CHAPTER 18

Bott-Samelson varieties and Schubert varieties

Schubert varieties in G/P admit explicit equivariant desingularizations by Bott-Samelson varieties. These are certain towers of $\mathbb{P}^1$-bundles, and their cohomology rings are relatively easy to compute.

In this chapter, we use the Bott-Samelson desingularization to obtain a positive formula for restricting a Schubert class to a fixed point. This, in turn, leads to a criterion for a point of a Schubert variety to be nonsingular.

1. Definitions, fixed points, and tangent spaces

Let $G \supset B \supset T$ be as usual: G is a semisimple (or reductive) group, with Borel subgroup B and maximal torus T. For each simple root α, we have a minimal parabolic subgroup P_α, and the corresponding projection of flag varieties is a $\mathbb{P}^1$-bundle, $G/B \to G/P_\alpha$. These spaces occur frequently in this chapter, so we will write

$$X = G/B \quad \text{and} \quad X_\alpha = G/P_\alpha$$

from now on.

For any sequence of simple roots $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d)$, we have a *big Bott-Samelson variety* $Z(\underline{\alpha}) = Z(\alpha_1, \ldots, \alpha_d)$, defined by

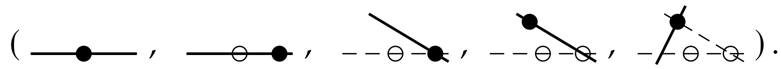
$$Z(\underline{\alpha}) = X \times_{X_{\alpha_1}} X \times_{X_{\alpha_2}} \cdots \times_{X_{\alpha_d}} X.$$

Since each projection $X \to X_{\alpha_i}$ is a $\mathbb{P}^1$-bundle, $Z(\underline{\alpha})$ is a tower of $\mathbb{P}^1$-bundles over X. In particular, it is a nonsingular projective variety of dimension $\dim X + d$. The group G acts diagonally on $Z(\underline{\alpha})$, equivariantly for each projection $pr_i \colon Z(\underline{\alpha}) \to X$. (We index these projections from left to right by $0 \leq i \leq d$.)

Example 1.1. For $G = SL_n$, so $X = Fl(\mathbb{C}^n)$, a Bott-Samelson variety can be described as a sequence of flags, with the ith differing from the $(i-1)$st only in position j, if $\alpha_i = t_j - t_{j+1}$. That is,

$$Z(\underline{\alpha}) = \left\{ (F_\bullet^{(0)}, \dots, F_\bullet^{(d)}) \,\middle|\, \begin{array}{c} E_k^{(i)} = E_k^{(i-1)} \text{ for all } k \neq j, \\ \text{where } \alpha_i = t_j - t_{j+1} \end{array} \right\}.$$

When $n = 3$, these can be represented as configurations of points and lines in $\mathbb{P}^2$. For instance, suppose $\alpha = t_1 - t_2$ and $\beta = t_2 - t_3$. Then a general point of $Z(\alpha, \beta, \alpha, \beta)$ looks like a quintuple of flags:



So from left to right, consecutive flags differ by moving the point, then the line, then the point, and finally the line again.

The T-fixed points of $Z(\underline{\alpha})$ are easily described. An *$\underline{\alpha}$-chain* (or simply *chain*) of elements of W is a sequence

$$\underline{v} = (v_0, v_1, \dots, v_d)$$

such that for each i, either $v_i = v_{i-1}$ or $v_i = v_{i-1} \cdot s_{\alpha_i}$.

EXERCISE 1.2. Show that the T-fixed points of $Z(\underline{\alpha})$ are the $2^d \cdot |W|$ points

$$Z(\underline{\alpha})^T = \left\{ p_{\underline{v}} = (p_{v_0}, p_{v_1}, \dots, p_{v_d}) \right\}$$

where each $\underline{v}$ is an α-chain.

The *(small) Bott-Samelson variety* is the fiber $X(\underline{\alpha}) = pr_0^{-1}(p_e)$, that is,

$$X(\underline{\alpha}) = \{p_e\} \times_{X_{\alpha_1}} X \times_{X_{\alpha_2}} \cdots \times_{X_{\alpha_d}} X.$$

The projection $X(\alpha_1, \dots, \alpha_d) \to X(\alpha_1, \dots, \alpha_{d-1})$ is a $\mathbb{P}^1$-bundle, so $X(\underline{\alpha})$ is a nonsingular projective variety of dimension d. Since p_e is fixed by B, the Bott-Samelson variety $X(\underline{\alpha})$ comes with an action of B (but not G, in general).

The Bott-Samelson variety $X(\underline{\alpha})$ has 2^d T-fixed points $p_{\underline{v}}$, for chains $\underline{v} = (e, v_1, \dots, v_d)$. We will index these in two ways: using the chain $\underline{v}$, and using the subset $I = \{i_1 < \cdots < i_\ell\} \subseteq \{1, \dots, d\}$ defined by

$$I = \left\{ i \,\middle|\, v_i = v_{i-1} \cdot s_{\alpha_i} \right\}.$$

We often use the notation interchangeably, writing $p_{\underline{v}} = p_I$. Sometimes we write $I = I^{\underline{v}}$ and $\underline{v} = \underline{v}^I$ to indicate the bijection between chains and subsets.

