CHAPTER 5

Localization I

The possibility of restricting attention to fixed points is a key feature of equivariant cohomology. The technique works best when the group is a torus T, and we will see some examples indicating why. There are three basic pieces of the *localization package*:

(1) the *main localization theorem*, which says when the restriction homomorphism $\iota^*\colon H_T^*X \to H_T^*X^T$ is injective, or an isomorphism after inverting elements of Λ_T;

(2) the *integration formula*, which computes a Gysin homomorphism $f_*\colon H_T^*X \to H_T^*Y$ in terms of a corresponding map on fixed loci; and

(3) The *image theorem*, describing the image of ι^* as a subring of $H_T^*X^T$ defined by divisibility conditions.

We will return to the third component in Chapter 7, and focus on the first two pieces here.

1. The main localization theorem (first approach)

The main theorem says the restriction homomorphism

$$\iota^*\colon H_T^*X \to H_T^*X^T$$

becomes an isomorphism after inverting classes in $\Lambda_T = \operatorname{Sym}^* M$, coming from characters $\chi \in M$. This is true for any algebraic variety X, as we will see later. A very simple proof can be given for nonsingular varieties, though, so we consider that case first. The main idea is to prove this statement about restriction to the fixed locus by considering the Gysin pushforward from the fixed locus.

Example 1.1. Let T act on $\mathbb{P}(V) = \mathbb{P}^{n-1}$ by characters $\chi_1, \ldots, \chi_n$. We have computed

$$H_T^* \mathbb{P}^{n-1} = \Lambda_T[\zeta] / \prod (\zeta + \chi_i),$$

where $\zeta = c_1^T(\mathscr{O}(1))$. If the characters $\chi_1, \ldots, \chi_n$ are distinct, the fixed points are the coordinate lines $p_i = [0, \ldots, 0, 1, 0, \ldots, 0]$, for $i = 1, \ldots, n$. The tangent spaces are

$$T_{p_i}\mathbb{P}^{n-1} = \mathrm{Hom}(L_i, V/L_i) \cong \bigoplus_{j \neq i} L_i^\vee \otimes L_j,$$

where L_i is the coordinate line, isomorphic to $\mathbb{C}_{\chi_i}$ as a T-representation. In coordinates, one sees this by computing

$$\begin{aligned} z \cdot [a_1, \ldots, a_{i-1}, 1, a_{i+1}, \ldots, a_n] &= [\chi_1(z)\, a_1, \ldots, \chi_i(z), \ldots, \chi_n(z)\, a_n] \\ &= [\frac{\chi_1(z)}{\chi_i(z)} a_1, \ldots, 1, \ldots, \frac{\chi_n(z)}{\chi_i(z)} a_n]. \end{aligned}$$

So $c_{n-1}^T(T_{p_i}\mathbb{P}^{n-1}) = \prod_{j \neq i}(\chi_j - \chi_i)$.

The self-intersection formula then says $(\iota_{p_i})^*(\iota_{p_i})_*$ is multiplication by $\prod_{j\neq i}(\chi_j - \chi_i)$. One can also see this directly. The Gysin pushforward $(\iota_{p_i})_* \colon H_T^*(p_i) \to H_T^*\mathbb{P}^{n-1}$ sends 1 to $[p_i]^T = \prod_{j \neq i}(\zeta + \chi_j)$, and the restriction of the tautological bundle is $\mathscr{O}(-1)|_{p_i} = L_i$, so ζ restricts to $c_1^T(L_i^\vee) = -\chi_i$.

Exercise 1.2. Using the basis $\{1, \zeta, \ldots, \zeta^{n-1}\}$ for $H_T^*\mathbb{P}^{n-1}$ and the standard basis for $\Lambda^{\oplus n}$, compute the matrix of the restriction homomomorphism

$$\iota^* \colon H_T^*\mathbb{P}^{n-1} \to H_T^*(\mathbb{P}^{n-1})^T \cong \Lambda^{\oplus n}.$$

Compute its determinant, and conclude that the map is injective.[1]

Exercise 1.3. If the characters $\chi_1, \ldots, \chi_n$ are not distinct, the fixed locus $(\mathbb{P}^{n-1})^T$ has positive-dimensional components. Identify the fixed locus, and show that the restriction homomorphism is still injective.

The slice theorem provides a useful tool for linearizing group actions near fixed points or orbits: For any reductive (or compact) group G acting on X, there is an invariant neighborhood of p in X which is equivariantly isomorphic to an invariant neighborhood of 0 in T_pX. More generally, we have the following:

THEOREM 1.4 (SLICE THEOREM). *Let X be a nonsingular complex alegbraic variety.*

(1) Suppose K is a compact Lie group acting on X, with an orbit $O = K \cdot x \subseteq X$. Then there is a K-invariant open neighborhood $U \subseteq X$ of O which is equivariantly isomorphic to an open neighborhood of the zero section in the normal bundle $N_{O/X}$.

(2) Suppose X is affine, and G is a reductive group acting on X, with a closed orbit $O = G \cdot x$. Then there is a G-equivariant étale neighborhood $U \to X$ of O which is equivariantly isomorphic to an étale neighborhood of the zero section of the normal bundle $N_{O/X}$.

The first statement, for compact groups, is easily proved: by averaging any hermitian metric over K, one can find a K-invariant hermitian metric on X. A tubular neighborhood of the orbit $K \cdot x$ with respect to this metric provides the desired K-invariant open neighborhood. References with more details can be found in the Notes.

