EQUIVARIANT COHOMOLOGY IN ALGEBRAIC GEOMETRY ERRATA VERSION 2024.10.3

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- (p. 166, §10.6) "...is independent [of] choices."
- (p.294-5, §16.2) The symbol "X" should be "G/B" (5 instances).
- (p.383-7, §A.8) This section contains an inaccuracy about the cohomology of inverse limits, and should be rewritten as below.

A.8. Limits

In this section, we consider cohomology rings associated to direct and inverse systems and their limits.

A directed poset is one with the property that for each pair n, n' in the poset there is an n'' greater than both. (In examples, the poset will be the natural numbers.) A directed system is a collection of objects and maps $\{X_n, \alpha_{n,n'}\}$, where $\alpha_{n,n'}\colon X_n \to X_{n'}$ for $n \leq n'$. An inverse system is the same, but with $\alpha_{n,n'}\colon X_n \to X_{n'}$ for $n \geq n'$.

Given a directed system of topological spaces $X = \{X_n, \alpha_{n,n'}\}$ with cohomology rings H^*X_n, we define the cohomology ring of the system as $H^*X = \varprojlim H^*X_n$. Similarly, given an inverse system of topological spaces $X = \{X_n, \alpha_{n,n'}\}$ with cohomology rings H^*X_n, we define its cohomology by $H^*X = \varinjlim H^*X_n$. In either case, the limit is taken in the category of graded rings. (We sometimes abuse notation by writing X for the corresponding limit space. The question of when H^*X is the cohomology of the limit space is discussed in remarks at the end of the section.)

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Suppose we have a map of direct systems $\{f_n \colon X_n \to Y_n\}$, so that each diagram

$$\begin{array}{ccc} X_n & \longrightarrow & X_{n'} \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_{n'}} \\ Y_n & \longrightarrow & Y_{n'} \end{array}$$

commutes. Then the induced homomorphism $\varprojlim H^*Y_n \to \varprojlim H^*X_n$ is the pullback homomorphism $f^* \colon H^*Y \to H^*X$. The same construction produces pullbacks for maps of inverse systems of spaces.

Under further conditions, one can define Gysin pushforwards. Given a map of direct or inverse systems $\{f_n \colon X_n \to Y_n\}$, suppose each $f_n \colon X_n \to Y_n$ is a proper map of complex manifolds, with $d = \dim Y_n - \dim X_n$ constant for all n. Furthermore, assume each square

$$\begin{array}{ccc} X_n & \xrightarrow{\alpha_{n,n'}} & X_{n'} \\ {\scriptstyle f_n}\downarrow & & \downarrow{\scriptstyle f_{n'}} \\ Y_n & \xrightarrow{\beta_{n,n'}} & Y_{n'} \end{array}$$

is a fiber square, so that $\beta^*_{n,n'}(f_{n'})_* = (f_n)_*\alpha^*_{n,n'}$ by naturality of Gysin homomorphisms. Then the pushforwards $(f_n)_*$ define a Gysin homomorphism

$$f_* \colon H^iX \to H^{i+2d}Y$$

between the cohomology rings of the systems.

One also has fundamental classes of subvarieties. First we consider a direct system of embeddings of complex manifolds $\{X_n\}$. Suppose $V_n \subseteq X_n$ is a direct system of closed subvarieties, such that $V_{n'} \cap X_n = V_n$ for all $n \leq n'$; suppose also that this intersection is transverse, so that $\alpha^*_{n,n'}[V_{n'}] = [V_n]$, and each $V_n \subseteq X_n$ has the same codimension, say d. Then the classes $[V_n] \in H^{2d}X_n$ define an element $([V_n])$ in $\varprojlim H^{2d}X_n$. We take this as a definition of $[V] \in H^{2d}X$.

Next we consider an inverse system $\{X_n\}$ of spaces. For fixed n, suppose a closed subspace $V_n \subseteq X_n$ has a fundamental class $[V_n] \in H^{2d}X_n$. This determines a class in the limit, by the canonical homomorphism $H^{2d}X_n \to H^{2d}X$. For any $n' \geq n$, let $V_{n'} = \alpha^{-1}_{n',n}V_n \subseteq X_{n'}$. If, for all $n' \geq n$, the maps $\alpha_{n',n} \colon X_{n'} \to X_n$ are such that

$\alpha^*_{n',n}[V_n] = [V_{n'}]$ in $H^{2d}X_{n'}$, then the classes $[V_{n'}]$ all determine the same class $[V]$ in $H^{2d}X$. For example, this holds if all $\alpha_{n',n} : X_{n'} \to X_n$ are smooth maps of complex manifolds (Proposition A.3.2).

