CHAPTER 9

Schubert calculus on Grassmannians

In Chapter 4, we computed $H_G^*Gr(d,V)$ in terms of Schur polynomials, using the tautological bundles on $Gr(d,V)$. Here we will study the geometry of this space in more detail. Our main focus is on Schubert varieties, especially ways of describing and multiplying their classes in equivariant cohomology.

1. Schubert cells and Schubert varieties

As in Chapter 4, we fix $d+e=n$, and consider the Grassmannian $Gr(d,V)=Gr(V,e)$. Now we also fix a flag

$$E_\bullet : E_1 \subset E_2 \subset \cdots \subset E_n = V,$$

with $\dim E_q = q$. Often we will write $E^q = E_{n-q}$, so subscripts indicate dimension, and superscripts indicate codimension in V.

Given a partition $\lambda = (e \geq \lambda_1 \geq \cdots \geq \lambda_d \geq 0)$, the *Schubert cell* $\Omega_\lambda^\circ = \Omega_\lambda^\circ(E_\bullet)$ is the set of subspaces $F \subset V$ satisfying the conditions

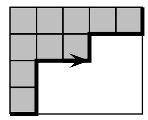
$$\dim(F \cap E_q) = k \ \text{ for } \ q \in [e+k-\lambda_k, e+k-\lambda_{k+1}],\ k=0,\dots,d.$$

Equivalently, given a subset $I=\{i_1<\cdots<i_d\} \subset \{1,\dots,n\}$, this is the same as defining

$$\Omega_I^\circ = \left\{F \,\middle|\, \dim(F\cap E^{q-1}) = d-k \text{ for } q \in (i_k, i_{k+1}],\ k=0,\dots,d\right\}.$$

(By convention, we set $\lambda_0=e$, $\lambda_{d+1}=0$, $i_0=0$, $i_{d+1}=n+1$.) The equivalence is by $i_k = k+\lambda_{d+1-k}$. The bijection between partitions λ inside the $d\times e$ rectangle and d-element subsets $I \subseteq \{1,\dots,n\}$ can be seen graphically by recording the vertical steps when walking SW to NE along the border of λ, as shown in Figure 1.

Let us choose a standard basis $e_1,\dots,e_n$ so that the fixed subspace E^q is the span of $e_{q+1},\dots,e_n$. The Borel subgroup $B^- \subseteq GL(V)$



$d = 4,\ e = 5,\ n = 9$
$\lambda = (5, 3, 1, 1)$
$I = \{2, 3, 6, 9\}$

Figure 1. Partitions and k-subsets of $\{1, \ldots, n\}$

preserving the flag $E_\bullet$ gets identified with lower-triangular matrices. For each partition λ (or subset I), there is a point $p_\lambda = p_I \in Gr(d, V)$, corresponding to the subspace $E_I \subseteq V$ spanned by standard basis vectors $\{e_i \mid i \in I\}$. The cell Ω°_λ can then be described as the B^--orbit of this point, so

$$\Omega^\circ_\lambda = B^- \cdot p_\lambda.$$

A simple exercise in Gaussian elimination shows that points in Ω°_λ are uniquely represented as column spans of matrices in "column echelon form" as

$$\Omega^\circ_\lambda = \Omega^\circ_I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & 1 & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

with the pivots appearing in rows I. This shows that Ω°_λ is an affine space of codimension $|\lambda|$ in $Gr(d, V)$, that is, $\Omega^\circ_\lambda \cong \mathbb{C}^{de-|\lambda|}$. It also shows that the Schubert cells decompose $Gr(d, V)$, that is,

$$Gr(d, V) = \coprod_\lambda \Omega^\circ_\lambda, \quad \text{union over } \lambda \subseteq \overset{e}{\boxed{}}\, d.$$

The *Schubert varieties* are

$$\Omega_\lambda = \Omega_\lambda(E_\bullet) = \overline{\Omega^\circ_\lambda} \subseteq Gr(d, V).$$

They can be described by replacing equalities by inequalities in the dimension conditions:

$$\Omega_\lambda = \left\{F \,\middle|\, \dim(F \cap E_{e+k-\lambda_k}) \geq k \text{ for } k = 1, \ldots, d\right\}.$$

(This is not difficult to see, but it is not obvious either!) It follows that each Schubert variety decomposes into Schubert cells:

$$\Omega_\lambda = \coprod_{\mu \supseteq \lambda} \Omega^\circ_\mu,$$

the union over partitions μ in the $d \times e$ rectangle which contain λ. As with Schubert cells, we will often write $\Omega_I = \Omega_\lambda$, when I is the d-element subset corresponding to the partition λ.

Not all the inequalities are needed to define a Schubert variety:

EXERCISE 1.1. Show that the inequalities in the above definition of Ω_λ are equivalent to

$$\Omega_\lambda = \left\{F \,\middle|\, \dim(F \cap E_{e+k-\lambda_k}) \geq k \text{ for } k \text{ such that } \lambda_k > \lambda_{k+1}\right\}.$$

That is, the conditions coming from corners of the Young diagram suffice to define Ω_λ.

For example, if $\lambda = (p, \ldots, p, 0, \ldots, 0)$, with p occurring q times (so the Young diagram is a $q \times p$ rectangle), then Ω_λ is defined by the single condition $\dim(F \cap E_{e+q-p}) \geq q$.

2. Schubert classes and the Kempf-Laksov formula

The Schubert varieties $\Omega_\lambda(E_\bullet)$ are evidently B^--invariant subvarieties, and the Schubert cell decomposition implies their classes form a basis for cohomology.

PROPOSITION 2.1. *The classes* $[\Omega_\lambda]^{B^-}$ *form a basis for* $H^*_{B^-}Gr(d, V)$ *over* $\Lambda = \Lambda_{B^-}$.

PROOF. Let $X_i \subseteq Gr(d, V)$ be the union of all Ω_λ with $|\lambda| = de - i$, i.e., all Schubert varieties of dimension i. Then $X_i \smallsetminus X_{i-1}$ is the disjoint union of Schubert cells Ω°_λ of dimension i, so the statement follows from the equivariant cell decomposition lemma (Chapter 4, Proposition 7.1). □

In Chapter 4, we saw a presentation

$$H^*_{B^-}Gr(d,V) = \Lambda_{B^-}[c_1,\dots,c_e]/(s_{d+1},\dots,s_n),$$

with $c_k = c_k^{B^-}(\mathbb{Q})$ and $s_k = c_k^{B^-}(V-\mathbb{Q})$. Now that we have a basis of Schubert classes, a question naturally arises: How does one express $[\Omega_\lambda]^{B^-}$ in terms of the presentation, i.e., as polynomials in the Chern classes $c_k^{B^-}(\mathbb{Q})$?

To give such a formula for $[\Omega_\lambda]^{B^-}$, we introduce some notation. Given a partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_d \geq 0)$ and elements $c(1), c(2), \dots, c(d)$, where $c(i)$ is a graded series $1 + c_1(i) + c_2(i) + \cdots$, the *multi-Schur determinant* is

$$\begin{aligned}\Delta_\lambda(\boldsymbol{c}) = \Delta_\lambda(c(1),\dots,c(d)) &:= \det\left(c_{\lambda_i+j-i}(i)\right)_{1\leq i,j\leq d}\\ &= \begin{vmatrix} c_{\lambda_1}(1) & c_{\lambda_1+1}(1) & \cdots & \\ c_{\lambda_2-1}(2) & c_{\lambda_2}(2) & \cdots & \\ \vdots & & \ddots & \\ & & & c_{\lambda_d}(d)\end{vmatrix}.\end{aligned}$$

(To remember this formula, write $c_{\lambda_i}(i)$ down the diagonal, and make the subscripts increase by 1 across rows.) One may truncate zeroes in λ without changing the determinant $\Delta_\lambda(\boldsymbol{c})$. The Schur determinant considered in Chapter 4 is the case where $c = c(1) = \cdots = c(d)$.

Our first formula for equivariant Schubert classes was proved by Kempf and Laksov in the context of degeneracy loci. Special cases were found much earlier by Giambelli.

THEOREM 2.2 (KEMPF-LAKSOV). *For λ in the $d \times e$ rectangle, we have*

$$[\Omega_\lambda]^{B^-} = \Delta_\lambda(c(1),\dots,c(d)),$$

where $c(i) = c^{B^-}(\mathbb{Q} - E_{e+i-\lambda_i})$.