For each subset $I \subseteq \{1, \dots, d\}$, there is a B-invariant subvariety $X(I) \subseteq X(\underline{\alpha})$, defined by

$$X(I) = \left\{(x_1, \dots, x_d) \in X(\underline{\alpha}) \,\middle|\, x_j = x_{j-1} \text{ for } j \notin I\right\}.$$

In fact, this is canonically isomorphic to another Bott-Samelson variety. Each subset $I = \{i_1 < \cdots < i_\ell\}$ corresponds to a subword $\underline{\alpha}(I) = (\alpha_{i_1}, \dots, \alpha_{i_\ell})$, and we have

$$X(I) \cong X(\underline{\alpha}(I)).$$

(Use a diagonal embedding of $X^{\ell+1}$ in X^{d+1}.) Containment among these subvarieties corresponds to containment of subsets:

$$X(J) \subseteq X(I) \quad \text{iff} \quad J \subseteq I.$$

For example, $X(\{1, \dots, d\}) = X(\underline{\alpha})$, and $X(\emptyset)$ is the point $p_\emptyset$.

Each $X(I)$ is the closure of a locally closed set $X(I)^\circ$, consisting of the points where $x_i \neq x_{i-1}$ for $i \in I$. In fact, these are cells.

LEMMA 1.3. *We have $X(I)^\circ \cong \mathbb{A}^\ell$, where $\ell = \#I$.*

PROOF. It suffices to consider $I = \{1, \dots, d\}$. Here one has the $\mathbb{P}^1$-bundle $X(\alpha_1, \dots, \alpha_d) \to X(\alpha_1, \dots, \alpha_{d-1})$. The complement of the locus where $x_{d-1} = x_d$ is an $\mathbb{A}^1$-bundle over $X(\alpha_1, \dots, \alpha_{d-1})$, so the claim follows by induction on d. □

The subvarieties $X(I)$ therefore determine a cell decomposition of $X(\underline{\alpha})$, and their classes $x(I) = [X(I)]^T$ form a basis for $H_T^* X(\underline{\alpha})$, as I varies over subsets of $\{1, \dots, d\}$. It also follows that

$$p_J \in X(I) \quad \text{iff} \quad J \subseteq I.$$

We will need a description of the tangent spaces.

LEMMA 1.4. *Let $\underline{v} = (e, v_1, \dots, v_d)$ be an $\underline{\alpha}$-chain. The torus weights on $T_{p_{\underline{v}}} X(\underline{\alpha})$ are $\{-v_1(\alpha_1), \dots, -v_d(\alpha_d)\}$.*

More generally, for $K \subseteq I$, with corresponding chains $\underline{v}^K$ and $\underline{v}^I$, the weights on $T_{p_K} X(I)$ are $-v_i^K(\alpha_i)$ for $i \in I$.

Proof. We will find the weights at any fixed point of the big Bott-Samelson variety . For a chain $\underline{v} = (v_0, v_1, \ldots, v_d)$, consider the point $p = p_{\underline{v}} \in Z(\underline{\alpha})$. The tangent space to $Z(\underline{\alpha})$ at p is the fiber product of vector spaces

$$T_{p_0}X \underset{T_{p[1]}X_{\alpha_1}}{\times} T_{p_1}X \underset{T_{p[2]}X_{\alpha_2}}{\times} \cdots \underset{T_{p[d]}X_{\alpha_d}}{\times} T_{p_d}X,$$

where we have written $p_i = p_{v_i} \in X$ and $p_{[i]} = p_{[v_i]} \in X_{\alpha_i}$ to economize on subscripts. (Note $[v_i] = [v_{i-1}]$ for each i, since $\underline{v}$ is an $\underline{\alpha}$-chain.) We have seen descriptions of each of these spaces in Chapter 15. The weights are $v_0(R^-)$, from the first factor, together with weights $-v_i(\alpha_i)$ for $1 \leq i \leq d$, since $\mathfrak{g}_{-v_i(\alpha_i)}$ is the kernel of $T_{p_i}X \to T_{p_{[i]}}X_{\alpha_i}$.

When $v_0 = e$, the variety $X(\underline{\alpha})$ is the fiber over p_e in the first factor, so the weights $R^- = v_0(R^-)$ are omitted, proving the first claim. The second claim follows from the first, using $X(I) \cong X(\alpha_{i_1}, \ldots, \alpha_{i_\ell})$. □

2. Desingularizations of Schubert varieties

Let $f \colon X(\underline{\alpha}) \to X$ be the projection onto the last factor; that is, f is the restriction of $pr_d \colon Z(\underline{\alpha}) \to X$. For each $I \subseteq \{1, \ldots, d\}$, with corresponding $\underline{\alpha}$-chain $\underline{v} = (e, v_1, \ldots, v_d)$, we have $f(p_I) = p_{v_d}$. The subset I corresponds to the subword $(\alpha_{i_1}, \ldots, \alpha_{i_\ell})$ of $\underline{\alpha}$, and

$$v_d = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_\ell}}.$$

Since f is proper and B-equivariant, $f(X(I))$ contains the Schubert variety $X(v_d) \subseteq X$. However, if $(\alpha_{i_1}, \ldots, \alpha_{i_\ell})$ is not a reduced word for v_d, the image of f may be larger.

Lemma 2.1. *Let $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d)$ be a sequence of simple roots. The set of products $s_{\alpha_{i_1}} \cdots s_{\alpha_{i_\ell}}$ over subwords contains a unique maximum element $w(\underline{\alpha}) \in W$ in Bruhat order, and*

$$f(X(\underline{\alpha})) = X(w(\underline{\alpha})).$$

We have $w(\underline{\alpha}) = s_{\alpha_1} \cdots s_{\alpha_d}$ if and only if the word $\underline{\alpha}$ is reduced.

