CHAPTER 1

Preview

Before beginning in earnest, we offer a taste of the themes and topics this book will explore. Although we give some definitions and sketches of arguments here, the reader should rest assured that later chapters will provide more detail.

1. The Borel construction

Suppose a Lie group G acts on a space X (on the left). The standard definition of the G-equivariant cohomology of X, written H_G^*X, goes like this. Find a contractible space $\mathbb{E}G$ with G acting freely (on the right), and form the quotient

$$\mathbb{E}G \times^G X := (\mathbb{E}G \times X)/(e \cdot g, x) \sim (e, g \cdot x).$$

Then define

$$H_G^i X := H^i(\mathbb{E}G \times^G X).$$

The idea behind this definition is to have $H_G^i X = H^i(G\backslash X)$ when the action on X is free; replacing X by $\mathbb{E}G \times X$ leaves the homotopy type unchanged, but produces a free action, with quotient $\mathbb{E}G \times^G X$. This construction first appeared (unnamed) in Borel's 1958-9 seminar on transformation groups, so the space $\mathbb{E}G \times^G X$ is often called the *Borel construction*.

The space $\mathbb{B}G := \mathbb{E}G/G$ is a classifying space for G and it, along with the quotient map $\mathbb{E}G \to \mathbb{B}G$, is universal in an appropriate category, so this definition is independent of choices. We will not need this general topological machinery, though.

The case when X is a point is important. Here we are looking at

$$\Lambda_G := H_G^*(\text{pt}) = H^*\mathbb{B}G.$$

Since $\mathbb{B}G$ usually has nontrivial cohomology, $H^*_G(\mathrm{pt}) \neq \mathbb{Z}$ in general! This is an essential feature of equivariant cohomology.

EXAMPLE 1.1. For the multiplicative group $G = \mathbb{C}^*$, we can take $\mathbb{E}G = \mathbb{C}^\infty \smallsetminus \{0\}$. Certainly G acts freely, and it is a pleasant exercise to prove this space is contractible.[1] The quotient is $\mathbb{B}G = \mathbb{CP}^\infty$. This lets us compute our first equivariant cohomology ring:

$$\Lambda_{\mathbb{C}^*} = H^*_{\mathbb{C}^*}(\mathrm{pt}) = H^*\mathbb{CP}^\infty = \mathbb{Z}[t],$$

where t is the Chern class of the tautological line bundle on $\mathbb{CP}^\infty$.

For the circle group $G = S^1$, regarded as the unit complex numbers, we can use $\mathbb{E}G = S^\infty$, regarded as the unit sphere in $\mathbb{C}^\infty$. This is contractible, since $\mathbb{C}^\infty \smallsetminus \{0\}$ retracts onto it, and we obtain the same quotient space $\mathbb{B}G = \mathbb{CP}^\infty$ as for $\mathbb{C}^*$. Alternatively, we could use the same space $\mathbb{E}G = \mathbb{C}^\infty \smallsetminus \{0\}$, since the subgroup $G = S^1 \subseteq \mathbb{C}^*$ acts freely here. Either way, we obtain

$$\Lambda_{S^1} = \Lambda_{\mathbb{C}^*} = \mathbb{Z}[t].$$

This is an instance of a general phenomenon: cohomology for a complex group is the same as for a maximal compact subgroup.

EXAMPLE 1.2. Elaborating on the previous example, for the torus $T = (\mathbb{C}^*)^n$ we can take $\mathbb{E}T = (\mathbb{C}^\infty \smallsetminus \{0\})^n$ to get $\mathbb{B}T = (\mathbb{CP}^\infty)^n$. We find

$$\Lambda_T = \mathbb{Z}[t_1, \ldots, t_n],$$

where t_i comes from the tautological bundle on the ith factor of $(\mathbb{P}^\infty)^n$. As before, we get the same result for the compact torus $(S^1)^n \subseteq (\mathbb{C}^*)^n$.

Early applications of equivariant cohomology were topological, focusing on questions about how the cohomology of a space constrains the group actions it admits. Algebraic geometers were slower to realize its utility, perhaps because the spaces $\mathbb{E}G \times^G X$ are generally infinite-dimensional (as we've already seen). However, some of the core ideas of equivariant cohomology had been used in algebraic geometry for quite a while. The space $\mathbb{E}G \times^G X$ is a fiber bundle over the classifying space $\mathbb{B}G$, with fiber X, and the study of such bundles goes back at least to Ehresmann in the 1940's. In algebraic geometry, fiber bundle constructions are familiar and ubiquitous—we are used to going from a vector space to a vector bundle, projective space to projective bundle, or Grassmannian to Grassmann bundle. A key theme for us is that equivariant cohomology is intimately linked to the study of general fiber bundles.