Often we will assume that T acts with finitely many fixed points. This has a characterization in terms of tangent spaces. A fixed point $p \in X^T$ is *isolated* if it is a connected component of X^T.

LEMMA 1.5. *Let G be a connected reductive linear algebraic group (or compact connected Lie group) acting on a nonsingular algebraic variety X, with a fixed point $p \in X^G$. The point p is isolated if and only if the trivial representation does not occur in T_pX.*

PROOF. By the slice theorem, we can reduce to the case where $X = V$ is a representation of G, and $p = 0$ is the origin. In this case, the lemma is immediate, since for any representation V of a connected group, the origin $0 \in V$ is an isolated fixed point if and only if V contains no copy of the trivial representation. □

The reductive (or compact) hypothesis is necessary.

EXAMPLE 1.6. Let the additive group $G = \mathbb{C}$ act on $\mathbb{C}^2$ by the matrix $\left[\begin{smallmatrix} 1 & a \\ 0 & 1 \end{smallmatrix}\right]$, inducing an action on $\mathbb{P}^1$. The point $p = [1, 0]$ is the unique fixed point, but the representation on $T_p\mathbb{P}^1$ is trivial.

When $G = T$ is a torus and $\dim X = n$, the lemma says that $p \in X^T$ is isolated if and only if $c_n^T(T_pX) \neq 0$. This formulation is particular to tori, and is not true for other reductive groups.

EXAMPLE 1.7. Consider $G = SL_n = X$ acting on itself by conjugation. The fixed points are the center of G, so there are finitely many; in particular, the identity element $e \in G$ is isolated. The action of G on $T_eG = \mathfrak{sl}_n$ is the adjoint representation. Restricting this to the diagonal torus $T \subset SL_n$ one sees an $(n-1)$-dimensional space of weight zero, namely $\mathfrak{t} \subseteq \mathfrak{sl}_n$, so $c_{top}^T(T_eG) = 0$. Since this is the image of $c_{top}^G(T_eG)$ under the injective map $\Lambda_G \to \Lambda_T$, it follows that $c_{top}^G(T_eG) = 0$, as well.

We can now state our first localization theorem.

THEOREM 1.8 (LOCALIZATION THEOREM, FINITE FIXED LOCUS). *Consider a d-dimensional nonsingular variety X with finitely many fixed points. Let*

$$c = \prod_{p \in X^T} c_d^T(T_pX) \in \Lambda,$$

and let $S \subseteq \Lambda$ be a multiplicative set containing c (which is nonzero, since all fixed points are isolated). Assume there are $m \leq \#X^T$ classes in H_T^*X restricting to a basis of H^*X.

Then $m = \#X^T$, the homomorphisms

$$S^{-1}H_T^*X \xrightarrow{S^{-1}\iota^*} S^{-1}H_T^*X^T \quad \text{and} \quad S^{-1}H_T^*X^T \xrightarrow{S^{-1}\iota_*} S^{-1}H_T^*X$$

are isomorphisms, and $\iota^* \colon H_T^*X \to H_T^*X^T$ is injective.

Most of hypotheses can be omitted, and we will see a stronger form of the localization theorem in Chapter 7. However, this simple version suffices for all the examples we will study, and it has the advantage of being very easy to prove. The main idea is to use the Gysin pushforward, as we saw for projective space.

PROOF. Let us temporarily write $n = \#X^T$, so we have

$$\Lambda^{\oplus n} = H_T^* X^T \xrightarrow{\iota_*} H_T^* X \xrightarrow{\iota^*} H_T^* X^T = \Lambda^{\oplus n}.$$

By basic properties of Gysin maps, the composition $\iota^* \circ \iota_* \colon \Lambda^{\oplus n} \to \Lambda^{\oplus n}$ is diagonal, and on the summand corresponding to $p \in X^T$ it is multiplication by $c_d^T(T_p X)$. So $\det(\iota^* \iota_*) = c$, and the cokernel of ι^* is annihilated by c. In particular, $S^{-1} H_T^* X \to S^{-1} H_T^* X^T$ is surjective.

The assumption that m elements restrict to a basis of $H^* X$ means that $H_T^* X$ is a free Λ-module of rank m (by Leray-Hirsch or graded Nakayama). Since Λ is noetherian, we conclude that $m = n$ and $S^{-1} H_T^* X \to S^{-1} H_T^* X^T$ is an isomorphism. Injectivity of ι^* follows from the fact that $H_T^* X$ is free over the domain Λ. □

EXAMPLE 1.9. When T acts on $V \cong \mathbb{C}^n$ by distinct characters $\chi_1, \ldots, \chi_n$, the localization theorem for $X = \mathbb{P}(V) = \mathbb{P}^{n-1}$ is simply the Chinese Remainder Theorem. Indeed, with

$$A = S^{-1} H_T^* \mathbb{P}^{n-1} = (S^{-1}\Lambda)[\zeta] / \left(\prod (\zeta + \chi_i)\right),$$

the localization theorem says that the homomorphism

$$A \to A/(\zeta + \chi_1) \times \cdots \times A/(\zeta + \chi_n)$$

is an isomorphism. Algebraically, this is true because the ideals $(\zeta + \chi_i)$ are pairwise comaximal.