The entries $c(i)$ of the Schur determinant may be replaced by $c'(i) = \cdots = c'(k) = c^{B^-}(\mathbb{Q} - E_{e+k-\lambda_k})$ if $\lambda_i = \cdots = \lambda_k > \lambda_{k+1}$. This follows from an easy property of multi-Schur determinants. Suppose $c(i-1) = c(i)\cdot(1+a)$, for some element a of degree 1. If $\lambda_{i-1} = \lambda_i$, then

$$\Delta_\lambda(\dots, c(i-1), c(i), \dots) = \Delta_\lambda(\dots, c(i), c(i), \dots).$$

(The left-hand side comes by adding a times the ith row to the $(i-1)$st row of the matrix on the right-hand side, and this operation leaves the determinant unchanged.)

One can prove the Kempf-Laksov formula by finding a desingularization of the locus Ω_λ and computing pushforwards; this was Kempf and Laksov's approach. In §4, we will give a different proof in T-equivariant cohomology (which is the same as B^--equivariant cohomology) via combinatorics of symmetric functions. In the next section, we establish some localization formulas which we will need.

EXAMPLE 2.3. The formula says

$$[\Omega_{\square}(E_\bullet)]^{B^-} = c_1^{B^-}(\mathbb{Q} - E_e).$$

This is easy to see directly, since

$$\begin{aligned}\Omega_{\square}(E_\bullet) &= \left\{F \,\middle|\, \dim(E_e \cap F) \geq 1\right\} \\ &= \left\{F \,\middle|\, \mathrm{rk}(E_e \to \mathbb{Q}) < e\right\} \\ &= \left\{F \,\middle|\, \textstyle\bigwedge^e E_e \to \bigwedge^e \mathbb{Q} \text{ is } 0\right\},\end{aligned}$$

so its cohomology class is the first Chern class of the line bundle $\mathrm{Hom}(\bigwedge^e E_e, \bigwedge^e \mathbb{Q})$, which is equal to $c_1^{B^-}(\mathbb{Q} - E_e)$. (We saw this for $Gr(2, \mathbb{C}^4)$ in Example 2.7 of Chapter 5.)

EXAMPLE 2.4. Other instances of the Kempf-Laksov formula are

$$\begin{aligned}[\Omega_{\ydiagram{2,1}}(E_\bullet)]^{B^-} &= \begin{vmatrix} c_2^{B^-}(\mathbb{Q} - E_{e-1}) & c_3^{B^-}(\mathbb{Q} - E_{e-1}) \\ 1 & c_1^{B^-}(\mathbb{Q} - E_{e+1}) \end{vmatrix} \\ &= c_2(1)\, c_1(2) - c_3(1)\end{aligned}$$

and

$$\begin{aligned}[\Omega_{\ydiagram{1,1}}(E_\bullet)]^{B^-} &= \begin{vmatrix} c_1^{B^-}(\mathbb{Q} - E_e) & c_2^{B^-}(\mathbb{Q} - E_e) \\ 1 & c_1^{B^-}(\mathbb{Q} - E_{e+1}) \end{vmatrix} \\ &= c_1(1)\, c_1(2) - c_2(1),\end{aligned}$$

or alternatively,

$$[\Omega_{\square}(E_\bullet)]^{B^-} = \begin{vmatrix} c_1^{B^-}(\mathbb{Q} - E_{e+1}) & c_2^{B^-}(\mathbb{Q} - E_{e+1}) \\ 1 & c_1^{B^-}(\mathbb{Q} - E_{e+1}) \end{vmatrix}$$
$$= c_1(2)\, c_1(2) - c_2(2).$$

3. Tangent spaces and normal spaces

In the rest of this chapter, we study the equivariant geometry of $Gr(d, \mathbb{C}^n)$ with respect to an action of a torus T by characters $\chi_1, \ldots, \chi_n$. Let $e_1, \ldots, e_n$ be a weight basis, so $z \cdot e_i = \chi(z)\, e_i$ for all $z \in T$. As we have seen before (Chapter 5, Example 1.10), $Gr(d, \mathbb{C}^n)$ has an open cover by T-invariant open sets, one for each $I = \{i_1 < \cdots < i_d\} \subset \{1, \ldots, n\}$.

$$\begin{bmatrix} * & * & * & * \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 1 & 0 \\ * & * & * & * \\ * & * & * & * \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In matrix form, these open sets are represented by $n \times d$ matrices so that the submatrix on rows I is the identity matrix, and the remaining entries are free. For example, if $d = 4$, $n = 9$, and $I = \{2, 3, 6, 9\}$, the corresponding open set is shown at left. This illustrates a natural identification with the de-dimensional affine space $E_I^\vee \otimes E_{\{1,\ldots,n\} \smallsetminus I}$, where T acts by the characters $\chi_j - \chi_i$ for $i \in I$ and $j \notin I$. There is an equivariant isomorphism between this open affine and the tangent space $T_{p_I} Gr(d, \mathbb{C}^n)$, identifying p_I with the origin $0 \in T_{p_I} Gr(d, \mathbb{C}^n)$. If all characters $\chi_1, \ldots, \chi_n$ are distinct, then all characters on each tangent space $T_{p_I} Gr(d, \mathbb{C}^n)$ are nonzero, and the fixed locus consists of the finitely many points p_I.

Comparing this with our description of the Schubert cell Ω_I°, we see that the weights of T on $T_{p_I}\Omega_I$ are $\{\chi_j - \chi_i \mid i \in I, j \notin I, i < j\}$. It follows that the weights of T on the normal space N_I to Ω_I at p_I are $\{\chi_j - \chi_i \mid i \in I, i \notin I, i > j\}$. From the self-intersection formula, this means

$$(*) \qquad [\Omega_I]^T|_{p_I} = c_{top}^T(N_I) = \prod_{\substack{i \in I, j \notin I \\ i > j}} (\chi_j - \chi_i).$$

From the cell decomposition, we know

$$\begin{aligned} \Omega_\mu \subseteq \Omega_\lambda \quad &\text{iff} \quad \mu \supseteq \lambda \quad \text{(as diagrams)} \\ &\text{iff} \quad J \geq I \quad \text{(i.e., } j_k \geq i_k \text{ for all } k), \end{aligned}$$

where I is the subset corresponding to the partition λ, and J is the one corresponding to μ. This means that $p_\mu = p_J$ lies in $\Omega_\lambda = \Omega_I$ iff $\mu \supseteq \lambda$ iff $J \geq I$, so

$$(**) \qquad [\Omega_I]^T|_{p_J} = 0 \quad \text{unless} \quad J \geq I.$$

Now let us assume the characters χ_i are all distinct, so the fixed points are isolated. It turns out that the two conditions $(*)$ and $(**)$ uniquely determine the class $[\Omega_I]^T$.

LEMMA 3.1. *If a homogeneous element* $\alpha \in H_T^*Gr(d, \mathbb{C}^n)$ *satisfies* $(*)$ *and* $(**)$*, then* $\alpha = [\Omega_I]^T$.

PROOF. We have seen that $[\Omega_I]^T$ satisfies the conditions, so it is enough to show that if two classes α and α' satisfy $(*)$ and $(**)$, then they must be the same. Equivalently, we will show that $\beta = \alpha - \alpha'$ is zero.

We know $\beta|_{p_J} = 0$ unless $J \geq I$ by $(**)$. Let $K \geq I$ be a minimal element such that $\beta|_{p_K} \neq 0$. It must be that $K > I$, since $\alpha|_{p_I} = \alpha'|_{p_I}$ by $(*)$. The "GKM" divisibility conditions (Chapter 7, Corollary 4.3) force $\beta|_{p_K}$ to be a multiple of

$$\prod_{\substack{i \in K, j \notin K \\ i > j}} (\chi_j - \chi_i),$$

which is equal to $[\Omega_K]^T|_{p_K}$. So $\beta|_{p_K}$ has degree at least $|\nu|$, where ν is the partition corresponding to K. Since $K > I$, we have $|\nu| > |\lambda|$, contradicting the homogeneity of the classes α and α'. □

The conditions $(*)$ and $(**)$ can be regarded as an interpolation problem; the lemma says this problem has a unique solution. We will next tie this to symmetric functions, making it an explicit problem of polynomial interpolation.

4. Double Schur polynomials

We consider functions of two sets of variables, $x = (x_1, x_2, \dots, x_d)$ and $y = (y_1, y_2, \dots)$. (One can extend the y variables to be doubly infinite, allowing non-positive indices, but in practice only finitely many appear.) We define the "double monomial"

$$(x_i|y)^p = (x_i - y_1)(x_i - y_2)\cdots(x_i - y_p).$$

There are several equivalent definitions of *double Schur functions* $s_\lambda(x|y)$, generalizing corresponding definitions of the single Schur polynomials, which are recovered by setting $y = 0$.

Bialternants. Generalizing Cauchy's functions, we set

$$s_\lambda(x|y) = \frac{\left|(x_i|y)^{\lambda_j+d-j}\right|_{1\le i,j\le d}}{\left|(x_i|y)^{d-j}\right|_{1\le i,j\le d}},$$

where both determinants are $d \times d$. The numerator is an alternating function of x, and a pleasant exercise shows that the denominator is the Vandermonde

$$\left|(x_i|y)^{d-j}\right| = \left|x_i^{d-j}\right| = \prod_{i<j}(x_i - x_j),$$

so the ratio is a polynomial.