In fact, we will work with an alternative (but equivalent) definition of H_G^* which stays within the realm of finite-dimensional spaces. This involves using "approximations" $\mathbb{E}_m$ to $\mathbb{E}G$, and each $\mathbb{E}_m \times^G X$ will be a finite-dimensional algebraic manifold whenever X is. (A technical assumption on G or X may be necessary, to guarantee algebraicity of the quotient, but it will be automatic in most applications.) For instance, we'll use $\mathbb{E}_m = \mathbb{C}^m \smallsetminus 0 \to \mathbb{B}_m = \mathbb{P}^{m-1}$ to approximate $\mathbb{B}\mathbb{C}^*$. In the next chapter, we'll prove lemmas that show this leads to a well-defined theory.

As we'll see, equivariant cohomology shares many familiar properties with ordinary (singular) cohomology: it is functorial (contravariant for equivariant maps), has Chern classes (for equivariant vector bundles), and fundamental classes (for invariant subvarieties of a nonsingular variety). Most of these properties are verified by doing the analogous construction for ordinary cohomology on $\mathbb{E}G \times^G X$ (or an approximation).

2. Fiber bundles

The Borel construction produces a certain fiber bundle from the action of G on X: the fiber is X, and the base is $\mathbb{B}G$ (or an approximation). It is helpful to think of the diagram

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{E} \times^G X \\ \downarrow & & \downarrow \\ \mathrm{pt} & \hookrightarrow & \mathbb{B} \end{array}$$

with the vertical arrow on the right coming from the projection on the first factor. Pullback along the horizontal arrows—i.e., restriction to a fiber—defines a forgetful homomorphism $H_G^*X \to H^*X$, from equivariant to ordinary cohomology. Pullback along the vertical arrows gives homomorphisms

$$\mathbb{Z} = H^*(\mathrm{pt}) \to H^*X \quad \text{and} \quad \Lambda_G = H^*_G(\mathrm{pt}) \to H^*_G X.$$

The first of these is trivial, but the second endows $H^*_G X$ with the richer structure of a Λ_G-algebra, at least when this ring is commutative. In some cases, this structure is rich enough to determine X itself! (For example, this happens when X is a toric manifold.)

Example 2.1. The standard action of $T = (\mathbb{C}^*)^n$ on $\mathbb{C}^n$ (by scaling coordinates) defines an action on $\mathbb{P}^{n-1} = \mathbb{P}(\mathbb{C}^n)$, giving the universal quotient line bundle $\mathscr{O}(1)$ an equivariant structure. Writing $\zeta = c_1^T(\mathscr{O}(1))$ for its equivariant Chern class in $H^2_T\mathbb{P}^{n-1}$, we have

$$H^*_T\mathbb{P}^{n-1} = \Lambda_T[\zeta]/\prod_{i=1}^{n}(\zeta + t_i).$$

We will work this out in detail soon; it follows easily from the general formula for the cohomology of a projective bundle. Note that sending $t_i \mapsto 0$ for all i defines a surjection $H^*_T\mathbb{P}^{n-1} \to H^*\mathbb{P}^{n-1} = \mathbb{Z}[\overline{\zeta}]/(\overline{\zeta}^n)$, where $\overline{\zeta} = c_1(\mathscr{O}(1))$ is the ordinary Chern class.

3. The localization package

The possibility of carrying out global computations using only local information at fixed points provides one of the most powerful applications of equivariant cohomology. This works best when $G = T$ is a torus. Two notions underwrite this technique. The first is that *equivariant cohomology should determine ordinary cohomology*: in good situations,

> $H^*_T X \to H^*X$ is *surjective*, with kernel generated by the kernel of $\Lambda_T \to \mathbb{Z}$.

The second notion is that *equivariant cohomology should be determined by fixed points*: in good situations, for $\iota\colon X^T \hookrightarrow X$,

> $H^*_T X \xrightarrow{\iota^*} H^*_T X^T$ is *injective*, and becomes an isomorphism after inverting enough elements of Λ_T.

We will also see theorems characterizing the image of ι^*.

Both of these are desired properties, but they can certainly fail for a given action of T on X. (For example, if X has no fixed points, it will be difficult for the second notion to hold; if $X = T$, acting on itself by translation, then both properties fail. A useful exercise is to look for other examples where one or both of these properties fails.) In plenty of common situations, though, both do hold true: for example, whenever X is a nonsingular projective variety with finitely many fixed points. Theorems about when these properties hold form the core of the *localization package*.