EXAMPLE 1.10. Again suppose T acts on $V \cong \mathbb{C}^n$ by distinct characters $\chi_1, \ldots, \chi_n$. Then $X = Gr(d, V)$ has finitely many fixed points, corresponding to coordinate subspaces:

$$X^T = \left\{ p_I \mid I = \{i_1 < \cdots < i_d\} \subseteq \{1, \ldots, n\} \right\},$$

where $p_I = [E_I]$ is the subspace $E_I = \langle e_{i_1}, \ldots, e_{i_d} \rangle = \langle e_i \mid i \in I \rangle$.

Indeed, each tangent space

$$T_{p_I} X = \mathrm{Hom}(E_I, V/E_I) \cong \bigoplus_{\substack{i \in I \\ j \notin I}} L_i^\vee \otimes L_j$$

has weights $\chi_j - \chi_i$, for $i \in I$ and $j \notin I$, which are all nonzero. We see

$$c^T_{top}(T_{p_I}X) = \prod_{\substack{i\in I\\ j\notin I}} (\chi_j - \chi_i).$$

There are $\binom{n}{d}$ fixed points, and we know bases of $H^*_T X$ with $\binom{n}{d}$ elements, restricting to bases of H^*X. So $H^*_T X \hookrightarrow H^*_T X^T \cong \Lambda^{\oplus\binom{n}{d}}$.

An explicit coordinate description of this action is as follows. Given a subset I, let $J = \{1, \dots, n\} \setminus I$ be the complement, so $V/E_I \cong E_J$ and there is a decomposition $V = E_I \oplus E_J$. As in Chapter 4, §2, corresponding to this decomposition there is an open neighborhood $U \cong \mathrm{Hom}(E_I, E_J)$ of p_I. For instance, let us take $n = 6$, $d = 3$, and $I = \{2, 4, 5\}$, and the standard action of $T = (\mathbb{C}^*)^6$ on $V = \mathbb{C}^6$. The induced action on U can be represented in matrix form as

$$z \cdot \begin{bmatrix} * & * & * \\ 1 & 0 & 0 \\ * & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ * & * & * \end{bmatrix} = \begin{bmatrix} z_1 * & z_1 * & z_1 * \\ z_2 & 0 & 0 \\ z_3 * & z_3 * & z_3 * \\ 0 & z_4 & 0 \\ 0 & 0 & z_5 \\ z_6 * & z_6 * & z_6 * \end{bmatrix} = \begin{bmatrix} \frac{z_1}{z_2} * & \frac{z_1}{z_4} * & \frac{z_1}{z_5} * \\ 1 & 0 & 0 \\ \frac{z_3}{z_2} * & \frac{z_3}{z_4} * & \frac{z_3}{z_5} * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ \frac{z_6}{z_2} * & \frac{z_6}{z_4} * & \frac{z_6}{z_5} * \end{bmatrix}.$$

This description makes the tangent weights visible.

With only a little more care, we can relax the hypothesis that the fixed locus be finite. We still assume that X is nonsingular. A basic fact is that X^T is always nonsingular. In fact, this is true more generally of fixed loci for actions by *diagonalizable groups*, i.e., $G \cong (\mathbb{C}^*)^r \times A$ for some finite abelian group A:

LEMMA 1.11. *When a diagonalizable group G acts on a nonsingular variety X, the fixed locus X^G is nonsingular.*

(In topology, this can be deduced easily from the slice theorem, and it holds more generally for the action of any compact group G. A stronger version of this lemma in algebraic geometry was proved by Iversen.)

We will need another lemma about the characters of the torus acting on the normal bundle to a fixed component.

LEMMA 1.12. *Let X be a nonsingular variety, and let $Z \subseteq X^T$ be a connected component of the fixed locus, of codimension d in X. Write $N = N_{Z/X}$ for its normal bundle, an equivariant vector bundle of rank d on Z. Then there are nonzero characters $\chi_1, \ldots, \chi_d$ so that for any point $p \in Z$, the fiber $N_p = T_pX/T_pZ$ has T acting by these weights. The action of T on T_pZ is trivial.*

PROOF. Use the slice theorem to find a neighborhood $U \subseteq X$ of p which is equivariantly isomorphic to a neighborhood of 0 in T_pX. Then $Z \cap U$ maps to an open subset of the 0-weight space of T_pX (where T acts trivially), since this is $T_pZ \subseteq T_pX$. It follows that the characters on $N_p = T_pX/T_pZ$ are all nonzero. Since Z is connected, these characters are the same for any other point $q \in Z$. □

For any connected component $Z \subseteq X^T$ of codimension d, the self-intersection formula says that the composition

$$H_T^*Z \xrightarrow{\iota_*} H_T^*X \xrightarrow{\iota^*} H_T^*Z$$

is multiplication by the top Chern class $c_d^T(N_{Z/X})$. In $H_T^*Z = \Lambda \otimes_{\mathbb{Z}} H^*Z$, this class can be written as

$$c_d^T(N_{Z/X}) = \chi_1 \cdots \chi_d + \sum_{i=1}^{d} a_{d-i} c_i,$$

for some classes $a_j \in \Lambda^{2j}$ and $c_i \in H^{2i}Z$, where $\chi_1, \ldots, \chi_d$ are the characters of T on the normal bundle, as in the previous lemma. Since H^*Z is a finite-dimensional ring, the elements c_i are nilpotent, so $c_d^T(N_{Z/X})$ becomes invertible in $S^{-1}H_T^*Z$, for any multiplicative set S containing $\chi_1 \cdots \chi_d$.

With these observations, the proof of the following goes just as in the case where X^T is finite.