Tableaux. There is a formula in terms of semistandard Young tableaux:

$$s_\lambda(x|y) = \sum_{\mathcal{T}\in SSYT(\lambda)} \prod_{(i,j)\in\lambda} (x_{\mathcal{T}(i,j)} - y_{\mathcal{T}(i,j)+j-i}),$$

the sum over SSYT tableaux $\mathcal{T}$ of shape λ with entries in $\{1, \dots, d\}$.

For example, if $d = 3$, there are 8 semistandard Young tableaux of shape $\lambda = \begin{array}{|c|c|}\hline & \\ \hline & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}$,

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & 1 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2 & 2 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}\quad \begin{array}{|c|c|}\hline 2 & 3 \\ \hline 3 & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array},$$

so the double Schur function is

$$\begin{aligned} s_\lambda(x|y) = &(x_1 - y_1)(x_1 - y_2)(x_2 - y_1) + (x_1 - y_1)(x_1 - y_2)(x_3 - y_2) \\ &+ (x_1 - y_1)(x_2 - y_3)(x_2 - y_1) + (x_1 - y_1)(x_2 - y_3)(x_3 - y_2) \\ &+ (x_1 - y_1)(x_3 - y_4)(x_2 - y_1) + (x_1 - y_1)(x_3 - y_4)(x_3 - y_2) \\ &+ (x_2 - y_2)(x_2 - y_3)(x_3 - y_2) + (x_2 - y_2)(x_3 - y_4)(x_3 - y_2). \end{aligned}$$

Jacobi-Trudi. The Jacobi-Trudi determinantal formula generalizes to

$$s_\lambda(x|y) = \left| h_{\lambda_i+j-i}(x|\tau^{1-j}y) \right|_{1\le i,j\le d},$$

where

$$h_p(x|y) = s_{(p)}(x|y) = \sum_{1\le i_1\le\cdots\le i_p\le d} (x_{i_1} - y_{i_1})(x_{i_2} - y_{i_2+1})\cdots(x_{i_p} - y_{i_p+p-1}),$$

and τ is the shift operator defined by $(\tau^j y)_i = y_{i+j}$.

The main fact we need is a *vanishing theorem* for double Schur functions. Let $y^\lambda_k = y_{\lambda_{d+1-k}+k}$; or in terms of the corresponding subset I, $y^I_k = y_{i_k}$.

LEMMA 4.1. *We have*

$$s_\lambda(y^\lambda|y) = \prod_{\substack{i\in I, j\notin I \\ i>j}} (y_j - y_i),$$

where $I \subseteq \{1, \dots, n\}$ is the subset corresponding to λ, and

$$s_\lambda(y^\mu|y) = 0 \quad \text{if } \mu \not\supseteq \lambda.$$

EXERCISE 4.2. Prove Lemma 4.1.[1]

After appropriately identifying the variables, the lemma says that double Schur functions satisfy conditions $(*)$ and $(**)$ from §3—that is, the same interpolation problem solved uniquely by $[\Omega_\lambda]^T$! To make this precise, let $x_1, \dots, x_d$ be (equivariant) Chern roots of the dual tautological bundle $\mathbb{S}^\vee$ on $Gr(d, \mathbb{C}^n)$, so

$$c^T_k(\mathbb{S}^\vee) = e_k(x_1, \dots, x_d)$$

Then, specializing the y variables by $y_i = -\chi_i$,

$$c_k^T(\mathbb{S}^\vee)|_{p_\lambda} = c_k^T(E_I^\vee) = e_k(y_1^\lambda, \dots, y_d^\lambda).$$

In other words, there is a commuting diagram

$$\begin{tikzcd}
c_k^T(\mathbb{S}^\vee) & H_T^* Gr(d, \mathbb{C}^n) \arrow[dr, "\iota^*_{p_\lambda}"] & \\
c_k \arrow[u, mapsto] \arrow[d, mapsto] & \Lambda[c_1, \dots, c_d] \arrow[u, two heads] \arrow[d, hook] \arrow[r] & \Lambda \\
e_k(x) & \Lambda[x_1, \dots, x_d]. \arrow[ur, "x_k \mapsto y_k^\lambda"'] &
\end{tikzcd}$$

The polynomials $s_\lambda(x|y)$ are symmetric in the x variables, so they lie in $\Lambda[c_1, \dots, c_d]$. They satisfy (*) and (**) by Lemma 4.1, so it follows from Lemma 3.1 that $s_\lambda(x|y) \mapsto [\Omega_\lambda]^T$.

Invoking the Jacobi-Trudi formula, we obtain:

Corollary 4.3. *Evaluating $x_1, \dots, x_d$ as equivariant Chern roots of $\mathbb{S}^*$, and $y_i = -\chi_i$, we have*

$$\begin{aligned}
[\Omega_\lambda]^T &= s_\lambda(x|y) \\
&= \left| h_{\lambda_i + j - i}(x|\tau^{1-j} y) \right|_{1 \le i,j \le d}.
\end{aligned}$$

This proves the Kempf-Laksov formula (Theorem 2.2), once one knows the entries of the matrices are identical.

Exercise 4.4. With the specializations as in Corollary 4.3, show that $c^T_{\lambda_i+j-i}(\mathbb{Q} - E_{e+i-\lambda_i}) = h_{\lambda_i+j-i}(x|\tau^{1-j}y)$.[2]

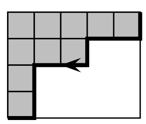
5. Poincaré duality

We have seen one basis for $H_T^* Gr(d, \mathbb{C}^n)$, the Schubert classes $\sigma_\lambda = [\Omega_\lambda]^T$. Our next goal is to describe the Poincaré dual basis. Let $\widetilde{E}_\bullet$ be the *opposite flag* to $E_\bullet$, so if E_k is spanned by $e_{n+1-k}, \dots, e_n$, then $\widetilde{E}_k$ is spanned by $e_1, \dots, e_k$. The flag $\widetilde{E}_\bullet$ is fixed by the Borel group B, which in this basis is the set of upper-triangular matrices in GL_n.

The *opposite Schubert cells* and *varieties* are defined as before, but with respect to the flag $\widetilde{E}_\bullet$:

$$\widetilde{\Omega}^\circ_\lambda := \Omega^\circ_\lambda(\widetilde{E}_\bullet) \qquad \text{and} \qquad \widetilde{\Omega}_\lambda := \Omega_\lambda(\widetilde{E}_\bullet).$$

These are B-invariant, so also T-invariant. To identify the T-fixed points contained in $\widetilde{\Omega}_\lambda$, it will help to introduce some more notation. Let $\lambda^\vee$ be the complement to λ in the $d \times e$ rectangle, also called the *dual partition*. In formulas, this is $\lambda^\vee_k = e - \lambda_{d+1-k}$. Let $I^\vee \subseteq \{1, \ldots, n\}$ be the corresponding d-element subset, so $I^\vee = \{i^\vee_1 < \cdots < i^\vee_d\}$, with $i^\vee_k = n + 1 - i_{d+1-k}$. This can be seen by reading the border of $\lambda \subseteq \overset{e}{\boxed{}}\, d$ in the opposite direction, from NE to SW, as illustrated below.



$$\lambda = (5,3,1,1) \qquad \lambda^\vee = (4,4,2,0)$$
$$I = \{2,3,6,9\} \qquad I^\vee = \{1,4,7,8\}$$

EXERCISE 5.1. Verify that $p_I = p_\lambda$ is the unique T-fixed point in $\widetilde{\Omega}^\circ_{I^\vee} = \widetilde{\Omega}^\circ_{\lambda^\vee}$, so $\widetilde{\Omega}^\circ_{\lambda^\vee} = B \cdot p_\lambda$.