Example A.8.1. Let $X_n = \mathbb{P}^{n-1} = \mathbb{P}(\mathbb{C}^n)$, with $X_n \hookrightarrow X_{n+1}$ given by the linear embedding of $\mathbb{C}^n$ as the span of the first n standard basis vectors in $\mathbb{C}^{n+1}$. The limit space is $\bigcup X_n \cong \mathbb{P}^\infty$, and we have

$$\begin{aligned} H^*X &= \varprojlim H^*\mathbb{P}^{n-1} \\ &= \varprojlim \mathbb{Z}[t]/(t^n) \\ &= \mathbb{Z}[t]. \end{aligned}$$

Let $H_n \subseteq \mathbb{P}^{n-1}$ be the hyperplane spanned by the last $n-1$ standard basis vectors. Then $H = \bigcup H_n \subseteq \mathbb{P}^\infty$ is the subspace where the first coordinate is zero, and $H \cap \mathbb{P}^{n-1} = H_n$, transversely for all n. Thus we can identify $t = [H]$ in $H^*\mathbb{P}^\infty$.

Example A.8.2. Fix a basepoint $p \in \mathbb{P}^\infty$. Let $X_n = \prod_{k=1}^n \mathbb{P}^\infty$, embedded in X_{n+1} by $(p_1, \ldots, p_n) \mapsto (p_1, \ldots, p_n, p)$. The direct limit is the *restricted product* of projective spaces,

$$\varinjlim X_n \cong \prod_{k\geq 1}{}' \mathbb{P}^\infty,$$

whose points are countable tuples $(p_1, p_2, \ldots)$ such that $p_i = p$ is the basepoint for all but finitely many coordinates. The inverse limit of cohomology rings is

$$H^*X = \varprojlim H^*X_n = \varprojlim \mathbb{Z}[t_1, \ldots, t_n] = \mathbb{Z}[\![t_1, t_2, \ldots]\!]_{\mathrm{gr}}.$$

Here the notation $\mathbb{Z}[\![t]\!]_{\mathrm{gr}} = \mathbb{Z}[\![t_1, t_2, \ldots]\!]_{\mathrm{gr}}$ is used for the *graded formal series* ring. This ring consists of formal sums $\sum c_\alpha t^\alpha$, where each $t^\alpha = t_1^{\alpha_1} t_2^{\alpha_2} \cdots$ is a monomial in finitely many t-variables, and $c_\alpha \in \mathbb{Z}$; the sum may have infinitely many terms but the total degree must be bounded. (For example, $t_1 + t_2 + \cdots$ is an element of this ring.)

Note that H^2X has uncountable rank as a $\mathbb{Z}$-module, and it is not free. (It is isomorphic to the direct product of countably many copies of $\mathbb{Z}$. This is the dual of H_2X, which is isomorphic to the direct sum of countably many copies of $\mathbb{Z}$.)

Example A.8.3. Consider $X_n = \prod_{k=1}^n \mathbb{P}^\infty$ as in the previous example, but as an inverse system via the projection $X_n \to X_{n-1}$ on the first $n-1$ factors. The limit space is

$$\varprojlim X_n = \prod_{k\geq 1} \mathbb{P}^\infty$$

is just the usual product of countably many projective spaces. Its cohomology ring is

$$H^*X = \varinjlim H^*X_n = \varinjlim \mathbb{Z}[t_1, \dots, t_n] = \mathbb{Z}[t_1, t_2, \dots],$$

the polynomial ring in countably many variables.

Fix i, and let $V(i)_n \subseteq X_n = \prod_{k=1}^n \mathbb{P}^\infty$ be the subspace where the first coordinate of the ith factor is zero; in the notation of Example A.8.1, this is

$$V(i)_n = \prod_{k=1}^{i-1} \mathbb{P}^\infty \times H \times \prod_{k=i+1}^{n} \mathbb{P}^\infty.$$

Then $[V(i)_n] = t_i$ in $H^*X_n = \mathbb{Z}[t_1, \dots, t_n]$ for all n, so $[V(i)] = t_i$ in H^*X.