THEOREM 1.13 (LOCALIZATION THEOREM, NONSINGULAR VARIETIES). *Let X be a nonsingular variety, and $S \subseteq \Lambda$ a multiplicative set containing all nonzero characters appearing in T_pX, for all $p \in X^T$. Write $X^T = \coprod Z_\alpha$*

as a union of connected components. Assume there are m elements of H_T^*X restricting to a basis of H^*X, with $m \leq \sum_\alpha \operatorname{rk} H^*Z_\alpha$.

Then $m = \sum \operatorname{rk} H^*Z_\alpha$, the homomorphisms

$$S^{-1}H_T^*X \xrightarrow{S^{-1}\iota^*} S^{-1}H_T^*X^T \quad \text{and} \quad S^{-1}H_T^*X^T \xrightarrow{S^{-1}\iota_*} S^{-1}H_T^*X$$

are isomorphisms, and $\iota^* \colon H_T^*X \to H_T^*X^T$ is injective.

EXERCISE 1.14. Prove Theorem 1.13, using the Gysin homomorphism as before.

EXERCISE 1.15. Consider T acting on $X = \mathbb{P}^2$ by characters $0, \chi, \chi$. What is X^T? Work out the weights on each tangent space.

EXERCISE 1.16. Suppose the T-action on V decomposes as $V = \bigoplus_{i=1}^m V_i$, where V_i is the χ_i-isotypic component, and $\chi_1, \ldots, \chi_m$ are distinct. Say $\dim V_i = n_i$. Show that $X = Gr(d, V)$ has fixed locus

$$X^T \cong \coprod_{\substack{d_1+\cdots+d_m=d \\ 0\leq d_i\leq n_i}} Gr(d_1, V_1) \times \cdots \times Gr(d_m, V_m).$$

Note that $\operatorname{rk} H^*X = \binom{n}{d} = \sum \prod_{i=1}^m \binom{n_i}{d_i} = \operatorname{rk} H^*X^T$. The normal bundle to a component $Z_{\mathbf{d}} = Gr(d_1, V_1) \times \cdots \times Gr(d_m, V_m)$ is

$$N_{\mathbf{d}} = \bigoplus_{j\neq i} \operatorname{Hom}(\mathbb{S}_i, \mathbb{Q}_j).$$

What are the characters of T acting on the restriction of $N_{\mathbf{d}}$ to a fixed point?[2]

2. Integration formula

From now on, we will assume that $S \subseteq \Lambda$ is a multiplicative set such that the maps

$$S^{-1}H_T^*X^T \xrightarrow{S^{-1}\iota_*} S^{-1}H_T^*X \xrightarrow{S^{-1}\iota^*} S^{-1}H_T^*X^T$$

are isomorphisms. (We have proved this in the case where X is nonsingular, with H_T^*X free over Λ of rank equal to that of H^*X^T. In fact, $S^{-1}\iota^*$ is an isomorphism for any X, for a suitable S, as we will see in Chapter 7.)

Consider a proper T-equivariant map of nonsingular varieties $f: X \to Y$. For each connected component $P \subseteq X^T$, $f(P)$ is contained in a unique connected component $Q \subseteq Y^T$; let $f_P: P \to Q$ be the restriction of f. For any class $u \in H_T^* X$, we will write $u|_P \in H_T^* P$ for the restriction of this class to P, and similarly for the restriction classes in $H_T^* Y$ to Q.

Being components of the fixed locus for actions on nonsingular varieties, both P and Q are nonsingular, and the map f_P is proper, so both vertical maps in the diagram

$$\begin{tikzcd} P \arrow[r, hook] \arrow[d, "f_P"'] & X \arrow[d, "f"] \\ Q \arrow[r, hook] & Y \end{tikzcd}$$

have associated Gysin homomorphisms. Our goal is to compute f_* in terms of $(f_P)_*$. More precisely, we compute the restrictions $f_*(u)|_Q$, for any $u \in H_T^* X$.

THEOREM 2.1 (INTEGRATION FORMULA). *For any $u \in H_T^* X$ and any connected component $Q \subseteq Y^T$, we have*

$$f_*(u)|_Q = c^T_{top}(N_{Q/Y}) \cdot \sum_{P : f(P) \subseteq Q} (f_P)_* \left(\frac{u|_P}{c^T_{top}(N_{P/X})} \right).$$

In general, the formula takes place in the image of $\Lambda \otimes H^* Q = H_T^* Q$ in $S^{-1} H_T^* Q = S^{-1}\Lambda \otimes H^* Q$. When $H^* Q$ is free over $H^*(\mathrm{pt})$—for example, if Q is a point, or if one uses field coefficients for cohomology—the homomorphism $\Lambda \otimes H^* Q \to S^{-1}\Lambda \otimes H^* Q$ is injective, and the formula holds in $H_T^* Q = \Lambda \otimes H^* Q$. This will be the case in all our applications.