For example, with $d = 4$, $n = 9$, and $I = \{2,3,6,9\}$, we have

$$\Omega^\circ_I = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ * & * & 0 & 0 \\ * & * & 0 & 0 \\ 0 & 0 & 1 & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \widetilde{\Omega}^\circ_{I^\vee} = \begin{bmatrix} * & * & * & * \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

both inside the T-invariant affine neighborhood of p_I. The pivot 1's in $\widetilde{\Omega}^\circ_{I^\vee}$ are in the rows indicated by $I^\vee$, but read from bottom to top. So

$$\operatorname{codim} \Omega_\lambda = \dim \widetilde{\Omega}_{\lambda^\vee} = |\lambda|,$$

while

$$\dim \Omega_\lambda = \operatorname{codim} \widetilde{\Omega}_{\lambda^\vee} = |\lambda^\vee|.$$

The correspondence $\lambda \leftrightarrow \lambda^\vee$ reverses inclusions. Since the opposite Schubert variety decomposes as $\widetilde{\Omega}_\lambda = \coprod_{\mu\supseteq\lambda} \widetilde{\Omega}^\circ_\mu$ and $p_{\mu^\vee} \in \widetilde{\Omega}^\circ_\mu$, we see

$$p_\mu \in \widetilde{\Omega}_\lambda \quad \text{iff} \quad \mu^\vee \supseteq \lambda \quad \text{iff} \quad \mu \subseteq \lambda^\vee.$$

Proposition 5.2. *Let $\sigma_\lambda = [\Omega_\lambda]^T$ and $\widetilde{\sigma}_\lambda = [\widetilde{\Omega}_\lambda]^T$. Then $\{\widetilde{\sigma}_{\lambda^\vee}\}$ is the Poincaré dual basis to $\{\sigma_\lambda\}$.*

Proof. We must show

$$\rho_*(\sigma_\lambda \cdot \widetilde{\sigma}_\mu) = \begin{cases} 1 & \text{if } \mu = \lambda^\vee; \\ 0 & \text{otherwise.} \end{cases}$$

When $\mu = \lambda^\vee$, the varieties Ω_λ and $\widetilde{\Omega}_{\lambda^\vee}$ meet transversally in the single point p_λ. In general, the above analysis of fixed points shows that

$$(\Omega_\lambda \cap \widetilde{\Omega}_\mu)^T = \left\{ p_\nu \mid \mu^\vee \supseteq \nu \supseteq \lambda \right\}.$$

The fact that $\Omega_\lambda \cap \widetilde{\Omega}_{\lambda^\vee}$ is transverse is apparent from a computation of tangent spaces—say, by using matrix descriptions of Ω°_I and $\widetilde{\Omega}^\circ_{I^\vee}$. One sees as before that $T_{p_\lambda}\widetilde{\Omega}_{\lambda^\vee}$ has weights $\{\chi_j - \chi_i \mid i \in I, j \notin I, i > j\}$, so that

$$T_{p_\lambda} Gr(d, \mathbb{C}^n) = T_{p_\lambda}\Omega_\lambda \oplus T_{p_\lambda}\widetilde{\Omega}_{\lambda^\vee}.$$

This shows $\rho_*(\sigma_\lambda \cdot \widetilde{\sigma}_{\lambda^\vee}) = 1$.

If $\mu \neq \lambda^\vee$, there are two possibilities to consider. First, suppose $\mu \not\subseteq \lambda^\vee$, so $\mu^\vee \not\supseteq \lambda$. Then $(\Omega_\lambda \cap \widetilde{\Omega}_\mu)^T = \emptyset$, so the intersection is empty. (Any nonempty projective variety has a T-fixed point, by Borel's fixed point theorem.) So $\rho_*(\sigma_\lambda \cdot \widetilde{\sigma}_\mu) = 0$ in this case.

On the other hand, suppose $\mu \subsetneq \lambda^\vee$, so $\mu^\vee \supsetneq \lambda$. Then $|\mu^\vee| - |\lambda| > 0$. But this means $de - |\mu| - |\lambda| > 0$, that is, $|\mu| + |\lambda| - de < 0$. Since $\rho_*(\sigma_\lambda \cdot \widetilde{\sigma}_\mu) \in \Lambda_T^{2(|\mu|+|\lambda|-de)} = 0$, we are done. □

We obtain a formula for the class of an opposite Schubert variety by replacing $E_\bullet$ by $\widetilde{E}_\bullet$ in the Kempf-Laksov formula.

Theorem 5.3. *In $H_T^*Gr(d,\mathbb{C}^n)$, we have*

$$\begin{aligned}[\widetilde{\Omega}_\lambda]^T &= \Delta_\lambda(\widetilde{c}(1),\dots,\widetilde{c}(s)) \\ &= s_\lambda(x|\widetilde{y}),\end{aligned}$$

where $\widetilde{c}(k) = c^T(\mathbb{Q} - \widetilde{E}_{e+k-\lambda_k})$, $x_1,\dots,x_d$ are equivariant Chern roots of $\mathbb{S}^\vee$, and $\widetilde{y}_k = y_{n+1-k} = -\chi_{n+1-k}$.

6. Multiplication

A major goal of equivariant Schubert calculus is to describe the coefficients $c_{\lambda\mu}^\nu$ appearing in the expansion

$$\sigma_\lambda \cdot \sigma_\mu = \sum_\nu c_{\lambda\mu}^\nu \, \sigma_\nu,$$

in $H_T^*Gr(d,\mathbb{C}^n)$. The same problem can be posed for other flag varieties, but Grassmannians are one of the very few cases where a complete and satisfying answer is known.

We will prove some basic facts about these coefficients, assuming throughout that the characters $\chi_1,\dots,\chi_n$ are distinct. (The general case can be obtained from this one, by specializing the χ_i's.) Evidently, $c_{\lambda\mu}^\nu$ is a homogeneous polynomial of degree $|\lambda|+|\mu|-|\nu|$.

Lemma 6.1. *The coefficients $c_{\lambda\mu}^\nu$ satisfy the following properties:*

(i) $c_{\lambda\mu}^\nu = 0$ *unless $\lambda \subseteq \nu$ and $\mu \subseteq \nu$.*

(ii) $c_{\lambda\mu}^\mu = \sigma_\lambda|_\mu = [\Omega_\lambda]^T|_{p_\mu}$.

(iii) $c_{\lambda\lambda}^\lambda = \displaystyle\prod_{\substack{i\in I, j\notin I \\ i>j}} (\chi_j - \chi_i)$.

Proof. For (i), the cell decomposition lemma shows that restrictions of classes σ_α for $\alpha \not\supseteq \lambda$ form a basis for $H^*(X \smallsetminus \Omega_\lambda)$, because these are classes of the Schubert varieties Ω_α not contained in Ω_λ. The class σ_λ maps to 0 under $H_T^*X \to H_T^*(X \smallsetminus \Omega_\lambda)$—as one can see by using the long exact sequence of Borel-Moore homology—so $\sigma_\lambda \cdot \sigma_\mu$ also maps to 0, for any μ. It follows that $\sigma_\lambda \cdot \sigma_\mu = \sum_\nu c_{\lambda\mu}^\nu \sigma_\nu$ involves only those ν such that $\nu \supseteq \lambda$. By symmetry, one concludes that $\nu \supseteq \mu$, as well.

For (ii), we restrict the equation defining $c_{\lambda\mu}^{\nu}$ to p_μ, obtaining

$$\sigma_\lambda|_\mu \cdot \sigma_\mu|_\mu = \sum_\nu c_{\lambda\mu}^{\nu}\, \sigma_\nu|_\mu .$$

We know $\sigma_\nu|_\mu = 0$ unless $p_\mu \in \Omega_\nu$, that is, $\mu \supseteq \nu$; but by (i), we also know $c_{\lambda\mu}^{\nu} = 0$ unless $\mu \subseteq \nu$. The only term surviving on the right-hand side is $\nu = \mu$, so we find $\sigma_\lambda|_\mu \cdot \sigma_\mu|_\mu = c_{\lambda\mu}^{\mu}\, \sigma_\mu|_\mu$. We found the formula for $\sigma_\mu|_\mu$ in §3, and this is not a zerodivisor since all χ_i are distinct. Canceling these factors gives the claimed formula for $c_{\lambda\mu}^{\mu}$.

Formula (iii) follows from (ii), using the formula for $\sigma_\lambda|_\lambda$. $\square$

The *Chevalley-Pieri formula* gives a rule for multiplication by a divisor class $\sigma_\square$. The classical (non-equivariant) version says that in H^*X,

$$\sigma_\square \cdot \sigma_\lambda = \sum_{\lambda^+} \sigma_{\lambda^+},$$

the sum over all partitions λ^+ obtained from λ by adding one box. For example, in $H^*Gr(3, \mathbb{C}^7)$ we have

$$\sigma_\square \cdot \sigma_{(3,2)} = \sigma_{(4,2)} + \sigma_{(3,3)} + \sigma_{(3,2,1)} .$$

We will prove a very general version of this formula in Chapter 19. The reader may enjoy proving the corresponding formula for multiplying a Schur polynomial by $h_1 = x_1 + x_2 + \cdots + x_d$. (As usual, references are in the Notes.)