Comparing with the previous example, the embedding of $\prod'_{k\geq 1} \mathbb{P}^\infty$ in $\prod_{k\geq 1}\mathbb{P}^\infty$ induces an inclusion of rings $\mathbb{Z}[t_1, t_2, \dots] \hookrightarrow \mathbb{Z}[\![t_1, t_2, \dots]\!]_{\mathrm{gr}}$.

To conclude, we describe some situations where the formally-defined cohomology rings of systems are related to cohomology rings of the corresponding limit spaces.

Remark A.8.4. The relationship between limits and cohomology depends on the cohomology theory, so in this remark, the notation $H^*(X)$ will depend on the theory to be specified.

For direct limits $\varinjlim X_n$, there is always a natural homomorphism

$$H^*(\varinjlim X_n) \to \varprojlim H^*X_n.$$

We will use singular cohomology and consider CW complexes $\{X_n\}$, where for each $n \leq n'$, the map $X_n \to X_{n'}$ is a closed embedding of complexes. The direct limit is the union

$$\varinjlim X_n = \bigcup X_n,$$

and is also a CW complex. Then for each i, there is a natural exact sequence

$$0 \to \varprojlim_n{}^1 H^{i-1} X_n \to H^i(\varinjlim X_n) \to \varprojlim_n H^i X_n \to 0,$$

where $\varprojlim^1$ is the derived functor of $\varprojlim$. In particular, if $H^* X_n$ vanishes in odd degrees for all n, then there is a natural isomorphism $H^*(\varinjlim X_n) \cong \varprojlim H^* X_n$. See [2, Theorem 3F.8].

For instance, the singular cohomology rings of the spaces $\mathbb{P}^\infty$ and $\prod'_{k\geq 1} \mathbb{P}^\infty$ are the rings computed in Examples A.8.1 and A.8.2 above.

Turning to inverse systems, we use Čech-Alexander-Spanier cohomology, which satisfies the *continuity axiom*: if all the spaces X_n are compact Hausdorff (and hence so is the limit), the natural homomorphism

$$\varinjlim H^* X_n \to H^*(\varprojlim X_n)$$

is an isomorphism [3, Ch. 6, Sec. 6]; see also [4] for relaxations of the compactness requirement as well as examples showing that some conditions are needed.

Čech-Alexander-Spanier cohomology agrees with singular cohomology for locally contractible paracompact Hausdorff spaces, but not in general, as the following simple example shows.

For example, let $X_n^{(q)} = \prod_{k=1}^n \mathbb{P}^q$ form an inverse system (with respect to n) via projections, with limit $\prod_{k\geq 1} \mathbb{P}^q$. The cohomology ring is a truncated polynomial ring in countably many variables:

$$H^*\left(\prod_{k\geq 1} \mathbb{P}^q\right) = \varinjlim H^* X_n^{(q)} = \mathbb{Z}[t_1, t_2, \ldots]/(t_1^{q+1}, t_2^{q+1}, \ldots)$$

In particular, for $q > 0$, $H^2\left(\prod_{k\geq 1} \mathbb{P}^q\right)$ is a free $\mathbb{Z}$-module of countably infinite rank, so by the universal coefficient theorem, it cannot be the singular cohomology of any space. (The space $\prod_{k\geq 1} \mathbb{P}^q$ is compact Hausdorff, but not locally contractible.)

The space $\prod_{k\geq 1} \mathbb{P}^\infty$ of Example A.8.3 is not compact, so the continuity axiom does not directly compute its cohomology. One can show that the natural map $\varinjlim H^* X_n = \mathbb{Z}[t_1, t_2, \ldots] \to H^*\left(\prod_{k\geq 1} \mathbb{P}^\infty\right)$ is an isomorphism of Čech-Alexander-Spanier cohomology rings, but the techniques go beyond the scope of this appendix.

References

[1] V. Bartik, 1968. "Aleksandrov-Čech cohomology and mappings into Eilenberg-MacLane polyhedra," *Math. USSR Sb.* **5**, 221–228.

[2] A. Hatcher, 2002. *Algebraic Topology*, Cambridge.

[3] E. Spanier, 1966. *Algebraic Topology*, McGraw-Hill.

[4] T. Watanabe, 1987. "The continuity axiom and the Čech homology," *Geometric topology and shape theory (Dubrovnik, 1986)*, 221–239, Lecture Notes in Math. **1283**, Springer, Berlin.