PROOF. Since the Gysin map $S^{-1}\iota_*: S^{-1} H_T^* X^T \to S^{-1} H_T^* X$ is an isomorphism, it suffices to prove the formula for $u = (\iota_P)_*(z)$, for some component $P \subseteq X^T$ and $z \in H_T^* P$. By functoriality and the self-intersection formula, the left-hand side is

$$\begin{aligned}(\iota_Q^* \circ f_* \circ (\iota_P)_*)(z) &= (\iota_Q^*(\iota_Q)_*(f_P)_*)(z) \\ &= \begin{cases} c_{top}^T(N_{Q/Y}) \cdot (f_P)_*(z) & \text{if } f(P) \subseteq Q; \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

On the right-hand side, using the same properties of Gysin maps, we have

$$u|_P = (\iota_P^* \circ (\iota_P)_*)(z) = c_{top}^T(N_{P/X}) \cdot z,$$

and $u|_{P'} = 0$ for $P' \neq P$. So the sum on this side reduces to the single term

$$c_{top}^T(N_{Q/Y}) \cdot (f_P)_* \left(\frac{c_{top}^T(N_{P/X}) \cdot z}{c_{top}^T(N_{P/X})} \right) = c_{top}^T(N_{Q/Y}) \cdot (f_P)_*(z),$$

agreeing with the left-hand side. □

EXAMPLE 2.2. When Y is a point, we get an integration formula for $\rho\colon X \to \mathrm{pt}$:

$$\rho_*(u) = \sum_{P \subseteq X^T} (\rho_P)_* \left(\frac{u|_P}{c_{top}^T(N_{P/X})} \right),$$

where $(\rho_P)_*\colon H_T^* P \to \Lambda$ is integration over P.

EXAMPLE 2.3. Suppose X and Y have finitely many fixed points, and $f\colon X \to Y$ is a smooth morphism with relative tangent bundle $T_{X/Y}$. For each $q \in Y^T$ we have

$$f_*(u)|_q = \sum_{p \in f^{-1}(q)^T} \frac{u|_p}{c_{top}^T(T_{X/Y}|_p)},$$

since each f_p is an isomorphism.

When $P = \{p\}$ is a point, the Chern class appearing in the corresponding summand is $c_d^T(T_p X) = \chi_1(p) \cdots \chi_d(p)$, where $d = \dim X$ and the $\chi_i(p)$ are the characters of T acting on the tangent space $T_p X$. Combining the two previous examples gives a particularly useful and simple case:

COROLLARY 2.4. *Let X be a d-dimensional nonsingular compact algebraic variety with finitely many fixed points. Then*

$$\rho_*(u) = \sum_{p \in X^T} \frac{u|_p}{c_d^T(T_pX)}$$

for any class $u \in H_T^*X$.

EXAMPLE 2.5. For T acting on $\mathbb{P}^{n-1}$ via distinct characters $\chi_1, \dots, \chi_n$, with $\zeta = c_1^T(\mathscr{O}(1))$, we know

$$\rho_*(\zeta^k) = \begin{cases} 0 & \text{if } k < n-1, \\ 1 & \text{if } k = n-1, \end{cases}$$

by degree considerations in the first case, and by the classical fact that $n-1$ hyperplanes intersect in a point in the second case. On the other hand, the integration formula computes this as

$$\rho_*(\zeta^k) = \sum_{i=1}^{n} \frac{(-\chi_i)^k}{\prod_{j \neq i}(\chi_j - \chi_i)}.$$

Comparing the two yields a nontrivial algebraic identity!

EXAMPLE 2.6. Consider $T = \mathbb{C}^*$ acting on $\mathbb{P}^2$ by the characters $0, t, 2t$, so $z \cdot [a, b, c] = [a, zb, z^2c]$. The fixed points are the usual coordinate points p_1, p_2, p_3. For $u \in H_T^*\mathbb{P}^2$, let $u_i = u|_{p_i}$. The integration formula says

$$\rho_*(u) = \frac{u_1}{2t^2} + \frac{u_2}{-t^2} + \frac{u_3}{2t^2} = \frac{u_1 - 2u_2 + u_3}{2t^2}.$$

This must be a class in $\Lambda = \mathbb{Z}[t]$, so the integration formula implies a *divisibility condition* relating the restrictions to the three fixed points: $2t^2$ must divide the polynomial $u_1 - 2u_2 + u_3$.

When computing via localization, it is often convenient to represent the fixed points of X as the vertices of a graph, with edges connecting vertices when the corresponding fixed points are connected by a T-invariant curve. This graph is called the *moment graph* of X, and we will see several examples in the next few chapters. (Symplectic geometry explains the way these graphs are drawn; see

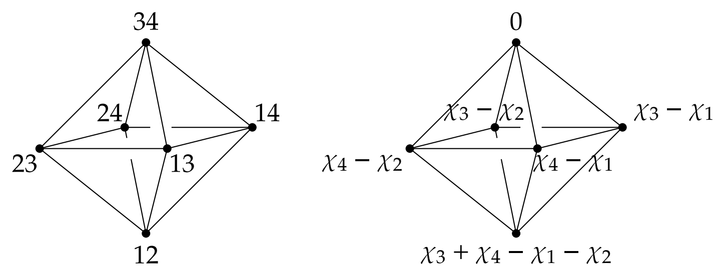

Figure 1. The fixed points in $X = Gr(2, \mathbb{C}^4)$, and the class $[\Omega]^T$ restricted to $H_T^* X^T$.

the Notes in Chapter 7.) The image of a class under the restriction $H_T^* X \hookrightarrow H_T^* X^T$ is given by labelling the vertices of the moment graph with characters.

Example 2.7. We will compute the number of lines meeting four general lines in $\mathbb{P}^3$. Let $X = Gr(2, \mathbb{C}^4)$ be the space of lines on $\mathbb{P}^3$, with an action of T induced by characters $\chi_1, \dots, \chi_4$.

