Preface

Given a Lie group G acting on a space X, the *equivariant cohomology ring* H_G^*X packages information about the interaction between the topology of X and the representation theory of G. On one hand, it provides a way of exploiting the symmetry of X, as manifested by the G-action, to understand H^*X; on the other hand, appropriate choices of X are useful in studying representations of G.

Defined by A. Borel in his 1958–9 seminar on transformation groups, equivariant cohomology arose in the context of a problem of interest to topologists: given some cohomological information about X, what can be said about the group actions X admits? Must there be fixed points? How many? By constructing an auxiliary space, Borel built a framework for answering these questions in special situations, e.g., when G is a torus and X is a compact manifold satisfying a technical hypothesis (now known as *equivariant formality*).

It took several decades for ideas of equivariant cohomology to enter mainstream algebraic geometry. By 2000, though, localization had become a standard technique in Gromov-Witten theory and applications to enumerative geometry. Equivariant methods were also used in producing degeneracy locus formulas and in proving Littlewood-Richardson rules in Schubert calculus.

One reason for the lag may be the role of infinite-dimensional spaces. Indeed, Borel's construction produces a certain fiber bundle over the classifying space $\mathbb{B}G$, with fiber X. Classifying spaces are almost always infinite-dimensional, so they are certainly not algebraic varieties. However, for the groups appearing most frequently in applications to algebraic geometry—linear algebraic groups, and especially torus groups—these spaces can be "approximated" by familiar finite-dimensional varieties. Such approximation spaces were introduced by Totaro in the late 1990s, building on ideas of Bogomolov, and they were incorporated into a theory of equivariant Chow groups by Edidin and Graham. The same ideas work equally well for cohomology, and in fact, some of the foundational notions are simpler for cohomology than for Chow groups.

Our aim in this text is to introduce the main ideas of equivariant cohomology to an audience with a general background in algebraic geometry. We therefore avoid using infinite-dimensional spaces in any essential way, relying instead on finite-dimensional approximations. A recurring theme is that studying the equivariant geometry of X is essentially the same as studying fiber bundles with fiber X. The fiber bundle point of view has a long tradition in algebraic geometry, and by emphasizing this, we hope that newcomers to equivariant cohomology will find that many of the constructions are already familiar.

In our choice of topics, we were guided by a desire to keep preqisites minimal. Apart from a "Leray-Hirsch" type lemma, and a few basic facts about Chern classes and cohomology classes of subvarieties, all that we need is standard material from first courses in algebraic topology and algebraic geometry. Projective spaces and Grassmannians are usually familiar to beginners, and they suffice to illustrate a broad range of equivariant phenomena. Toric varieties and homogeneous spaces are natural next steps, and here one already encounters the frontiers of current research.

On the other hand, this introductory text is not an all-inclusive reference, and we have left out many exciting topics, inevitably including ones which some researchers (even ourselves!) might consider essential. Readers will have to look elsewhere for the construction of equivariant cohomology via differential forms; for a detailed discussion of the moment map and the symplectic point of view; for applications to the cohomology of finite or discrete groups; and for equivariant K-theory and more exotic cohomology theories. Part of our aim is to prepare and encourage readers to explore the many excellent sources for learning about such things.

The book grew out of lectures, and we have tried to blend some of the organic character of a series of lectures with the logical organization of a textbook. The first six chapters cover the basics, including a simple version of the localization theorem and an illustration of its application to the space of conics. This material is important for most users of equivariant cohomology. Refinements of the localization theorem, including the "GKM" description of equivariant cohomology, are given in Chapter 7. Here we employ some more technical arguments, and for the most part the results are not logically required elsewhere in the book.

The remainder of the text consists of examples and applications—to toric varieties, Grassmannians, flag varieties, and general homogeneous spaces.

Grassmannians and flag varieties are fascinating objects of study in their own right, and we give an account of their combinatorial structure and equivariant geometry in Chapters 9 and 10. These spaces also form part of the link between equivariant cohomology and degeneracy locus formulas: in a precise way, a formula for the cohomology class of a degeneracy locus is equivalent to one for the equivariant class of a certain Schubert variety. This connection motivated much of our perspective, and it is the subject of Chapter 11.

Projective spaces, Grassmannians, and flag varieties are examples of homogeneous spaces for the general linear group. Other classical groups—the symplectic and orthogonal groups—appear in a similar way, and their corresponding flag varieties are related to refined degeneracy locus problems. The problem of extending what is known for GL_n ("type A") to the other classical types has received much attention over the last few decades. For a complete telling of this story, putting all classical groups on equal footing, we must refer elsewhere. Chapters 13 and 14 provide a sample, describing the equivariant cohomology of symplectic flag varieties ("type C").

The type C degeneracy locus formulas require a new coefficient ring, and this raises a question: where is the analogous coefficient ring in type A? The answer has become clear only in very recent work, involving a certain infinite-dimensional Grassmannian. (As usual, and in keeping with our general theme, it can also be understood via appropriate finite-dimensional approximations.) To provide a bridge between type A and type C, this is discussed rather briefly in Chapter 12.

Once one understands something about flag varieties for symplectic and orthogonal groups, it is natural to ask about general homogeneous spaces. These spaces play a key role in the story of equivariant cohomology, too: thanks to a theorem of Borel, if G is a reductive group with Borel subgroup B and maximal torus T, then the G-equivariant cohomology of any space on which G acts is related to its T-equivariant cohomology through the flag variety G/B. This is explained in Chapter 15, and further developed in Chapter 16.

There are several possible approaches to defining equivariant *homology*. One which is well-suited to our theme of finite-dimensional approximation is presented in Chapter 17, based on ideas of Edidin, Graham, and Totaro. Equivariant Segre classes appear naturally in this context, as do the equivariant multiplicities introduced by Rossmann and Brion.

In Chapters 18 and 19, we conclude with a study of Schubert varieties in homogeneous spaces. Highlights include a formula for the restriction of a Schubert class to a fixed point (due to Andersen-Jantzen-Soergel and Billey), a criterion for a Schubert variety to be nonsingular at a fixed point (following Kumar and Brion), and some formulas for multiplying equivariant Schubert classes, along with a theorem of Graham which asserts that such products always expand positively, in a suitable sense.

Each chapter ends with a “Notes” section, providing some limited historical and mathematical context, as well as references for material in the text. We have also included hints for many of the exercises, and complete solutions in a few cases.

Appendix A is a brief summary of basic results from algebraic topology which we need in the text; much of this material is essential, and we advise the reader to review it before embarking on the main text. The other appendices may be perused as needed.

Early drafts of what became this book began with WF's Eilenberg Lectures at Columbia University in 2007, and DA's notes have been available online since then. In the meantime, both authors have given lectures augmenting and improving on these notes—in courses at the University of Michigan, the University of Washington, and the Ohio State University, and in lecture series at the Institute for Advanced Study in Princeton in 2007, at IMPANGA in Bedłewo in 2010, and at IMPA in Rio de Janeiro in 2014. We heartily thank the many students, friends, and colleagues who attended these lectures and gave feedback on the notes. Special thanks go to P. Achinger, I. Cavey, J. de Jong, D. Genlik, O. Lorscheid, D. Speyer, and A. Zinger for their detailed comments, and to M. Brion, D. Edidin, W. Graham, and B. Totaro for their influence on our understanding of the subject.