PROOF. Since $X(\underline{\alpha})$ is irreducible, the image of the B-equivariant morphism $f\colon X(\underline{\alpha}) \to X$ must be some Schubert variety $X(w)$. It follows that $w = w(\underline{\alpha})$ satisfies the asserted properties. □

In fact, the maximal element $w(\underline{\alpha})$ can be easily computed. Let "$*$" be the associative product on W defined by

$$w * s_\alpha = \begin{cases} ws_\alpha & \text{if } \ell(ws_\alpha) > \ell(w); \\ w & \text{otherwise.} \end{cases}$$

This product is called the *Demazure product*.

EXERCISE 2.2. Show that $w(\underline{\alpha}) = s_{\alpha_1} * \cdots * s_{\alpha_d}$, i.e., it is the Demazure product of reflections from $\underline{\alpha}$.[1]

LEMMA 2.3. *The map $f\colon X(\underline{\alpha}) \to X(w)$ is birational if and only if α is a reduced word for $w = w(\underline{\alpha})$.*

PROOF. If $\underline{\alpha}$ is not a reduced word, then $w(\underline{\alpha})$ is the product of reflections for a proper subword, so it has length $\ell(w(\underline{\alpha})) < d$. In this case, f cannot be birational by dimension.

If $\underline{\alpha}$ is reduced, then $w = w(\underline{\alpha}) = s_{\alpha_1} \cdots s_{\alpha_d}$, and $f(p_{\{1,\dots,d\}}) = p_w$. The map $f\colon X(\underline{\alpha})^\circ \to X(w)^\circ$ is B-equivariant, and therefore also equivariant for the subgroup $U(w) = \dot{w}U\dot{w}^{-1} \cap U$. Since the map $u \mapsto u \cdot p_w$ defines an isomorphism $U(w) \xrightarrow{\sim} X(w)^\circ$, it follows that $f\colon X(\underline{\alpha})^\circ \to X(w)^\circ$ is an isomorphism. □

For a reduced word $\underline{\alpha}$, one can also establish the birationality of $f\colon X(\underline{\alpha}) \to X(w)$ by examining tangent weights. The tangent space to $X(\underline{\alpha})$ at $p = p_{\{1,\dots,d\}}$ has weights

$$\alpha_1,\ s_{\alpha_1}(\alpha_2), \dots,\ s_{\alpha_1} \cdots s_{\alpha_{d-1}}(\alpha_d),$$

using Lemma 1.4, for $v_i = s_{\alpha_1} \cdots s_{\alpha_i}$. These are precisely the weights on $T_{p_w}X(w)$ (see Chapter 15, Lemma 2.2).

Given a Schubert variety $X(w) \subseteq G/B$, one obtains a B-equivariant desingularization $f\colon X(\underline{\alpha}) \to X(w)$ by choosing a reduced word for w. For a parabolic subgroup P, the projection $G/B \to G/P$ maps $X(w^{\min})$ birationally onto $X[w]$, so we obtain desingularizations of these varieties, too.

Corollary 2.4. *For a Schubert variety* $X[w] \subseteq G/B$*, and any reduced word* $\underline{\alpha}$ *for* $w^{\min}$*, one obtains a desingularization* $X(\underline{\alpha}) \to X[w]$ *by composing* f *with the projection* $G/B \to G/P$*.* □

These statements have evident analogues for the subvarieties $X(I) \subseteq X(\underline{\alpha})$. If I is a subset, with subword $\underline{\alpha}(I)$, we will write $w(I) = w(\underline{\alpha}(I))$ for the corresponding Demazure product.

Corollary 2.5. *Let* I *be a subset, and let* $\underline{v} = (v_1, \dots, v_d)$ *be the corresponding chain. The following are equivalent:*

(i) The map $X(I) \to X(w(I))$ *is birational.*

(ii) $w(I) = v_d$*.*

(iii) $\ell(v_d) = \#I$*.*

(iv) The subword $\underline{\alpha}(I)$ *is a reduced word for* v_d*.* □

Example 2.6. Let $\underline{\alpha} = (\alpha, \alpha)$, for some simple root α. Then $X(\underline{\alpha})$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The Demazure product is $s_\alpha * s_\alpha = s_\alpha$, and the map $f\colon X(\alpha, \alpha) \to X(s_\alpha)$ is identified with the second projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. The subvarieties $X(I) = X(\underline{v})$ are

$$\begin{aligned} X(\{1,2\}) &= X(s_\alpha, e) = X(\underline{\alpha}), \\ X(\{1\}) &= X(s_\alpha, s_\alpha) = \delta(\mathbb{P}^1) \text{ (the diagonal in } \mathbb{P}^1 \times \mathbb{P}^1\text{)}, \\ X(\{2\}) &= X(e, s_\alpha) = \{p_e\} \times \mathbb{P}^1, \text{ and} \\ X(\emptyset) &= X(e,e) = \{(p_e, p_e)\}. \end{aligned}$$

While $X(\underline{\alpha})$ always has finitely many fixed points, it often has infinitely many invariant curves—even when $\underline{\alpha}$ is a reduced word.

Exercise 2.7. The following are equivalent, for a sequence of simple roots $\underline{\alpha} = (\alpha_1, \dots, \alpha_d)$:

(a) $X(\underline{\alpha})$ has finitely many T-curves.

(b) The roots $\alpha_1, \dots, \alpha_d$ are distinct.

(c) $X(\underline{\alpha})$ is a toric variety for the quotient of T whose character lattice has basis $\alpha_1, \dots, \alpha_d$.

(d) The map $f\colon X(\underline{\alpha}) \to X(w)$ is an isomorphism.

(Use the description of weights on tangent spaces.)[2]

Another construction of the Bott-Samelson variety $X(\underline{\alpha})$ is sometimes useful.