To state the equivariant version precisely, we need another formula:

$$\begin{aligned} \sigma_\square|_\lambda &= \sum_{j \notin I} \chi_j - \sum_{i=d+1}^{n} \chi_i \\ &= \sum_{j=1}^{d} \chi_j - \sum_{i \in I} \chi_i . \end{aligned}$$

This follows from $\sigma_{\square} = c_1^T(\mathbb{Q} - E_e)$, using $c_1^T(\mathbb{Q})|_{p_\lambda} = \sum_{j \notin I} \chi_j$. The second line makes it clear that the formula is independent of n. Note that $\sigma_{\square}|_\lambda \neq \sigma_{\square}|_\mu$ if $\lambda \neq \mu$.

THEOREM 6.2 (EQUIVARIANT CHEVALLEY-PIERI). *In $H_T^* X$, we have*

$$\sigma_{\square} \cdot \sigma_\lambda = \sum_{\lambda^+} \sigma_{\lambda^+} + \left(\sum_{j=1}^{d} \chi_j - \sum_{i \in I} \chi_i \right) \sigma_\lambda,$$

the sum over λ^+ obtained from λ by adding one box.

PROOF. The sum is from the nonequivariant case; the equivariant coefficients must agree by degree. The other term has coefficient $c_{\square\lambda}^{\lambda} = \sigma_{\square}|_\lambda$ by Lemma 6.1(iii). No other terms appear, since $c_{\square\lambda}^{\nu}$ is nonzero only for $|\nu| \leq |\lambda| + 1$ and $\nu \supseteq \lambda$, by Lemma 6.1(i). □

Remarkably, the equivariant Chevalley rule determines all structure constants $c_{\lambda\mu}^{\nu}$ for $H_T^* Gr(d, \mathbb{C}^n)$, and hence also for $H^* Gr(d, \mathbb{C}^n)$. This is far from true of the non-equivariant rule: $H^* Gr(d, \mathbb{C}^n)$ is not generated by the divisor class. A general reason for this phenomenon was given in Chapter 7, §1. Here we will see an algorithmic proof.

First, we need some more formulas.

LEMMA 6.3. *We have*

(i) $$(\sigma_{\square}|_\lambda - \sigma_{\square}|_\mu)\, c_{\lambda\mu}^{\lambda} = \sum_{\mu^+} c_{\lambda\mu^+}^{\lambda},$$

the sum over μ^+ obtained by adding one box to μ, and

(ii) $$(\sigma_{\square}|_\nu - \sigma_{\square}|_\lambda)\, c_{\lambda\mu}^{\nu} = \sum_{\lambda^+} c_{\lambda^+\mu}^{\nu} - \sum_{\nu^-} c_{\lambda\mu}^{\nu^-},$$

the sums over λ^+ obtained by adding one box to λ, and ν^- obtained by removing one box from ν.

PROOF. For (i), using the formula $c_{\mu\lambda}^{\lambda} = \sigma_\mu|_\lambda$ together with commutativity, the left-hand side is $(\sigma_{\square}|_\lambda - \sigma_{\square}|_\mu)\sigma_\mu|_\lambda$, while the right-hand side is $\sum \sigma_{\mu^+}|_\lambda$. So (i) results from restricting the equivariant Chevalley-Pieri formula to p_λ.

For (ii), we will use associativity. By Chevalley-Pieri,

$$\begin{aligned}\sigma_{\square}\cdot(\sigma_\lambda\cdot\sigma_\mu) &= \sum_\nu c^\nu_{\lambda\mu}\sigma_{\square}\cdot\sigma_\nu \\ &= \sum_{\nu^+} c^\nu_{\lambda\mu}\sigma_{\nu^+} + \sum_\nu c^\nu_{\lambda\mu}(\sigma_{\square}|_\nu)\sigma_\nu,\end{aligned}$$

and

$$\begin{aligned}(\sigma_{\square}\cdot\sigma_\lambda)\cdot\sigma_\mu &= \sum_{\lambda^+}\sigma_{\lambda^+}\cdot\sigma_\mu + (\sigma_{\square}|_\lambda)\sigma_\lambda\cdot\sigma_\mu \\ &= \sum_{\lambda^+}\sum_\nu c^\nu_{\lambda^+\mu}\sigma_\nu + (\sigma_{\square}|_\lambda)\sum_\nu c^\nu_{\lambda\mu}\sigma_\nu.\end{aligned}$$

One obtains (ii) by equating coefficients of σ_ν. □

THEOREM 6.4 (MOLEV-SAGAN). *The polynomials $c^\nu_{\lambda\mu}$ in $\Lambda^{2(|\lambda|+|\mu|-|\nu|)}$ satisfy and are determined by the following properties:*

(i) $$c^\lambda_{\lambda\lambda} = \sigma_\lambda|_\lambda = \prod_{\substack{i\in I, j\notin I\\ i>j}}(\chi_j-\chi_i),$$

(ii) $$(\sigma_{\square}|_\lambda - \sigma_{\square}|_\mu)\, c^\lambda_{\lambda\mu} = \sum_{\mu^+} c^\lambda_{\lambda\mu^+},$$

and

(iii) $$(\sigma_{\square}|_\nu - \sigma_{\square}|_\lambda)\, c^\nu_{\lambda\mu} = \sum_{\lambda^+} c^\nu_{\lambda^+\mu} - \sum_{\nu^-} c^{\nu^-}_{\lambda\mu}.$$

PROOF. We have seen that (i)–(iii) hold. To prove that they uniquely characterize the coefficients $c^\nu_{\lambda\mu}$, we proceed by induction. Suppose $d^\nu_{\lambda\mu}$ are any polynomials satisfying (i)–(iii). We know $d^\lambda_{\lambda\lambda} = c^\lambda_{\lambda\lambda}$, since this is the explicit formula (i).

Next, $d^\lambda_{\lambda\mu} = \sigma_\mu|_\lambda = c^\lambda_{\lambda\mu}$, by induction on $|\lambda|-|\mu|$: the base case is where $\lambda=\mu$, and is done by (i); for $\lambda \supsetneq \mu$, use formula (ii) and induction. (On the left-hand side of (ii), the factor $(\sigma_{\square}|_\lambda - \sigma_{\square}|_\mu)$ is nonzero for $\lambda\neq\mu$. Terms on the right-hand side of (ii) have $|\lambda|-|\mu^+| < |\lambda|-|\mu|$.)

Finally, use (iii) and induction on $|\nu| - |\lambda|$ to get $d^{\nu}_{\lambda\mu} = c^{\nu}_{\lambda\mu}$. All terms on the right-hand side of (iii) have $|\nu| - |\lambda^+| < |\nu| - |\lambda|$ and $|\nu^-| - |\lambda| < |\nu| - |\lambda|$. This reduces to the base case $\lambda = \nu$, which was handled previously. □

Remark. By setting $\nu = \mu$ in (iii), one obtains

$$(\sigma_{\square}|_{\mu} - \sigma_{\square}|_{\lambda})\, c^{\mu}_{\lambda\mu} = \sum_{\lambda^+} c^{\mu}_{\lambda^+\mu}, \tag{ii'}$$

since the coefficients $c^{\mu^-}_{\lambda\mu}$ vanish. Using commutativity ($c^{\nu}_{\lambda\mu} = c^{\nu}_{\mu\lambda}$) and interchanging λ and μ, one recovers (ii) from (ii′). The conditions (i), (ii′), and (iii) also characterize $c^{\nu}_{\lambda\mu}$.

7. Grassmann duality

In Chapter 4 we noted the canonical isomorphisms

$$Gr(d, V) = Gr(V, e) = Gr(V^{\vee}, d) = Gr(e, V^{\vee}),$$

where $d + e = n = \dim V$, by identifications

$$[F \subseteq V] \leftrightarrow [V \twoheadrightarrow V/F] \leftrightarrow [V^{\vee} \twoheadrightarrow F^{\vee}] \leftrightarrow [(V/F)^{\vee} \subseteq V^{\vee}].$$

These are equivariant for any group G acting linearly on V, and by the dual representation on $V^{\vee}$.

To see this in matrices, we fix a basis, so $V \cong \mathbb{C}^n \cong V^{\vee}$. A point of $Gr(d, V)$ is the image of an embedding $[\mathbb{C}^d \hookrightarrow \mathbb{C}^n]$, so it is represented as the column span of a full-rank matrix A of size $n \times d$. A point of $Gr(V, e)$ is an isomorphism class of quotients $[\mathbb{C}^n \twoheadrightarrow \mathbb{C}^e]$, represented by a full-rank matrix B of size $e \times n$. Dually, a point of $Gr(V^{\vee}, d)$ is a quotient represented by the transposed matrix $A^{\dagger}$, and a point of $Gr(e, V^{\vee})$ is the column span of $B^{\dagger}$ (that is, the row span of B).