Fix the line ℓ_0 corresponding to the subspace $E_{12} = \operatorname{span}\{e_1, e_2\} \subseteq \mathbb{C}^4$, and consider the locus $\Omega \subseteq X$ of lines ℓ meeting ℓ_0, i.e.,

$$\Omega = \left\{ E \subseteq \mathbb{C}^4 \,\middle|\, \dim(E \cap E_{12}) \geq 1 \right\}.$$

This is defined by the condition that $\mathbb{S} \to \mathbb{C}^4/E_{12}$ has rank at most 1, where $\mathbb{S}$ is the tautological bundle on X. In other words, the determinant homomophism

$$\bigwedge^2 \mathbb{S} \to \bigwedge^2 (\mathbb{C}^4/E_{12})$$

is zero. So $\Omega = Z(s)$ is the zeroes of a section of the line bundle

$$\operatorname{Hom}(\bigwedge^2 \mathbb{S}, \bigwedge^2(\mathbb{C}^4/E_{12})) = \bigwedge^2 \mathbb{S}^\vee \otimes \mathbb{C}_{\chi_3+\chi_4},$$

and $[\Omega]^T$ is equal to its equivariant first Chern class. We will compute its restriction to the fixed points p_I.

We have $c_1^T(\bigwedge^2 \mathbb{S}^\vee \otimes \mathbb{C}_{\chi_3+\chi_4})|_{p_{ij}} = -\chi_i - \chi_j + \chi_3 + \chi_4$. The class $[\Omega]^T$ is shown as a labelled moment graph in Figure 1.

To address the four-lines problem, first note that the assumption that the given lines be general means that the intersection

$$\Omega_{\ell_1} \cap \Omega_{\ell_2} \cap \Omega_{\ell_3} \cap \Omega_{\ell_4} = \left\{ \ell \mid \ell \text{ meets } \ell_1, \ell_2, \ell_3, \text{ and } \ell_4 \right\}$$

is transverse and zero-dimensional, and we wish to compute the number of points—that is,

$$\int_X [\Omega_{\ell_1}] \cdot [\Omega_{\ell_2}] \cdot [\Omega_{\ell_3}] \cdot [\Omega_{\ell_4}],$$

where $\int_X$ is the (non-equivariant) pushforward $H^*X \to H^*(\mathrm{pt}) = \mathbb{Z}$.

Any line ℓ' in $\mathbb{P}^3$ can be translated to ℓ_0 by an element $g \in GL_4$. So

$$\Omega_{\ell'} = \left\{ \ell \mid \ell \cap \ell' \neq \emptyset \right\} = g^{-1}\Omega,$$

and since GL_4 is a connected group, we have $[\Omega_{\ell'}] = [\Omega]$ in H^*X. So it is equivalent to compute $\int_X [\Omega]^4$.

By basic properties of Gysin homomorphisms (Chapter 3, §6), $\int_X [\Omega]^4$ is equal to the image of $\rho_*(([\Omega]^T)^4)$ under $H_T^*(\mathrm{pt}) \to H^*(\mathrm{pt})$. The class is in degree 0, and $H_T^0(\mathrm{pt}) = H^0(\mathrm{pt}) = \mathbb{Z}$. So this non-equivariant pushforward is the same as the equivariant one, and we can compute it using the integration formula:

$$\begin{aligned}\rho_*(([\Omega]^T)^4) = {} & \frac{(\chi_3 + \chi_4 - \chi_1 - \chi_2)^4}{(\chi_3 - \chi_1)(\chi_3 - \chi_2)(\chi_4 - \chi_1)(\chi_4 - \chi_2)} \\ & + \frac{(\chi_4 - \chi_1)^4}{(\chi_2 - \chi_1)(\chi_2 - \chi_3)(\chi_4 - \chi_1)(\chi_4 - \chi_3)} \\ & + \text{(four more terms, one of which is zero).}\end{aligned}$$

This expression can be evaluated quickly by computer algebra, but to carry out the calculation by hand, it is useful to employ another simplification.

Let us write $\Omega_{ij} = \{E \mid \dim(E \cap E_{ij}) \geq 1\}$, so $\Omega = \Omega_{12}$. By the same reasoning as before, we can compute with any four choices of ij; in particular, we may choose them so that many terms in the integration formula are zero. For example, the product $[\Omega_{12}]^T \cdot [\Omega_{13}]^T \cdot [\Omega_{34}]^T \cdot [\Omega_{24}]^T$ has nonzero localizations at only two fixed points, p_{14} and p_{23}. (See Figure 2.) Using the integration formula for this product, one

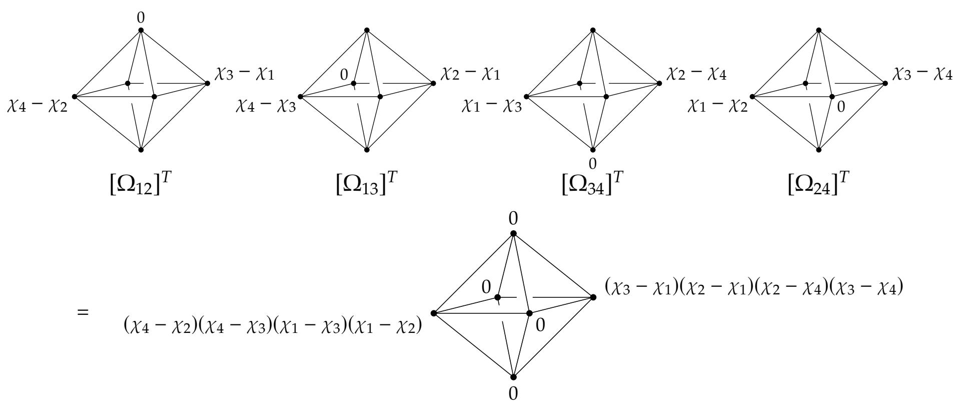

FIGURE 2. The product $[\Omega_{12}]^T \cdot [\Omega_{13}]^T \cdot [\Omega_{34}]^T \cdot [\Omega_{24}]^T$ in $H_T^*Gr(2, \mathbb{C}^4)$, represented by its localizations at fixed points.