Proposition 2.8. *For a word $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d)$, there is an isomorphism*

$$P_{\alpha_1} \times^B P_{\alpha_2} \times^B \cdots \times^B P_{\alpha_d}/B \to X(\underline{\alpha}),$$

given by $[p_1, \ldots, p_d] \mapsto (eB, p_1B, p_1p_2B, \ldots, p_1 \cdots p_dB)$. This is B-equivariant, where B acts via left multiplication on P_{α_1}. The subvarieties $X(I) \subseteq X(\underline{\alpha})$ are identified with

$$X(I) = \{[p_1, \ldots, p_d] \,|\, p_iB = eB \text{ for } i \notin I\},$$

and the point p_I corresponds to $[\varepsilon_1, \ldots, \varepsilon_d]$, where $\varepsilon_i = \dot{e}$ for $i \in I$, and $\varepsilon_j = \dot{s}_{\alpha_j}$ for $j \notin I$.

Exercise 2.9. Prove the proposition.[3]

Remark 2.10. Bott-Samelson varieties appear in the geometric construction of divided difference operators described in Chapter 16, §1. Let $\underline{\alpha}$ be a reduced word for w. The big Bott-Samelson variety $Z(\underline{\alpha})$ maps birationally to the double Schubert variety

$$Z(w) = \overline{G \cdot (p_e, p_w)} \subseteq X \times X$$

via the projection $pr_0 \times pr_d$. Using Chapter 16, Proposition 1.2, the operator $\mathrm{D}_{w^{-1}}$ on H_T^*X is identified with $pr_{d*}pr_0^*$.

On the other hand, these projections factor as iterated $\mathbb{P}^1$-bundles, and the diagram

$$\begin{array}{ccccccccc} & Z(\alpha_1) & & & Z(\alpha_2) & & \cdots & Z(\alpha_d) & \\ \swarrow & & \searrow & \swarrow & & \searrow & \swarrow & & \searrow \\ X & & & X & & & \cdots & & X \end{array}$$

shows that $\mathrm{D}_{w^{-1}} = \mathrm{D}_{\alpha_\ell} \circ \cdots \circ \mathrm{D}_{\alpha_1}$ is independent of the choice of reduced word. One can also see this by restricting the diagram

$$\begin{array}{ccc} & Z(\underline{\alpha}) & \\ {\scriptstyle pr_0} \swarrow & & \searrow {\scriptstyle pr_d} \\ X & & X \end{array}$$

to the fiber $pr_0^{-1}(p_e)$, obtaining

$$\begin{array}{ccc} & X(\underline{\alpha}) & \\ \swarrow & & \searrow^{f} \\ p_e & & X(w). \end{array}$$

Since f is birational, we have

$$\mathrm{D}_{w^{-1}}(x(e)) = pr_{d*}pr_0^*(x(e)) = f_*[X(\underline{\alpha})]^T = [X(w)]^T = x(w).$$

3. Poincaré duality and restriction to fixed points

We have seen that the classes $x(I) = [X(I)]^T$ form a Λ-module basis for $H_T^*X(\underline{\alpha})$. Next we will study their restrictions to fixed points, and determine the Poincaré dual basis.

Lemma 1.4 leads directly to a description of weights at the fixed points of $X(I) \subseteq X(\underline{\alpha})$. Suppose $K \subseteq I$, so $p_K \in X(I)$, and let $\underline{v}^K$ and $\underline{v}^I$ be the corresponding chains. The weights on $T_{p_K}X(I)$ are $-v_i^K(\alpha_i)$ for $i \in I$. This, in turn, gives a formula for restricting the classes $x(I) = [X(I)]^T$. For any $x \in H_T^*X(\underline{\alpha})$, its restriction to the fixed point p_I is denoted $x|_I$.

Corollary 3.1. *We have*

$$x(I)|_K = \begin{cases} \prod_{j \notin I} v_j^K(-\alpha_j) & \text{if } K \subseteq I; \\ 0 & \text{otherwise,} \end{cases}$$

Let $\{y(I)\}$ be the Poincaré dual basis to $\{x(I)\}$, meaning that $\rho_*(x(I) \cdot y(J)) = \delta_{I,J}$ in Λ, where $\rho\colon X(\underline{\alpha}) \to \mathrm{pt}$ is the projection. As we saw in Chapter 4, §6, such a basis always exists. It is natural to look for invariant subvarieties $Y(I)$ representing these Poincaré dual classes. However, no such algebraic subvarieties exist!

Example 3.2. Consider the variety $X(\alpha, \alpha) \cong \mathbb{P}^1 \times \mathbb{P}^1$ from Example 2.6. The basis $\{x(I)\}$ consists of the equivariant classes of

$$\begin{aligned} x(\emptyset) &= [(p_e, p_e)]^T, \\ x(\{1\}) &= [\delta(\mathbb{P}^1)]^T, \\ x(\{2\}) &= [\{p_e\} \times \mathbb{P}^1]^T, \text{ and} \\ x(\{1,2\}) &= [\mathbb{P}^1 \times \mathbb{P}^1]^T. \end{aligned}$$

Even non-equivariantly, the Poincaré dual basis cannot be represented by algebraic subvarieties: the class $y(\{2\})$ must have zero intersection with the diagonal class $x(\{1\})$, and no algebraic curve in $\mathbb{P}^1 \times \mathbb{P}^1$ can do this.