With this notation, the Grassmann duality isomorphism is

$$\begin{aligned} \gamma\colon Gr(d, \mathbb{C}^n) &\to Gr(e, \mathbb{C}^n), \\ F &\mapsto \ker(A^{\dagger}) = \operatorname{im}(B^{\dagger}). \end{aligned}$$

The group GL_n acts on $Gr(d, \mathbb{C}^n)$ by $F \mapsto g \cdot F$, which sends $A \mapsto g \cdot A$ and $B \mapsto B \cdot g^{-1}$. Grassmann duality is therefore equivariant with respect to the group automorphism $\varphi\colon GL_n \to GL_n$, $\varphi(g) = (g^\dagger)^{-1}$. If $T \to GL_n$ is a homomorphism given by characters $\chi_1, \ldots, \chi_n$, then the algebra homomorphism $H_T^* Gr(e, \mathbb{C}^n) \to H_T^* Gr(d, \mathbb{C}^n)$ intertwines the automorphism of $\Lambda_T = \mathrm{Sym}^* M$ induced by $\chi \mapsto -\chi$ for all $\chi \in M$.

Exercise 7.1. For λ in the $d \times e$ rectangle, show that $\gamma(\Omega_\lambda) = \widetilde{\Omega}_{\lambda'}$, where λ' is the conjugate partition (i.e., its diagram is the transpose of that of λ).[3]

Duality exchanges the exact sequences

$$0 \to \mathbb{S} \to \mathbb{C}^n_{Gr} \to \mathbb{Q} \to 0$$

and

$$0 \to \mathbb{Q}^\vee \to (\mathbb{C}^n_{Gr})^\vee \to \mathbb{S}^\vee \to 0.$$

Together with Exercise 7.1, this implies a dual Kempf-Laksov formula for Schubert classes:

Corollary 7.2. *Let $x_1, \ldots, x_d$ be equivariant Chern roots of $\mathbb{S}^\vee$, and $\widetilde{x}_1, \ldots, \widetilde{x}_e$ equivariant Chern roots of $\mathbb{Q}$. For a partition λ in the $d \times e$ rectangle, we have*

$$\begin{aligned}\sigma_\lambda &= s_\lambda(x|y) \\ &= s_{\lambda'}(\widetilde{x}|-\widetilde{y}),\end{aligned}$$

and

$$\begin{aligned}\widetilde{\sigma}_\lambda &= s_\lambda(x|\widetilde{y}) \\ &= s_{\lambda'}(\widetilde{x}|-y),\end{aligned}$$

where T acts on $Gr(d, \mathbb{C}^n)$ by characters $\chi_i = -y_i = -\widetilde{y}_{n+1-i}$.

Expressed as multi-Schur determinants in Chern classes of $\mathbb{S}^\vee$ and $\mathbb{Q}$, these formulas translate into

$$\sigma_\lambda = \Delta_\lambda(\boldsymbol{c}) = \Delta_{\lambda'}(\widetilde{\boldsymbol{c}}')$$

and

$$\widetilde{\sigma}_\lambda = \Delta_\lambda(\widetilde{\boldsymbol{c}}) = \Delta_{\lambda'}(\boldsymbol{c}'),$$

where

$$\begin{aligned} c(i) &= c^T(\mathbb{Q} - E_{e+i-\lambda_i}), \\ c'(i) &= c^T(\mathbb{S}^\vee - E^\vee_{d+i-\lambda'_i}), \\ \widetilde{c}(i) &= c^T(\mathbb{Q} - \widetilde{E}_{e+i-\lambda_i}), \quad \text{and} \\ \widetilde{c}'(i) &= c^T(\mathbb{S}^\vee - \widetilde{E}^\vee_{d+i-\lambda'_i}). \end{aligned}$$

(Recall that $\Delta_\lambda(c(E)) = s_{\lambda'}(x_1, \ldots, x_n)$ when $x_1, \ldots, x_n$ are Chern roots of E.)

This lets us prove a refinement of the Cauchy identity used in Chapter 4, §6.

Corollary 7.3. *Let* $\delta\colon Gr(d, \mathbb{C}^n) \to Gr(d, \mathbb{C}^n) \times Gr(d, \mathbb{C}^n)$ *be the diagonal embedding. Then*

$$\delta_*(1) = \sum_\lambda \Delta_\lambda(\boldsymbol{c}) \times \Delta_{(\lambda^\vee)'}(\boldsymbol{c}').$$

(The partition $(\lambda^\vee)'$ is what we called the *complement* to λ in Chapter 4.)

Proof. Use the Kempf-Laksov formulas for Schubert classes, together with the decomposition

$$\delta_*(1) = \sum_\lambda \sigma_\lambda \times \widetilde{\sigma}_{\lambda^\vee}$$

of the diagonal into Poincaré dual classes. □

The same statement holds, without change, for equivariant Grassmann bundles $\mathbf{Gr}(d, V) \to Y$, so long as the vector bundle $V \to Y$ admits opposite flags $E_\bullet$ and $\widetilde{E}_\bullet$.

Writing $(\mathbb{S}^\vee)^{(1)}$ and $\mathbb{Q}^{(2)}$ for the tautological bundles from the first and second factors of $Gr(d, \mathbb{C}^n) \times Gr(d, \mathbb{C}^n)$, and $x_1, \ldots, x_d$ and $\widetilde{x}_1, \ldots, \widetilde{x}_e$ for their respective Chern roots, the Corollary expresses an equality

$$\prod_{i=1}^d \prod_{j=1}^e (x_i + \widetilde{x}_j) = \sum_\lambda s_\lambda(x|y) \cdot s_{(\lambda^\vee)'}(\widetilde{x}|-y)$$

in $H_T^*(Gr(d, \mathbb{C}^n) \times Gr(d, \mathbb{C}^n))$, or in $H_T^*(\mathbf{Gr}(d, V) \times_Y \mathbf{Gr}(d, V))$.

Exercise 7.4. Let $c_{\lambda\mu}^{\nu} \in \Lambda$ be the coefficient defined by

$$\sigma_\lambda \cdot \sigma_\mu = \sum_{\nu} c_{\lambda\mu}^{\nu}\, \sigma_\nu \qquad\qquad \text{in } H_T^* Gr(d, \mathbb{C}^n),$$

as before, and consider similar coefficients defined by

$$\widetilde{\sigma}_\lambda \cdot \widetilde{\sigma}_\mu = \sum_{\nu} \widetilde{c}_{\lambda\mu}^{\nu}\, \widetilde{\sigma}_\nu \qquad\qquad \text{in } H_T^* Gr(d, \mathbb{C}^n),$$

and

$$\sigma_{\lambda'} \cdot \sigma_{\mu'} = \sum_{\nu'} c_{\lambda'\mu'}^{\nu'}\, \sigma_{\nu'} \qquad\qquad \text{in } H_T^* Gr(e, \mathbb{C}^n).$$

Show that $c_{\lambda\mu}^{\nu}$ maps to $\widetilde{c}_{\lambda\mu}^{\nu}$ under the substitution $\chi_i \mapsto \chi_{n+1-i}$, and $c_{\lambda\mu}^{\nu}$ maps to $c_{\lambda'\mu'}^{\nu'}$ under $\chi_i \mapsto -\chi_{n+1-i}$.

For example, using $\sigma_\lambda = s_\lambda(x|y)$ and $y_i = -\chi_i$, one computes the product

$$\begin{aligned}\sigma_{(2)} \cdot \sigma_{(3,1)} = \sigma_{(4,2)} + \sigma_{(3,3)} &+ (\chi_1 + \chi_3 - \chi_5 - \chi_6)\, \sigma_{(4,1)} \\ &+ (\chi_1 - \chi_5)\, \sigma_{(3,2)} + (\chi_1 - \chi_5)(\chi_3 - \chi_5)\, \sigma_{(3,1)}\end{aligned}$$

in $H_T^* Gr(2, \mathbb{C}^6)$. Compare this with

$$\begin{aligned}\widetilde{\sigma}_{(2)} \cdot \widetilde{\sigma}_{(3,1)} = \widetilde{\sigma}_{(4,2)} + \widetilde{\sigma}_{(3,3)} &+ (\chi_6 + \chi_4 - \chi_2 - \chi_1)\, \widetilde{\sigma}_{(4,1)} \\ &+ (\chi_6 - \chi_2)\, \widetilde{\sigma}_{(3,2)} + (\chi_6 - \chi_2)(\chi_4 - \chi_2)\, \widetilde{\sigma}_{(3,1)}\end{aligned}$$

and

$$\begin{aligned}\sigma_{(1,1)} \cdot \sigma_{(2,1,1)} = \sigma_{(2,2,1,1)} + \sigma_{(2,2,2)} &+ (\chi_1 + \chi_2 - \chi_4 - \chi_6)\, \sigma_{(2,1,1,1)} \\ &+ (\chi_2 - \chi_6)\, \sigma_{(2,2,1)} + (\chi_2 - \chi_4)(\chi_2 - \chi_6)\, \sigma_{(2,1,1)}\end{aligned}$$

in $H_T^* Gr(2, \mathbb{C}^6)$ and $H_T^* Gr(4, \mathbb{C}^6)$, respectively.