sees

$$\begin{aligned}
\int_X [\Omega]^4 &= \rho_*\big([\Omega_{12}]^T \cdot [\Omega_{13}]^T \cdot [\Omega_{34}]^T \cdot [\Omega_{24}]^T\big) \\
&= \frac{(\chi_3-\chi_1)(\chi_2-\chi_1)(\chi_2-\chi_4)(\chi_3-\chi_4)}{(\chi_2-\chi_1)(\chi_3-\chi_1)(\chi_2-\chi_4)(\chi_3-\chi_4)} \\
&\quad + \frac{(\chi_4-\chi_2)(\chi_4-\chi_3)(\chi_1-\chi_3)(\chi_1-\chi_2)}{(\chi_1-\chi_2)(\chi_4-\chi_2)(\chi_1-\chi_3)(\chi_4-\chi_3)} \\
&= 1 + 1 = 2,
\end{aligned}$$

so there are two lines through the four given lines.

EXERCISE 2.8. How many lines in $\mathbb{P}^4$ meet six general planes?[3]

3. Equivariant formality

There are general criteria which imply the hypotheses of the localization theorems—in particular, freeness of H_T^*X as a Λ-module. As noted earlier, we will be able to verify these hypotheses directly for our main examples and applications, so the results of this section are not logically necessary. However, it is sometimes useful to know when to expect the localization package to work, and the terminology appears frequently in the literature.

For a Lie group G acting on X, an integer $m > 0$, and a coefficient ring R (usually $\mathbb{Z}$ or a field), consider the following condition:

($*_m$) For $0 \leq i \leq m$, $H^i X$ is finitely generated and free over R, and there are elements $x_{ij} \in H^i_G X$ that restrict to a basis for $H^i X$.

The space X is called *(cohomologically) equivariantly formal* with respect to the action of G and the coefficient ring R if it satisfies ($*_m$) for all $m > 0$. The main reason for introducing this condition is the following direct consequence of the Leray-Hirsch theorem (Appendix A, §4):

Proposition 3.1. *Assume ($*_m$) holds for some $m > 0$.*

(1) Every element of $H^m_G X$ has a unique expression as $\sum_{i,j} c_{ij} x_{ij}$, for some $c_{ij} \in H^{m-i}\mathbb{B}G$.

(2) If X is equivariantly formal, then $H^*_G X$ is a free Λ_G-module with basis $\{x_{ij}\}$, and the forgetful homomorphism

$$H^*_G X \otimes_{\Lambda_G} R \to H^* X$$

is an isomorphism. In fact, for any G' acting on X through a homomorphism $G' \to G$, the corresponding homomorphism

$$H^*_G X \otimes_{\Lambda_G} \Lambda_{G'} \to H^*_{G'} X$$

is an isomorphism.

We are most interested in the case where $G = T$ is a torus. For nonsingular complete varieties with finitely many fixed points, a general theorem provides a cell decomposition.

Theorem 3.2 (Białynicki-Birula). *Suppose a torus T acts on a nonsingular complete variety X with finitely many fixed points. Then there is a filtration by T-invariant closed subsets $X = X_n \supseteq X_{n-1} \supseteq \cdots \supseteq X_0 \supseteq \emptyset$, with $X_i \smallsetminus X_{i-1} = \coprod U_{ij}$ and $U_{ij} \cong \mathbb{A}^i$. Moreover, the total number of cells U_{ij} is equal to $\# X^T$.*

This implies such varieties are always equivariantly formal, since the classes of the invariant subvarieties $\overline{U_{ij}}$ form bases for $H^*_T X$ and $H^* X$, over Λ and R, respectively.

COROLLARY 3.3. *Let a torus T act on a nonsingular complete variety X, with finitely many fixed points. Then X is equivariantly formal with integral coefficients. In particular,*

(1) $H_T^*X \to H^*X$ is surjective, with kernel generated by the kernel of $\Lambda_T \to \mathbb{Z}$; and

(2) $H_T^*X \to H_T^*X^T$ is injective, and becomes an isomorphism after inverting finitely many characters in Λ_T.

PROOF. With cells U_{ij} as in the Białynicki-Birula decomposition, the equivariant class $[\overline{U_{ij}}]^T$ restricts to the nonequivariant class $[\overline{U_{ij}}]$, so X is equivariantly formal. Injectivity of the restriction homomorphism comes from the diagram

$$\begin{tikzcd} H_T^*X \arrow[r, "\iota^*"] \arrow[d, hook] & H_T^*X^T \arrow[d, hook] \\ S^{-1}H_T^*X \arrow[r, "\sim"] & S^{-1}H_T^*X^T, \end{tikzcd}$$

for a suitable multiplicative set $S \subseteq \Lambda$, where the vertical arrows are injective since H_T^*X and $H_T^*X^T$ are free over Λ, and the bottom arrow is an isomorphism by the basic localization theorem (Theorem 1.8). □

Thus complete nonsingular varieties with finitely many fixed points give a large class of examples where one sees the "two notions" about equivariant cohomology described in Chapter 1.