Another way of phrasing the conclusion of Example 3.2 is this: we seek a curve $Y(\{2\}) \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ which consists of pairs (L, L') of lines in $\mathbb{C}^2$ such that $L \neq L'$—but the complement of the diagonal is affine, so it contains no complete curves. In fact, this observation indicates a solution. Using the standard Hermitian metric on $\mathbb{C}^2$, we may consider pairs of perpendicular lines (L, L'); in terms of a coordinate z on $\mathbb{P}^1$, this is the set of pairs $(z, -1/\overline{z})$. This set is a non-algebraic submanifold $Y(\{2\}) \subseteq \mathbb{P}^1 \times \mathbb{P}^1$, which we orient by projecting onto the first factor. (Projection onto the second factor would give the opposite orientation, as the coordinate description shows.) Fixing the metric amounts to reducing GL_2 to the maximal compact subgroup $U(2)$, and identifying $\mathbb{P}^1 = GL_2/B$ with $U(2)/(T \cap U(2))$.

The general situation is similar: we construct (non-algebraic) submanifolds $Y(I) \subseteq X(\alpha)$ whose classes represent the Poincaré dual classes $y(I)$. Let $K \subseteq G$ be a maximal compact subgroup, with maximal compact torus $S = T \cap K$, so we have a diffeomorphism $K/S \cong G/B$, and the Weyl group $W = N_K(S)/S$ acts on the right. For $I \subseteq \{1, \ldots, d\}$, we define

$$Y(I) = \{(e, x_1, \ldots, x_d) \in X(\underline{\alpha}) \mid x_i = x_{i-1} \cdot s_{\alpha_i} \text{ for } i \in I\}.$$

This is a C^∞ submanifold, of real codimension $2 \cdot \#I$ in $X(\underline{\alpha})$, invariant for the action of the compact torus S. Containment among these submanifolds reverses containment of subsets:

$$Y(K) \subseteq Y(I) \quad \text{iff} \quad p_K \in Y(I) \quad \text{iff} \quad K \supseteq I.$$

Lemma 3.3. *Giving each $Y(I)$ an appropriate orientation (to be specified in the proof), the classes $y(I) = [Y(I)]^S$ form the Poincaré dual basis to $x(I)$.*

For $K \supset I$, with corresponding α-chains $\underline{v}^K$ and $\underline{v}^I$, the normal space to $Y(I) \subseteq X(\underline{\alpha})$ at the fixed point p_K has characters $-v_i^K(\alpha_i)$, for $i \in I$.

Proof. To compute the tangent spaces of $Y(I)$, and to orient it, we work from the left, using induction on d. For $d = 1$, we have $Y(\{1\}) = \{\dot{s}_\alpha B\}$ (a point), and $Y(\emptyset) = X(\alpha) = \mathbb{P}^1$, so these are already oriented. Proceeding inductively, consider the projection $X(\alpha_1, \ldots, \alpha_d) \to X(\alpha_1, \ldots, \alpha_{d-1})$. If $d \in I$, this induces an isomorphism $Y(I) \to Y(I \smallsetminus \{d\})$. Otherwise, if $d \notin I$, it induces a $\mathbb{P}^1$-bundle, so there is a fiber square

$$\begin{array}{ccc} Y(I) & \hookrightarrow & X(\alpha_1, \ldots, \alpha_d) \\ \downarrow & & \downarrow \\ Y(\overline{I}) & \hookrightarrow & X(\alpha_1, \ldots, \alpha_{d-1}), \end{array}$$

where we have written $\overline{I} = I$ as a subset of $\{1, \ldots, d-1\}$. By the inductive assumption, we have an orientation of $Y(\overline{I})$. The canonical orientation of the $\mathbb{P}^1$ fiber then induces an orientation of $Y(I)$.

This construction also identifies the tangent spaces: assume $d \notin I$, and for $K \supseteq I$, write $p = p_K$ and $\overline{p}$ for the image of this point in $Y(\overline{I})$. The kernel of

$$T_p Y(I) \to T_{\overline{p}} Y(\overline{I})$$

is $\mathfrak{g}_\beta$, where $\beta = -v_d^K(\alpha_d)$.

It follows that $Y(I)$ meets $X(I)$ transversally in the point p_I. Indeed, we have weight decompositions of the tangent spaces as

$$T_{p_I} X(I) = \bigoplus_{i \notin I} \mathfrak{g}_{-v_i^I(\alpha_i)}$$

and

$$T_{p_I}Y(I) = \bigoplus_{i\in I} \mathfrak{g}_{-v_i^I(\alpha_i)}.$$

So these are complementary subspaces of $T_{p_I}X(\underline{\alpha})$. By considering fixed points, we see $X(I) \cap Y(J) = \emptyset$ unless $J \subseteq I$, and it follows that the classes $x(I)$ and $y(J)$ form Poincaré dual bases. □

This description of tangent spaces proves a formula for restricting the classes $y(I)$.

COROLLARY 3.4. *We have*

$$y(I)|_K = \begin{cases} \prod_{i\in I} v_i^K(-\alpha_i) & \textit{if } K \supseteq I; \\ 0 & \textit{otherwise.} \end{cases}$$

A more algebraic proof of Corollary 3.4 uses the localization formula. The dual classes $y(I)$ are uniquely determined by

$$\sum_{p_K \in X(J)} \frac{y(I)|_K}{c^T_{top}(T_{p_K}X(J))} = \delta_{I,J}, \tag{1}$$

for every subset $J \subseteq \{1,\ldots,d\}$. We know $p_K \in X(J)$ iff $K \subseteq J$, and in this case $c^T_{top}(T_{p_K}X(J)) = \prod_{j\in J}(-v_j^K(\alpha_j))$. To prove the claimed formula for $y(I)|_K$, it remains to establish the identity

$$\sum_{K: I\subseteq K\subseteq J} \frac{1}{\prod_{j\in J\smallsetminus I}(-v_j^K(\alpha_j))} = \delta_{I,J}. \tag{2}$$

This is clear if $I = J$, or if $I \not\subseteq J$. When $I \subsetneq J$, the terms cancel in pairs, as follows. Suppose j is the largest index in $J \smallsetminus I$; then for each $K \not\ni j$, there is $K' = K \cup \{j\}$, and the corresponding terms cancel. (Indeed, $s_{\alpha_j}(\alpha_j) = -\alpha_j$, so $v_j^{K'}(\alpha_j) = -v_j^K(\alpha)$ and the other factors in the product are equal.)

