Classification of groups of small(ish) order

Groups of order 12. There are 5 non-isomorphic groups of order 12. By the fundamental theorem of finitely generated abelian groups, we have that there are two abelian groups of order 12, namely

\[ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/6\mathbb{Z} \quad \text{and} \quad \mathbb{Z}/12\mathbb{Z}. \]

Let \( G \) be a non-abelian group of order 12. Let \( n_3 \) denote the number of Sylow-3 subgroups of \( G \). Then \( n_3 \) is either 1 or 4.

Suppose \( n_3 = 4 \). Let \( G \) act on the set of Sylow-3 subgroups by conjugation. This induces a homomorphism \( \varphi : G \to S_4 \). Suppose \( x \in \ker \varphi \). Then \( x \in N(P) \) for all Sylow-3 subgroups \( P \) where \( N(P) \) is the normalizer in \( P \). Now, by the orbit-stabilizer theorem, it follows that \( N(P) = P \) for all Sylow-3 subgroups \( P \). So \( x \) is an element of \( P \) for every Sylow-3 subgroup \( P \). Since \( |P| = 3 \) is prime, it follows that \( x = 1 \). Hence \( \varphi \) is an injection. It’s easy to see that \( \varphi G \) contains all 3 cycles of \( S_4 \). So it follows that \( \varphi G = A_4 \), the alternating group on 4 letters.

Now, suppose \( n_3 = 1 \). Then there is a single Sylow-3 subgroup of \( G \), say \( P \). Let \( Q \) be a Sylow-4 subgroup of \( G \). Since \( P \) is normal, the set \( PQ = \{ pq : p \in P, q \in Q \} \) is a subgroup of \( G \), in fact, \( PQ = G \). Now, let \( Q \) act on \( P \) by conjugation. This induces a homomorphism \( \varphi : Q \to \text{Aut}(P) \). Then \( G \cong P \ltimes \varphi Q \) where

\[ (p_1, q_1) \cdot (p_2, q_2) = (p_1\varphi(q_1)(p_2), q_1q_2). \]

Let \( V_4 \) be the Klein-4 group and \( C_4 \) the cyclic group of order 4. Then the 5 non-isomorphic groups of order 12 are

\[ \mathbb{Z}_2 \times \mathbb{Z}_6, \mathbb{Z}_{12}, A_4, P \ltimes \varphi V_4, P \ltimes \varphi C_4. \]

Groups of order 28. There are 4 non-isomorphic groups of order 28. By the Fundamental theorem for finite abelian groups, there are two abelian groups of order 28:

\[ \mathbb{Z}_2 \times \mathbb{Z}_{14} \quad \text{and} \quad \mathbb{Z}_{28}. \]

Now, let \( G \) be a non-abelian group of order 28, let \( P \) be the Sylow-7 subgroup, and let \( Q \) be a Sylow-2 subgroup. Then \( PQ = \{ pq : p \in P, q \in Q \} \) is a subgroup of \( G \) since \( P \) is normal (by Sylow):

\[ p_1q_1p_2q_2 = p_1(q_1p_2q_1^{-1})q_1q_2 \in PQ. \]

In fact, \( PQ = G \). Let \( \text{Aut}(P) \) denote the group of automorphisms of \( P \). Note that \( \text{Aut}(P) \) is cyclic of order 6 generated by \( \sigma : 1 \mapsto 3 \). Conjugation induces a map from \( \varphi : Q \to \text{Aut}(P) \). By order considerations, \( \ker \varphi \) is either equal to \( Q \) or of order 2. \( \ker \varphi = Q \) if and only if \( G \) is abelian.
So ker $\phi \neq Q$. Then im $\phi$ is a subgroup of Aut$(P)$ of order 2. It follows that the non-trivial elements of im $\phi$ act on $P$ by inversion. Now, $Q$ could be isomorphic to either $V_4$, the Klein-4 group, or $C_4$, the cyclic group of order 4. This gives us two possible groups:

$$P \rtimes_{\phi} V_4 \quad P \rtimes_{\phi} C_4,$$

where the group operation in $P \rtimes_{\phi} Q$ is

$$(p_1, q_1) \cdot (p_2, q_2) = (p_1\phi(q_1)(p_2), q_1q_2).$$

These two groups are non-isomorphic since they have different Sylow-2 subgroups. It’s easy to verify that the choice of $\phi$ is irrelevant.