8. Littlewood-Richardson rules

The ultimate goal is to find a positive formula for the coefficients $c_{\lambda\mu}^{\nu}$. Such a formula is often called a *Littlewood-Richardson rule*. Here we will state several of these rules, without proof.

In the nonequivariant case, the meaning of positivity is clear: $c_{\lambda\mu}^{\nu}$ is a nonnegative integer, and this Littlewood-Richardson rule is classical algebraic combinatorics. With $|\lambda| + |\mu| = |\nu|$, the coefficient $c_{\lambda\mu}^{\nu}$ is number of ways to fill the boxes of the skew diagram ν/λ with μ_1 1's, μ_2 2's, etc., so that

(a) the filling is weakly increasing along rows;
(b) the filling is strictly increasing down columns; and
(c) when the filling is read from right to left along rows, starting at the top, at each step one has

$$\#(1\text{'s}) \geq \#(2\text{'s}) \geq \cdots .$$

Conditions (a) and (b) say the filling is a semistandard Young tableau on the shape ν/λ. Condition (c), sometimes called the "Yamanouchi word" condition, means that the partition μ grows by reading the filling (in the indicated order), placing a box in the ith row when one reads an entry "i", and each intermediate step is also a partition.

Example 8.1. Let $\lambda = (2, 1, 1)$, $\mu = (3, 2, 1)$, $\nu = (4, 3, 2, 1)$. There are three fillings of ν/λ satisfying the conditions:

$$\begin{array}{cccc} & & \boxed{1} & \boxed{1} \\ & \boxed{1} & \boxed{2} & \\ & \boxed{2} & & \\ \boxed{3} & & & \end{array} \qquad \begin{array}{cccc} & & \boxed{1} & \boxed{1} \\ & \boxed{1} & \boxed{2} & \\ & \boxed{3} & & \\ \boxed{2} & & & \end{array} \qquad \begin{array}{cccc} & & \boxed{1} & \boxed{1} \\ & \boxed{2} & \boxed{2} & \\ & \boxed{3} & & \\ \boxed{1} & & & \end{array} .$$

So $c_{\lambda\mu}^{\nu} = 3$. The corresponding reading words—1 1 2 1 2 3, 1 1 2 1 3 2, and 1 1 2 2 3 1—satisfy the Yamanouchi condition.

There are many other versions of the Littlewood-Richardson rule. Some have equivariant analogues. In this context, a "positive" formula should express the polynomial $c_{\lambda\mu}^{\nu}$ as a weighted enumeration of some combinatorial set, with weights of the form $\prod(\chi_i - \chi_j)$ for $i < j$. Indeed, the formulas we have seen for special cases have this property, and a general theorem of Graham guarantees that this is always possible. We will return to this in Chapter 19.

Here is one version, due to Krieman and Molev (working independently). In the statement, reading in *column order* means that entries of a filling of λ are read along columns, from bottom to top, starting at the left.

Theorem 8.2. *The structure constants for multiplication in $H_T^*Gr(d,\mathbb{C}^n)$ are given by*

$$c_{\lambda\mu}^{\nu} = \sum_R \sum_{\mathcal{T}} \prod_{(i,j)\in\lambda} \left(\chi_{e+\mathcal{T}(i,j)-\rho(i,j)_{\mathcal{T}(i,j)}} - \chi_{e+\mathcal{T}(i,j)-(j-i)}\right),$$

where:

R runs over all sequences

$$\mu = \rho^{(0)} \subset \rho^{(1)} \subset \cdots \subset \rho^{(s)} = \nu,$$

such that $\rho^{(i)}$ is obtained by adding one box to $\rho^{(i-1)}$, in row r_i (so $s = |\nu/\mu|$).

$\mathcal{T}$ runs over "reverse barred ν-bounded tableaux" on the shape λ. This means:

- *$\mathcal{T}$ is a filling of the boxes of λ using entries from $\{1,\dots,d\}$, weakly decreasing along rows and strictly decreasing down columns;*
- *all entries in the jth column of λ are less than or equal to the number of boxes in the jth column of ν, that is, $\mathcal{T}(i,j) \le \nu'_j$;*
- *$s = |\nu/\mu|$ of the entries are marked with a bar. When these entries are read in column order, the resulting word is $\overline{r}_1\,\overline{r}_2 \cdots \overline{r}_s$. Thus each barred entry corresponds to a partition $\rho^{(i)}$.*

The product is over all boxes $(i,j) \in \lambda$ such that $\mathcal{T}(i,j)$ is unbarred. If (i,j) is a box with an unbarred entry, $\rho(i,j) = \rho^{(t)}$ is the partition corresponding to the previous barred entry (in column order). If there are no previous barred entries, $\rho(i,j) = \rho^{(0)} = \mu$.

Furthermore, $\rho(i,j)_{\mathcal{T}(i,j)} > j - i$ for all unbarred boxes (i,j).

EXAMPLE 8.3. For $d = 3$, $n = 6$ (so $e = 6 - 3 = 3$) and $\lambda = \mu = (2,1)$, $\nu = (3,1,1)$, there are two sequences R:

$$R_1 : \ydiagram{2,1} \subset \ydiagram{3,1} \subset \ydiagram{3,1,1} \qquad r_1 = 1,\ r_2 = 3$$

$$R_2 : \ydiagram{2,1} \subset \ydiagram{2,1,1} \subset \ydiagram{3,1,1} \qquad r_1 = 3,\ r_2 = 1$$

There is only one tableau for the sequence R_1:

$$\begin{ytableau} \bar{3} & 1 \\ \bar{1} \end{ytableau} \qquad \chi_{3+1-3} - \chi_{3+1+1-2} = \chi_1 - \chi_3.$$

(In this case, $\rho(1,2) = (3,1,1)$, so $\rho(1,2)_{\mathcal{T}(1,2)} = 3$.) For R_2, there are two tableaux:

$$\begin{ytableau} \bar{3} & \bar{1} \\ 1 \end{ytableau} \qquad \chi_{3+1-2} - \chi_{3+1+2-1} = \chi_2 - \chi_5 \quad \text{and}$$

$$\begin{ytableau} \bar{3} & \bar{1} \\ 2 \end{ytableau} \qquad \chi_{3+2-1} - \chi_{3+2+2-1} = \chi_4 - \chi_6.$$

(For the first of these, $\rho(2,1) = (2,1)$, so $\rho(2,1)_{\mathcal{T}(2,1)} = 2$. For the second, $\rho(2,1) = (2,1)$, and $\rho(2,1)_{\mathcal{T}(2,1)} = 1$.) So the rule says $c^{\nu}_{\lambda\mu} = \chi_4 - \chi_6 + \chi_2 - \chi_5 + \chi_1 - \chi_3$.

Historically, the first positive rule for $c^{\nu}_{\lambda\mu}$ was given by Knutson and Tao, and involves the combinatorics of *puzzles*. To describe it, we use another encoding of Schubert classes in $Gr(d, \mathbb{C}^n)$. Recall that a partition λ corresponds to a d-element subset $I \subseteq \{1, \ldots, n\}$. The *01-sequence* corresponding to λ has 1's in positions I, and 0's elsewhere. For example, with $d = 2$ and $n = 5$, the partition $\lambda = (2,0)$ has $I = \{1,4\}$ and 01-sequence 1 0 0 1 0.

To compute $c^{\nu}_{\lambda\mu}$, we label the boundary of an equilateral triangle by 01-sequences corresponding to three partitions λ, μ, and ν, oriented so that the sequence for λ appears along the NW side (from SW to NE), the sequence for μ appears along the NE side (from NW to SE) and the sequence for ν appears along the S side (from W to E). A *puzzle* of type $\Delta^{\nu}_{\lambda\mu}$ is a filling of the triangle by the pieces shown in Figures 2 and 3. All except the equivariant piece may be rotated; the equivariant piece must appear in its displayed orientation. (See Figure 5 for an example.)

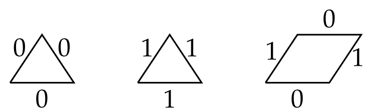


Figure 2. Classical puzzle pieces.

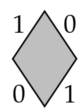


Figure 3. The equivariant puzzle piece.

Each equivariant piece contributes a factor $\chi_i - \chi_j$, computed from its position as shown in Figure 4. The *weight* $\mathrm{wt}(P)$ of a puzzle P is the product of all such factors; it is evidently an element of $\mathbb{Z}_{\geq 0}[\chi_1 - \chi_2, \ldots, \chi_{n-1} - \chi_n]$.