REMARK 3.5. The identification $X = G/B = K/S$ leads to a third description of the Bott-Samelson varieties. Each $K_\alpha = K \cap P_\alpha$ is a maximal compact subgroup of the minimal parabolic P_α, and the evident map

$$K_{\alpha_1} \times^S K_{\alpha_2} \times^S \cdots \times^S K_{\alpha_d}/S \to P_{\alpha_1} \times^B K_{\alpha_2} \times^B \cdots \times^B K_{\alpha_d}/B$$

is a diffeomorphism. The submanifolds $Y(I) \subseteq X(\underline{\alpha})$ are easy to identify from this point of view:

$$Y(I) = \Big\{[k_1, \dots, k_d] \,\Big|\, k_i S = \dot{s}_{\alpha_i} S \text{ for } i \in I\Big\}.$$

For the corresponding projection $f\colon X(\underline{\alpha}) \to X$, one sees

$$f(Y(\{1, \dots, k\})) = s_{\alpha_1} \cdots s_{\alpha_k} \cdot X(s_{\alpha_{k+1}} * \cdots * s_{\alpha_d})$$

and

$$f(Y(\{k+1, \dots, d\})) = X(s_{\alpha_1} * \cdots * s_{\alpha_k}) \cdot s_{\alpha_{k+1}} \cdots s_{\alpha_d},$$

where $w \cdot X(v)$ and $X(v) \cdot w$ denote the translations of Schubert varieties by the left and right W-actions.

4. A presentation for the cohomology ring

Multiplication in the basis $y(I)$ is particularly easy. To simplify the notation, we will write $p_i = p_{\{i\}}$, $p_{ij} = p_{\{i,j\}}$, $y_i = y(\{i\})$, and $y_{ij} = y(\{i, j\})$.

If $I \cap J = \emptyset$, then $Y(I)$ and $Y(J)$ meet transversally in $Y(I \cup J)$, so

$$y(I) \cdot y(J) = y(I \cup J) \quad \text{if } I \cap J = \emptyset. \tag{3}$$

In particular, $y_i \cdot y_j = y_{ij}$ if $i \neq j$, and $y(I) = y_{i_1} \cdots y_{i_\ell}$ if $I = \{i_1, \dots, i_\ell\}$. To determine the structure of $H_T^* X(\underline{\alpha})$, it suffices to give a formula for y_i^2.

PROPOSITION 4.1. *We have*

$$y_i^2 = \sum_{j<i} (-\langle \alpha_i, \alpha_j^\vee \rangle)\, y_{ij} + \alpha_i\, y_i, \tag{4}$$

where $\langle \alpha, \beta^\vee \rangle$ is the pairing between roots and coroots.

PROOF. By considering degrees and support, we have

$$y_i^2 = \sum_{j \neq i} c_{ij}\, y_{ij} + \lambda_i\, y_i, \tag{5}$$

for some $c_{ij} \in \mathbb{Z}$ and $\lambda_i \in M$. (Since $p_j \notin Y(\{i\})$ for $j \neq i$, we have $y_i|_{p_j} = 0$, so the classes y_j do not appear. Similarly, $p_\emptyset \notin Y(\{i\})$, so there is no "constant" term of degree 2 in Λ.) So we must determine these coefficients.

Using the restriction formula from Corollary 3.4, we have

$$y_i|_{p_i} = -v_i'(\alpha_i) = \alpha_i,$$

where the chain corresponding to $\{i\}$ is $\underline{v}' = (e, \ldots, e, s_{\alpha_i}, \ldots, s_{\alpha_i})$. Since $p_i \notin Y(\{i,j\})$ for $j \neq i$, restricting Equation (5) to this point gives

$$(\alpha_i)^2 = \lambda_i \, \alpha_i,$$

and it follows that $\lambda_i = \alpha_i$.

Similarly, we have

$$y_i|_{p_{ij}} = \begin{cases} \alpha_i & \text{if } i < j; \\ s_{\alpha_j}(\alpha_i) & \text{if } i > j. \end{cases}$$

(When $i < j$, the chain $\underline{v}'$ corresponding to $\{i,j\}$ has $v_i' = s_{\alpha_i}$, so $y_i|_{p_{ij}} = -s_{\alpha_i}(\alpha_i) = \alpha_i$. For $i > j$, the chain has $v_i' = s_{\alpha_j} s_{\alpha_i}$, so $y_i|_{p_{ij}} = -s_{\alpha_j} s_{\alpha_i}(\alpha_i) = s_{\alpha_j}(\alpha_i)$.) Likewise,

$$y_{ij}|_{p_{ij}} = \begin{cases} \alpha_i \, s_{\alpha_i}(\alpha_j) & \text{if } i < j; \\ \alpha_j \, s_{\alpha_j}(\alpha_i) & \text{if } i > j. \end{cases}$$

(For $i < j$, we have $v_j' = s_{\alpha_i} s_{\alpha_j}$, and $v_i' = s_{\alpha_i}$ as noted before, so Corollary 3.4 gives $y_{ij}|_{p_{ij}} = \alpha_i \cdot s_{\alpha_i}(\alpha_j)$. If $i > j$, swap the roles of i and j.)