Applying the general localization theorem to be proved in Chapter 7, similar reasoning shows that if a T-variety X is equivariantly formal, and H^*X^T is also free over R, then the restriction homomorphism $\iota^*\colon H_T^*X \to H_T^*X^T$ is injective.

Notes

Luna's étale slice theorem is explained in [**GIT**, p. 198]. The topological slice theorem is apparently due to Koszul [**Ko53**], and can be found in Audin's book [**Aud04**, Chapter I]. We learned Example 1.7 from Johan de Jong.

Iversen's theorem on the nonsingularity of the fixed locus (Lemma 1.11) applies more generally for actions of linearly reductive groups, i.e., those for which all finite-dimensional respresentations are completely reducible; in positive characteristic this amounts to considering diagonalizable groups [**Iv72**]. Iversen also includes a formula for Euler characteristics which gives $\operatorname{rk} H^*X = \#X^T$ in the case when X has finitely many fixed points and no odd-dimensional cohomology. Again, the novelty is mainly the algebraic proof and the application to positive characteristic; as Iversen points out, in topology it can be deduced from the Lefschetz trace formula.

The idea of proving localization theorems using Gysin pushforwards can be traced to Quillen [**Qn71a**] and Quart [**Qt79**], who used similar techniques in cobordism and K-theory, respectively.

The integration formula, especially in the finite fixed point case of Corollary 2.4, is known by many names in the literature. Names commonly attached to it include Atiyah-Bott (after their paper [**AtBo84**]), Berline-Vergne ([**BeVer82**]), Duistermaat-Heckman ([**DuHe82**]), and "stationary phase formula" (especially in the physics literature).

Example 2.5 is one case of a family of identities due to Sylvester, and rediscovered by many other mathematicians. A short review of the history, along with an elementary proof, can be found in [**Bh99**]. Many such identities can be obtained by equivariant localization on other spaces.

The usage of the term "equivariantly formal" in the sense of §3 appears to originate in the seminal article of Goresky-Kottwitz-MacPherson [**GKM98**]. In this paper (and in much of the literature stemming from it), an equivariantly formal space is defined to be one for which the Serre spectral sequence for the fibration $\mathbb{E}G \times^G X \to \mathbb{B}G$,

$$E_2^{p,q} = H^p(\mathbb{B}G; H^q X) \Rightarrow H_G^{p+q} X,$$

degenerates at the E_2 term. This condition was considered earlier by Borel [**Bor60**, §XII.3–6].

Using coefficients in $\mathbb{Q}$, nine sufficient conditions for equivariant formality are given in [**GKM98**, Theorem 14.1], including the following.

- $H^*(X;\mathbb{Q})$ vanishes in odd degrees, and G is a connected linear algebraic group or compact Lie group.

- X is a nonsingular projective variety, and $G = T$ is a torus.

- X is a possibly singular projective algebraic variety, $G = T$ is a torus, and for all $q \geq 0$, $H^q(X;\mathbb{Q})$ is pure of weight q (in the sense of mixed Hodge theory).

The last condition includes all toric varieties. The use of field coefficients is essential in all of these conditions.

A different notion of equivariant formality is used in rational homotopy theory, where it involves an isomorphism between $H^*(X;\mathbb{Q})$ and a certain differential graded algebra. In order to disambiguate the terminology, Franz and Puppe propose to add the modifier "cohomological" to the equivariant formality we consider. (They also point out that the abbreviation CEF also stands for "cohomology extension of the fiber", which nicely captures the geometry.)

Białynicki-Birula proved a stronger version of Theorem 3.2, where the fixed locus X^T may have positive-dimensional components [**BB73**]; see also [**Bri97b**, §3.1].

Hints for exercises

[1]Here is another way to prove injectivity. The composition

$$\Lambda^{\oplus n} = H_T^*(\mathbb{P}^{n-1})^T \xrightarrow{\iota_*} H_T^*\mathbb{P}^{n-1} \xrightarrow{\iota^*} H_T^*(\mathbb{P}^{n-1})^T = \Lambda^{\oplus n}$$

is diagonal. What is its determinant? (This shows the maps $S^{-1}H_T^*(\mathbb{P}^{n-1})^T \to S^{-1}H_T^*\mathbb{P}^{n-1} \to S^{-1}H_T^*(\mathbb{P}^{n-1})^T$ are isomorphisms, for an appropriate multiplicative set S.)

[2]The tangent bundle TX restricts to $Z_{\mathbf{d}}$ as $\mathrm{Hom}(\mathbb{S},\mathbb{Q}) = \bigoplus_{i,j}\mathrm{Hom}(\mathbb{S}_i,\mathbb{Q}_j)$, and $TZ_{\mathbf{d}}$ accounts for the diagonal summands; this explains the computation of $N_{\mathbf{d}}$. The characters are $\chi_j - \chi_i$ for $i \neq j$, appearing with multiplicity $d_i(n_j - d_j)$.

[3]The locus $\Omega \subseteq Gr(2,\mathbb{C}^5)$ of lines meeting the plane $\mathbb{P}(E_{123})$ is given by the vanishing of $\bigwedge^2 \mathbb{S} \to \bigwedge^2(\mathbb{C}^5/E_{123})$. So its class is $[\Omega]^T = c_1^T(\bigwedge^2 \mathbb{S}^\vee \otimes \mathbb{C}_{\chi_4+\chi_5})$.