Another component of this package is an *integration formula* which computes the pushforward along a proper map of nonsingular varieties, $f\colon X \to Y$, via restriction to fixed points. In the especially useful case of $\rho\colon X \to \mathrm{pt}$, with X^T finite, this takes the form

$$\int_X \alpha := \rho_*(\alpha) = \sum_{p \in X^T} \frac{\alpha|_p}{c^T_{\mathrm{top}}(T_p X)},$$

where the right-hand side is a finite sum of elements of the fraction field of Λ_T—that is, rational functions in the variables t_i.

All of this package consists of essentially equivariant phenomena: for any space X with nontrivial cohomology and finitely many fixed points, you could never have an injection $H^*X \to H^*X^T$, by degree! Similarly, the right-hand side of the integration formula is only defined equivariantly, since the denominators are positive-degree elements of $H^*_T(\mathrm{pt})$.

Localization fits into the fiber bundle picture via sections: in terms of the previous diagram, we have

$$\begin{array}{ccc} X & \hookrightarrow & \mathbb{E} \times^G X \\ {\scriptstyle \iota_p}\Big\uparrow\Big\downarrow & & \Big\downarrow\Big\uparrow \\ \mathrm{pt} & \hookrightarrow & \mathbb{B} \end{array}$$

with the inclusion $\iota_p\colon \{p\} \hookrightarrow X$ of a fixed point inducing a section of the fiber bundle $\mathbb{E} \times^G X \to \mathbb{B}$. Pulling back along this section gives the restriction homomorphism $\iota_p^*\colon H^*_G X \to H^*_G(p) = \Lambda_G$.

4. Schubert calculus and Schubert polynomials

Our two main thematic strands—fiber bundles and localization—braid together nicely in modern Schubert calculus. Here X is a projective homogeneous space, for example, $\mathbb{P}^{n-1}$, $Gr(d,\mathbb{C}^n)$, $Fl(\mathbb{C}^n)$, or more generally, G/P for a reductive group G and parabolic subgroup P. The cohomology ring has a basis of Schubert classes $[\Omega_w]$, where $\Omega_w \subseteq X$ is a Schubert variety, defined by certain incidence conditions. These subvarieties are invariant for the action of a torus, and in fact their equivariant classes $\sigma_w = [\Omega_w]^T$ form a Λ_T-basis for H_T^*X. (The set W indexing the Schubert basis is a quotient of the Weyl group of G. For $G = GL_n$, this is the symmetric group S_n.)

A central problem is to understand these classes σ_w. In particular, one would like expressions for them as polynomials in ring generators for H_T^*X; formulas for their restrictions to fixed points; and combinatorial rules for their multiplication. The last of these is a long-standing open problem: one can write

$$\sigma_u \cdot \sigma_v = \sum_w c_{uv}^w \, \sigma_w,$$

for some homogeneous polynomials c_{uv}^w in $\Lambda_T = \mathbb{Z}[t_1,\ldots,t_n]$. What are these polynomials?

The structure constants c_{uv}^w satisfy a positivity property: when written in appropriate variables, these polynomials have nonnegative coefficients. (When there is no torus, the structure constants are nonnegative integers, by an application of Kleiman-Bertini transversality.) The problem is to find a combinatorial formula for c_{uv}^w manifesting this positivity. Good answers are known for some spaces—Grassmannians, cominuscule varieties, 3-step flag varieties—but even the non-equivariant question remains open in most cases, despite much recent progress. A key theme in recent advances is that equivariant techniques aid in proving non-equivariant theorems.

One can say more about the other problems. There is an elegant formula for restricting σ_w to a fixed point p_u, expressed as a sum over certain reduced words in the Weyl group. And there are good formulas for representing σ_w as a polynomial in Chern classes, at least in classical types.

We will focus on "type A", and in particular the complete flag variety $Fl(\mathbb{C}^n)$. For each permutation $w \in S_n$, there are Schubert classes $[\Omega_w] \in H^*Fl(\mathbb{C}^n)$ and $[\Omega_w]^T \in H_T^*Fl(\mathbb{C}^n)$. In 1982, Lascoux and Schützenberger defined and initiated the study of *Schubert polynomials* $\mathfrak{S}_w(x) \in \mathbb{Z}[x_1, \ldots, x_n]$, which are homogeneous polynomials mapping to $[\Omega_w]$ under a ring presentation $\mathbb{Z}[x] \twoheadrightarrow H^*Fl(\mathbb{C}^n)$.