By substituting $\lambda_i = \alpha_i$ and restricting (5) to p_{ij}, we obtain

$$\alpha_i^2 = c_{ij} \, \alpha_j \, s_{\alpha_j}(\alpha_i) + \alpha_i^2,$$

for $i < j$, so $c_{ij} = 0$ in this case. Doing the same for $i > j$, we obtain

$$s_{\alpha_j}(\alpha_i)^2 = c_{ij} \, \alpha_j \, s_{\alpha_j}(\alpha_i) + \alpha_i \, s_{\alpha_j}(\alpha_i),$$

so $s_{\alpha_j}(\alpha_i) = c_{ij} \, \alpha_j + \alpha_i$. Since $s_{\alpha_j}(\alpha_i) = \alpha_i - \langle \alpha_i, \alpha_j^\vee \rangle \, \alpha_j$, the claim follows. □

As a consequence, we obtain a presentation for equivariant cohomology.

Corollary 4.2. *The map* $\eta_i \mapsto y_i$ *defines an isomorphism*

$$H_T^* X(\underline{\alpha}) = \Lambda[\eta_1, \ldots, \eta_d] \Big/ \left(\eta_i^2 + \sum_{j<i} \langle \alpha_i, \alpha_j^\vee \rangle \, \eta_i \eta_j - \alpha_i \, \eta_i \right)_{1 \leq i \leq d} .$$

Similar formulas determine multiplication in the $x(I)$ basis for $H_T^* X(\underline{\alpha})$.

Exercise 4.3. Writing $\beta_i = s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i)$, show that

$$x_i^2 = \sum_{j<i} (-\langle \beta_i, \beta_j^\vee \rangle) \, x_{ij} - \beta_i \, x_i ,$$

where $x_i = x(\{1, \ldots, d\} \smallsetminus \{i\})$ and $x_{ij} = x(\{1, \ldots, d\} \smallsetminus \{i, j\})$.

The equivariant cohomology of G/B embeds in that of a Bott-Samelson variety. Let $(\alpha_1, \ldots, \alpha_N)$ be a reduced word for the longest element $w_\circ$, so $f \colon X(\underline{\alpha}) \to G/B$ is birational. From the projection formula, the composition $f_* \circ f^*$ is the identity.

Corollary 4.4. *Let*

$$R = \Lambda[f^* y(s_\alpha) : \alpha \in \Delta] \subseteq H_T^* X(\underline{\alpha})$$

be the subalgebra generated by pullbacks of divisor classes. The pullback f^* *identifies* $H_T^*(G/B)$ *with the subalgebra of* $H_T^* X(\underline{\alpha})$ *consisting of elements* x *such that some integral multiple* $c \cdot x$ *lies in* R.

Proof. Using rational coefficients, we have seen that $H_T^*(G/B; \mathbb{Q})$ is generated over $\Lambda_{\mathbb{Q}} = H_T^*(\mathrm{pt}; \mathbb{Q})$ by the divisor classes $y(s_\alpha)$. (This follows from the Borel presentation given in Chapter 15, Corollary 6.6. It also follows from Chevalley's formula, which we will see in Chapter 19, §1.) Using the splitting $f_* \circ f^*$ and the fact that both $H_T^*(G/B)$ and $H_T^* X(\underline{\alpha})$ are free Λ-modules, it follows that

$$H_T^*(G/B) = H_T^*(X(\underline{\alpha})) \cap H_T^*(G/B; \mathbb{Q})$$

as submodules of $H_T^*(X(\underline{\alpha}); \mathbb{Q})$. □

5. A restriction formula for Schubert varieties

A remarkable formula for the restrictions $y(w)|_v$ was discovered by Andersen-Jantzen-Soergel, and in a different context, by Billey.

THEOREM 5.1 (ANDERSEN-JANTZEN-SOERGEL, BILLEY). *Fix a reduced word $(\alpha_1, \dots, \alpha_d)$ for $v \in W$. For any $w \in W$,*

$$y(w)|_v = \sum \beta_{i_1} \cdots \beta_{i_\ell}, \tag{6}$$

the sum over all subsets $I = \{i_1 < \cdots < i_\ell\} \subseteq \{1, \dots, d\}$ such that $\underline{\alpha}(I) = (\alpha_{i_1}, \dots, \alpha_{i_\ell})$ is a reduced word for w.

Here $\beta_i = s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i)$, as in Chapter 15, Lemma 1.6. By one of the many characterizations of Bruhat order there exists a subsequence $(\alpha_{i_1}, \dots, \alpha_{i_\ell})$ as in the theorem if and only if $w \leq v$, i.e., whenever $p_v \in Y(w)$.

Considered as a formula for $y(w)|_v$, one appealing feature is that the right-hand side is positive: the roots β_i which appear are all in R^+, and it follows that $y(w)|_v$ is nonzero whenever $v \geq w$. Another remarkable consequence of the formula is that the polynomial on the right-hand side is independent of the choice of reduced word.

We will give two proofs of this theorem: one based on the geometry of Bott-Samelson varieties, and another using induction and some algebra. We need an easy lemma.

LEMMA 5.2. *For any word $\underline{\alpha} = (\alpha_1, \dots, \alpha_d)$ and any $w \in W$, the pullback for $f\colon X(\underline{\alpha}) \to X$ is given by*

$$f^* y(w) = \sum y(I),$$

the sum over all subsets I such that $\#I = \ell(w)$ and the corresponding $\underline{\alpha}$-chain $\underline{v}$ has $v_d = w$.

