\), i.e., how does \(\delta\) depend on \(\epsilon\)?

**Example 1.** Prove \(\lim_{(x, y)\to(0, 0)} \frac{2x^2 y}{\sqrt{x^2 + y^2}} = 0\).

**Proof.** We want to show for all \(\epsilon > 0\), there exists \(\delta > 0\) such that if \(0 < \sqrt{x^2 + y^2} < \delta\), then \(|f(x, y)| < \epsilon\). So, we begin by fixing \(\epsilon > 0\). How does \(\delta\) depend on \(\epsilon\)? We begin by trying to find a “nice” upper bound for \(|f(x, y)|\). Note

\[
\left| \frac{2x^2 y}{\sqrt{x^2 + y^2}} \right| = \frac{2x^2 y}{x^2 + y^2} \cdot \frac{\sqrt{x^2 + y^2}}{x^2 + y^2} \leq 2|y| \sqrt{x^2 + y^2},
\]

since \(x^2 \leq x^2 + y^2\). Also \(|y| \leq \sqrt{x^2 + y^2}\). So

\[
\left| \frac{2x^2 y}{\sqrt{x^2 + y^2}} \right| \leq 2|y| \sqrt{x^2 + y^2} \leq 2 \left( \sqrt{x^2 + y^2} \right)^2. \quad (*)
\]

Inspired from the above inequality, we set \(\delta = \sqrt{\epsilon/2}\) and assume \(0 < \sqrt{x^2 + y^2} < \delta\). Then \(\left( \sqrt{x^2 + y^2} \right)^2 < \delta^2 = \epsilon/2\). That is, \(2 \left( \sqrt{x^2 + y^2} \right)^2 < \epsilon\). But we already know from \((*)\) that \(|f(x, y)| < \left( \sqrt{x^2 + y^2} \right)^2\). Putting everything together we get

\[
\left| \frac{2x^2 y}{\sqrt{x^2 + y^2}} \right| \leq 2 \left( \sqrt{x^2 + y^2} \right)^2 < \epsilon,
\]

as desired. \(\Box\)

**Example 2.** Prove \(\lim_{(x, y)\to(0, 0)} \frac{2x^2 y}{\sqrt{x^2 + y^2}} = 0\) using the squeeze theorem, and the theorem about which class of functions are continuous.

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\(^1\)The limit as \((x, y)\) → \((a, b)\) of \(f(x, y)\) doesn’t exist means for all \(L\), there exists \(\epsilon > 0\) such that for all \(\delta > 0\) there exists \((x, y)\) satisfying \(0 < \sqrt{(x-a)^2 + (y-b)^2} < \delta\) and \(|f(x, y) - L| > \epsilon\).
Example 3. Show that $\lim_{(x,y) \to (a,b)} \frac{x^2 y^3}{x^4 + y^6}$ does not exist. Hint: think of $\frac{x^3 y^4}{x^2 + y^2}$ as a function of $x$, say $f(x)$, where $y$ is constant, then try to find an upper bound for $f(x)$ (depending on $y$, of course) on $(-\infty, \infty)$ using methods from single variable calculus.

In order to show limits don’t exist, we can use the following

Proposition 1. If there exists paths $C_1$ and $C_2$ such that

$$\lim_{(x,y) \to (a,b)} f(x,y) \Bigg|_{C_1} \neq \lim_{(x,y) \to (a,b)} f(x,y) \Bigg|_{C_2},$$

then $\lim_{(x,y) \to (a,b)} f(x,y)$ does not exist.

This is the 2-dimensional analog of a familiar idea from single variable calculus, namely $\lim_{x \to a^-} f(x) \neq \lim_{x \to a^+} f(x)$ implies $\lim_{x \to a} f(x)$ does not exist.

Example 3. Show that $\lim_{(x,y) \to (0,0)} \frac{x^2 y^3}{x^2 + y^2}$ does not exist.

Solution. Let $C_1$ be the $x$-axis. Then $C_1$ is parametrized by $x(t) = t$ and $y(t) = 0$, moreover, $t = 0$ corresponds to $(0,0)$. So

$$\lim_{(x,y) \to (0,0)} \frac{x^2 y^3}{x^4 + y^6} \Bigg|_{C_1} = \lim_{t \to 0} \frac{t^2 \cdot 0^3}{t^4 + 0^6} = 0.$$

Now, let $C_2$ be the curve $y = x^{2/3}$. Then $C_2$ is parametrized by $x(t) = t$ and $y(t) = t^{2/3}$, and again, $t = 0$ corresponds to $(0,0)$. So

$$\lim_{(x,y) \to (0,0)} \frac{x^2 y^3}{x^4 + y^6} \Bigg|_{C_2} = \lim_{t \to 0} \frac{t^2 \cdot (t^{2/3})^3}{t^4 + (t^{2/3})^6} = \lim_{t \to 0} \frac{t^4}{2t^4} = \frac{1}{2}.$$

Since $1/2 \neq 0$, it must be that the limit does not exist.
(Challenge) Problem 2. Let

\[ f(x, y) = \begin{cases} \frac{y}{x}e^{-1/(x^2+y^2)} & x \neq 0 \\ 0 & \text{else.} \end{cases} \]

Show that

(a) \( \lim_{(x,y) \to (0,0)} f(x,y) \big|_C = 0 \) where \( C \) is any straight-line path through the origin.

(b) The limit as \( (x,y) \to (0,0) \) of \( f(x,y) \) does not exist.