There are also *double Schubert polynomials*

$$\mathfrak{S}_w(x; y) \in \mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n],$$

and it was later proved that these map to $[\Omega_w]^T$ in $H_T^*Fl(\mathbb{C}^n) = \mathbb{Z}[x, y]/I$. Since $H_T^*Fl(\mathbb{C}^n)$ is a quotient of a polynomial ring, there are necessarily many choices for polynomials representing $[\Omega_w]^T$, but it is generally agreed that $\mathfrak{S}_w(x; y)$ are the best ones. They have many wonderful combinatorial and geometric properties, and we will study them in detail later.

Briefly, here is a different way the polynomials $\mathfrak{S}_w(x; y)$ arise, which would have been familiar to mathematicians working over 100 years earlier. We will place rank conditions on $n \times n$ matrices, and compute the degree of the corresponding variety defined by the vanishing of certain minors. This sort of problem was studied by 19th century geometers, especially Cayley, Salmon, Roberts, and Giambelli. For a permutation $w \in S_n$, consider the (transposed) permutation matrix $A_w^\dagger$ having 1's in the $w(i)$th column of the ith row (position $(i, w(i))$) and 0's elsewhere. For example, the permutation $w = 2\,3\,1$ has matrix

$$A_{231}^\dagger = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Let $A[p, q]$ denote the upper-left $p \times q$ submatrix of any matrix A, and define

$$D_w = \left\{ A \in M_{n,n} \,\middle|\, \mathrm{rk}(A[p, q]) \le \mathrm{rk}(A_w^\dagger[p, q]) \text{ for all } 1 \le p, q \le n \right\}.$$

This is an irreducible subvariety of $M_{n,n} \cong \mathbb{A}^{n^2}$, of codimension $\ell(w) = \{i < j \,|\, w(i) > w(j)\}$. It is invariant for an action of $T = (\mathbb{C}^*)^n \times (\mathbb{C}^*)^n$ which scales rows and columns: viewing each factor of T as diagonal matrices, $(u,v) \cdot A = u\,A\,v^{-1}$. This means there is a class

$$[D_w]^T \in H_T^* M_{n,n} = \Lambda_T = \mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n],$$

homogeneous of degree $\ell(w)$ in the variables. (Since $M_{n,n}$ is contractible, its equivariant cohomology is that of a point.)

THEOREM. *This class equals the Lascoux-Schützenberger double Schubert polynomial:* $[D_w]^T = \mathfrak{S}_w(x;y)$.

This theorem is one piece of evidence of the naturality of Schubert polynomials, as well as the advantage of working equivariantly: there are no relations in the polynomial ring $H_T^* M_{n,n}$, so no choices.

EXAMPLE. The locus $D_{2\,3\,1}$ is defined by two equations, $a_{11} = a_{21} = 0$. Each coordinate a_{ij} comes with T-weight $x_i - y_j$, so Bézout's theorem implies $\mathfrak{S}_{2\,3\,1}(x;y) = (x_1 - y_1)(x_2 - y_1)$.

Notes

In addition to the original construction, many of the core ideas of equivariant cohomology appear in Borel's seminar on transformation groups [**Bor60**]. This includes the fiber bundle and localization perspective, as well as the idea of approximating by finite dimensional spaces. (They used CW complexes, not algebraic varieties.) In modern language, the Borel construction can be regarded as a "homotopy quotient" of X by G, since it is the homotopy colimit of a diagram $G \times X \rightrightarrows X$. Alternatively, one can view $H_G^* X$ as the cohomology of the quotient stack $[G\backslash X]$. See [**Beh04**] for an introduction to the stack perspective.

Much of the current work on equivariant cohomology in algebraic geometry has roots in the story of modern Schubert calculus. Recent breakthroughs in the structure constant problem begin with Knutson and Tao's puzzle rule for Grassmannians [**KnTao03**], which we will see in Chapter 9. Since then, formulas for two-step flag varieties have been found [**Co09, Buc15, BKPT16**], as well as very recent formulas for two- and three-step flags [**KnZJ20**]. There are also some rules for classical Schubert calculus on certain spaces G/P for groups G other than GL_n. Pragacz showed that a formula of Stembridge computes the structure constants of the Lagrangian Grassmannian [**Pra91**], and more generally, Thomas and Yong have found type-uniform formulas for all cominuscule flag varieties [**ThYo09**].

The restriction formula for equivariant Schubert classes at fixed points is due to Andersen-Jantzen-Soergel [**AJS94**] and Billey [**Bi99**]; we will prove it in Chapter 18. The relationship between double Schubert polynomials and equivariant classes was established in the 1990's [**Ful92, KnMi05, FeRi03**].

Hints for exercises

[1] A solution can be found in [**Hat02**, Ex. 1B.3]. See, e.g., [**MilSta74**, §14] for the computation of $H^*\mathbb{CP}^\infty$.