The puzzle rule for computing $c^{\nu}_{\lambda\mu}$ is this:

Theorem 8.4 (Knutson-Tao). *We have*

$$c^{\nu}_{\lambda\mu} = \sum_{\substack{\text{puzzles } P \\ \text{of type } \Delta^{\nu}_{\lambda\mu}}} \mathrm{wt}(P).$$

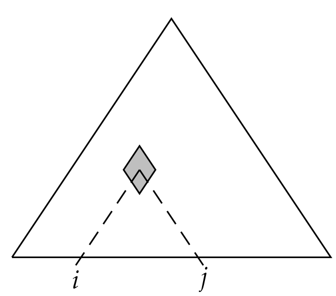


Figure 4. An equivariant piece in position (i, j).

For example, the puzzle in Figure 5 contributes $\chi_3 - \chi_5$ to the coefficient of $\sigma_{(3,1)}$ in $\sigma_{(2,1)} \cdot \sigma_{(2)}$. There are two other puzzles, computing $c_{(2,1),(2)}^{(3,1)} = (\chi_1 - \chi_2) + (\chi_2 - \chi_4) + (\chi_3 - \chi_5) = \chi_1 + \chi_3 - \chi_4 - \chi_5$.

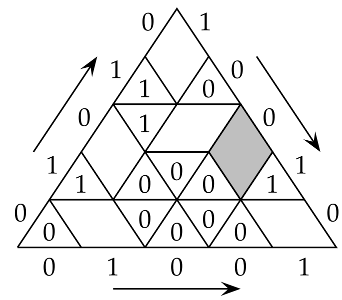


FIGURE 5. A puzzle of type $\Delta_{01010,10010}^{01001}$ and weight $\chi_3 - \chi_5$.

The commutativity property $c_{\lambda\mu}^{\nu} = c_{\mu\lambda}^{\nu}$ is not immediately obvious from the puzzle rule—in general, there is no bijection between puzzles of types $\Delta_{\lambda\mu}^{\nu}$ and $\Delta_{\mu\lambda}^{\nu}$, although the sums of their weights are equal. On the other hand, Grassmann duality (Exercise 7.4) is evident: one defines a bijection between puzzles of type $\Delta_{\lambda\mu}^{\nu}$ and those of type $\Delta_{\mu'\lambda'}^{\nu'}$ by reflecting a puzzle from left to right, and exchanging 0's and 1's.

EXERCISE 8.5. Using the puzzle rule, for λ corresponding to a subset I, show that

$$c_{\lambda\lambda}^{\lambda} = \prod_{\substack{i \in I, j \notin I \\ i > j}} (\chi_j - \chi_i)$$

recovering the formula we know for $c_{\lambda\lambda}^{\lambda} = \sigma_\lambda|_\lambda$.[4]

EXERCISE 8.6. Consider $\mathbb{P}^{n-1} = Gr(1, \mathbb{C}^n)$. The Schubert class $y_i \in H_T^*\mathbb{P}^{n-1}$ corresponds to the 01-sequence with a 1 in position $i+1$, and 0's elsewhere. Use the puzzle rule to recover the formula for c_{ij}^k given in Chapter 4, §7.

Notes

In the literature dealing with general Lie theory, B^--invariant subvarieties are usually called "opposite." Our conventions are reversed, but have the advantage of better stability properties. We will continue this usage through Chapter 13, and switch to the more standard convention when we discuss general Lie groups in Chapter 15.

The Kempf-Laksov theorem (Theorem 2.2) was originally stated in the context of degeneracy loci in Grassmann bundles [**KeLa74**]. Its appearance as an equivariant Schubert class represents an early instance of the connection between equivariant geometry and the geometry of fiber bundles, although it was not seen this way at the time.

Studying rank conditions on matrices of homogeneous polynomials, Giambelli proved the case of Theorem 2.2 corresponding to a rectangular partition [**Gi04**]. We will see more about this in Chapter 11.

In Chapter 4, we saw a basis of Schur determinants $\Delta_\lambda(c^{B^-}(\mathbb{Q}))$, for λ inside the $d \times e$ rectangle. Here we have studied the basis of Schubert classes $[\Omega_\lambda]^{B^-}$, which are expressed (via the Kempf-Laksov formula) as multi-Schur determinants. What is the transition matrix between these two bases? The answer, given in [**AF-ABCD**], involves certain *flagged Schur polynomials*, which are special cases of the *Schubert polynomials* to be studied in Chapter 10.

The argument for Lemma 3.1 comes from Knutson and Tao [**KnTao03**]. The same idea was axiomatized and applied to other settings by Guillemin and Zara [**GuZa01**] and Tymoczko [**Tym08b**]. An alternative framework for finding (unique) solutions to such interpolation problems was developed systematically by Fehér and Rimányi [**FeRi03**].

An excellent reference for double Schur polynomials (and their relatives) is Macdonald's note [**Mac92**]. In particular, he proves the equivalence of the three characterizations of $s_\lambda(x|y)$ we gave. (We mainly use his "Variation 6".)

Lemma 4.1 is due to Okounkov, who shows that these conditions characterize "shifted" Schur functions [**Ok96**]; see also [**OO97**] and [**MoSa99**].

Proofs of the classical (non-equivariant) case of the Chevalley-Pieri rule, Theorem 6.2, can be found in many sources, e.g., [**Ful-YT**, §9.4]. A proof of the equivariant version appears in [**KnTao03**, Appendix]. We will see a complete proof of the analogous formula for general homogeneous spaces G/P in Chapter 19. The characterization theorem (Theorem 6.4) is due to Molev and Sagan [**MoSa99**], and was used extensively by Knutson and Tao [**KnTao03**]. It also has an analogue for G/P, as we will see in Chapter 19.

There are many references for the classical Littlewood-Richardson rule and its variations; see, for example, [**Mac95**], [**Sta99**], [**Ful-YT**]. The first complete proof is due to Schützenberger, using a game called *jeu de taquin* [**Schü77**]. An equivariant *jeu de taquin* rule was given by Thomas and Yong [**ThYo18**].

A combinatorial rule for the multiplying double Schur polynomials $s_\lambda(x|y)$ was given by Molev and Sagan [**MoSa99**], but their original formula was not manifestly positive in the variables $y_i - y_j$, $i > j$. Molev later modified this to the positive formula described here [**Mo09**].

Knutson and Tao gave the first manifestly positive rule for the equivariant structure constants [**KnTao03**]. In fact, they computed $\widetilde{c}^{\nu}_{\lambda\mu}$, the structure constants for multiplying in the *opposite* Schubert basis $\{\widetilde{\sigma}_\lambda\}$. These are related to $c^{\nu}_{\lambda\mu}$ by the substitution $\chi_i \mapsto \chi_{n+1-i}$. This is realized by reflecting puzzles left-to-right, which has the effect of exhanging the equivariant and non-equivariant rhombi.

Hints for exercises

[1]Use the bialternant definition. The $(d-p,q)$-entry of the matrix in the numerator of $s_\lambda(y^\mu|y)$ is

$$(y_{\mu_p+d-p+1}|y)^{\lambda_q+d-q} = (y_{\mu_p+d-p+1} - y_1)\cdots(y_{\mu_p+d-p+1} - y_{\lambda_q+d-q}).$$

If $\mu \not\supseteq \lambda$, then some index k has $\mu_k < \lambda_k$ (so also $\mu_p < \lambda_q$ for $q \le k \le p$). But then for all $q \le k \le p$, we have

$$1 \le \mu_p + d - p + 1 \le \lambda_q + d - q,$$

so the above product vanishes, and it follows that the determinant is zero. If $\mu = \lambda$, the matrix is triangular because

$$1 \le \lambda_p + d - p + 1 \le \lambda_q + d - q$$

if $p > q$, so its determinant is the product

$$\prod_{p=1}^{d} \prod_{s=1}^{\lambda_p+d-p} (y_{\lambda_p+d-p+1} - y_s).$$

Dividing by the Vandermonde denominator gives the formula.

[2]The key identity is

$$h_p(x|\tau^{1-j}y) = \sum_{a+b=p} h_a(x_1,\ldots,x_d)(-1)^b e_b(y_1,\ldots,y_{d+p-j}),$$

where $y_m = 0$ for $m \leq 0$. (This is easy to prove from the tableau definition of $h_p = s_{(p)}$.) Then compare with the degree $p = \lambda_i - i + j$ term of $c^T(\mathbb{Q} - E_{e-p+j}) = c^T(\widetilde{E}_{d+p-j} - \mathbb{S})$.

[3]Let $E_\bullet$ and $\widetilde{E}_\bullet$ be the standard and opposite flags in $\mathbb{C}^n$. Under the identification $V \cong V^\vee$ by the chosen basis, we have $E_i \mapsto (V/E_i)^\vee \cong \widetilde{E}_{n-i}$.

[4]See [**KnTao03**, Proposition 3].