**Groups of order 45.** There are only 2 groups of order 45, and they are abelian. Let $G$ be a group of order $45 = 5 \cdot 3^2$. Let $n_5$ denote the number of Sylow-5 subgroups of $G$. Note that $n_5 \equiv 1 \mod 5$ and $n_5 | 9$. Hence $n_5 = 1$, thus $G$ contains a unique, normal Sylow-5 subgroup, say $Q$. Let $P$ be any Sylow-3 subgroup. Since $P \cap Q = \{\text{id}\}$, and since $Q$ is normal, we have that for every $g \in G$ there exists unique $p \in P$ and $q \in Q$ such that $g = pq$. Since

$$p_1q_1p_2q_2 = p_1p_2(p_2^{-1}q_1p_2)q_2,$$

we have that $G \simeq Q \times_p P$ where $\phi : P \to \text{Aut}(Q)$ defined by $p \mapsto (q \mapsto p^{-1}qp)$. But $|\text{Aut}(Q)| = 4$ whereas $|P| = 9$. Hence $\phi$ is the trivial map, that is, for all $q \in Q$, $p^{-1}qp = q$ for all $p \in P$.

Hence $G \simeq Q \times P$. Since any group of order $p$ or $p^2$ where $p$ is a prime must be abelian, we get that $G$ must be abelian. In fact, we have

$$G \simeq \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/9\mathbb{Z} \quad \text{or} \quad G \simeq \mathbb{Z}/5\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z} \times \mathbb{Z}/3\mathbb{Z}.$$

**Groups of order $pq$ where $p$ and $q$ are primes (not necessarily distinct).** Suppose $p = q$. Then $G$ is a $p$-group, so $G$ has a nontrivial center. So $|Z(G)| \geq p$, so $G/Z(G)$ is cyclic. Hence $G$ is abelian. By the fundamental theorem for finitely generated abelian groups, we have that $G$ is isomorphic to one of the following:

$$\mathbb{Z}_{p^2} \quad \text{or} \quad \mathbb{Z}_p \times \mathbb{Z}_p.$$

Now, suppose $p$ and $q$ are distinct, and without loss of generality that $p < q$. Let $n_q = \# \text{Syl}_q(G)$. Then $n_q \equiv 1 \mod q$ and $n_q | p$. Since $p \equiv 1 \mod q$ implies that $q \leq p - 1$, it must be that $n_q = 1$. Let $Q$ be the normal Sylow-q subgroup of $G$, and let $P \in \text{Syl}_p(G)$. Since $Q$ is normal in $G$, we have that $PQ \leq G$ is a subgroup. Since $P \cap Q = \{\text{id}\}$, we have that $G = PQ$, in fact,

$$G \simeq Q \rtimes_{\phi} P,$$

where $\phi : P \to \text{Aut}(Q)$ is defined by $\phi : p \mapsto (\sigma_p : q \mapsto pqp^{-1})$. 

Suppose \( q \not\equiv 1 \mod p \). Then \( \phi : P \rightarrow \text{Aut}(Q) \) must be trivial, and \( G \simeq Q \times P \simeq \mathbb{Z}_{pq} \).

Suppose \( q \equiv 1 \mod p \). Since \( P \) and \( Q \) are prime power ordered we have that \( P \) is cyclic generated by, say, \( g \), and \( \text{Aut}(Q) \) is cyclic, generated, say, by \( \sigma \). Since \( \phi \) is a homomorphism, we must have \( \phi = \phi_\alpha : g \mapsto \sigma^{(q-1)\alpha/p} \) where \( 0 < \alpha \leq p - 1 \), since elements of the form \( \sigma^{(q-1)\alpha/p} \) are precisely those elements of \( \text{Aut}(Q) \) that are order \( p \). We associate \( Q \simeq \mathbb{Z}_q \) and \( P \simeq \mathbb{Z}_p \). We take \( g = 1 \) and \( \sigma : 1 \mapsto 2 \). So, \( \sigma^{(q-1)\alpha/p} : 1 \mapsto 2^{(q-1)\alpha/p} \) and in general

\[
\sigma^{(q-1)\alpha/p} : a \mapsto a \cdot 2^{(q-1)\alpha/p}.
\]

Then

\[
\phi_\alpha : b \mapsto \sigma^{(q-1)\alpha b/p}.
\]

Let \( 0 < \alpha, \beta \leq p - 1 \). The map

\[
\psi : \mathbb{Z}_q \times_{\phi_\alpha} \mathbb{Z}_p \rightarrow \mathbb{Z}_q \times_{\phi_\beta} \mathbb{Z}_p,
\]

\[
(a, b) \mapsto \left( a, \frac{\alpha b}{\beta} \right)
\]

defines an isomorphism. Hence there are precisely 4 isomorphism classes of groups of order \( pq \):

\[
\mathbb{Z}_{p^2}, \quad \mathbb{Z}_p \times \mathbb{Z}_p, \quad \mathbb{Z}_{pq}, \quad \mathbb{Z}_q \times_{\phi_\alpha} \mathbb{Z}_p,
\]

where the first pair are when \( q = p \), and the second pair when \( q > p \).