PROOF. Let $\langle a, b \rangle$ denote the usual pairing in cohomology, given by pushforward of $a \cdot b$ to a point. By the projection formula, we have $\langle f^* y(w),\, x(I) \rangle = \langle y(w),\, f_* x(I) \rangle$. Since $f_* x(I) = x(v_d)$ when $X(I) \to X(v_d)$ is birational, and $f_* x(I) = 0$ otherwise, the lemma follows from Corollary 2.5. □

Remark 5.3. Applying the lemma to divisor classes, we have $f^*y(s_\alpha) = \sum y_i$, the sum over $1 \le i \le d$ such that $\alpha_i = \alpha$. Combining this with Proposition 4.1 gives a method for computing in $H_T^*(G/B)$.

First proof of Theorem 5.1. Let $f: X(\underline{\alpha}) \to X$ is the projection, and let $\underline{v} = (v_1, \dots, v_d)$ be the $\underline{\alpha}$-chain associated to $I = \{1, \dots, d\}$, so $v_i = s_{\alpha_1} \cdots s_{\alpha_i}$, and in particular $v = v_d$. Then $f(p_{\underline{v}}) = p_v$, so $y(w)|_v = (f^*y(w))|_{p_{\underline{v}}}$. By Lemma 5.2, this is $\sum y(K)|_I$, the sum over all K such that $\#K = \ell(w)$ and the corresponding $\underline{\alpha}$-chain $\underline{v}^K$ has $v_d^K = w$. On the other hand, by Corollary 3.4, we have $y(K)|_I = \prod_{i \in K}(-v_i(\alpha_i))$. Since $-v_i(\alpha_i) = \beta_i$, the theorem is proved. □

For the second proof, we use a variation on the functions ψ_v which we studied in Chapter 16. These were given by $\psi_v(w) = y(v)|_w$. Here we will use functions $\varphi_v: W \to \Lambda$, defined by

$$\varphi_v(w) = y(w)|_v = \psi_w(v).$$

Properties of these functions are immediate from the corresponding properties of ψ_w (Chapter 16, Proposition 2.5). We only need an inductive formula.

Lemma 5.4. *We have*

(7) $\varphi_v(w) = \varphi_{vs_\alpha}(w)$ *if* $\ell(ws_\alpha) > \ell(w)$;

(8) $\varphi_v(w) = \varphi_{vs_\alpha}(w) - v(\alpha)\,\varphi_{vs_\alpha}(ws_\alpha)$ *if* $\ell(ws_\alpha) < \ell(w)$.

Proof. Using the operators A_α from Chapter 16, Proposition 2.5, we have

$$\psi_w(vs_\alpha) - \psi_w(v) = v(\alpha)\,(\mathrm{A}_\alpha\psi_w)(v) = \begin{cases} 0 & \text{if } \ell(ws_\alpha) > \ell(w); \\ v(\alpha)\,\psi_{ws_\alpha}(v) & \text{if } \ell(ws_\alpha) < \ell(w). \end{cases}$$

This immediately proves (7), as well as (8) with $\varphi_v(ws_\alpha)$ appearing on the right-hand side in place of $\varphi_{vs_\alpha}(ws_\alpha)$. But by (7), we have $\varphi_v(ws_\alpha) = \varphi_{vs_\alpha}(ws_\alpha)$ (since $\ell(ws_\alpha) > \ell(ws_\alpha \cdot s_\alpha)$). □

Using the lemma, if we know the function φ_{vs_α}, for some α, then we know φ_v. For instance, we know

$$\varphi_e(w) = \begin{cases} 1 & \text{if } w = e; \\ 0 & \text{otherwise} \end{cases}$$

(since $p_e \notin Y(w)$ for $w \neq e$). This determines the rest!

Second proof of Theorem 5.1. We use induction on $\ell(v)$. For $\ell(v) = 0$, so $v = e$, this is the case observed above, so the theorem holds. In general, fix a reduced word for v as in the theorem. Set $f_v(w)$ to be the right-hand side of the formula (6), and let $\alpha = \alpha_d$. We assume the formula for φ_{vs_α} is known, using the reduced word $(\alpha_1, \ldots, \alpha_{d-1})$ for it.

If $\ell(ws_\alpha) > \ell(w)$, then no reduced word for w ends in α, and it follows that $f_v(w) = f_{vs_\alpha}(w)$. Since $\varphi_v(w) = \varphi_{vs_\alpha}(w)$ by Lemma 5.4, the formula holds in this case.

If $\ell(ws_\alpha) < \ell(w)$, then no reduced word for ws_α ends in α. Consider subsets $I = \{i_1 < \cdots < i_\ell\}$ corresponding to reduced words for w. For those I such that $i_\ell = d$, the sequence $(\alpha_{i_1}, \ldots, \alpha_{i_{\ell-1}})$ is a reduced word for ws_α, and $\beta_d = -v(\alpha) = (vs_\alpha)(\alpha)$. So the sum of such terms is

$$\sum_{I \text{ with } i_\ell = d} \beta_{i_1} \cdots \beta_{i_{\ell-1}} \beta_{i_\ell} = -v(\alpha)\, \varphi_{vs_\alpha}(ws_\alpha).$$

The other terms, where $i_\ell < d$, sum to $\varphi_{vs_\alpha}(w)$. Applying Lemma 5.4, the full sum is $\varphi_v(w)$, as required. □

Example 5.5. Theorem 5.1 includes a formula for the restrictions of divisor classes $y(s_\alpha)|_v$, as the sum of those β_i for which $\alpha_i = \alpha$. On the other hand, we saw $y(s_\alpha) = \varpi_\alpha - v(\varpi_\alpha)$ in Chapter 16, Lemma 2.6. The latter is often simpler to use in this case. For example, with $G = SL_n$ and $\alpha = t_1 - t_2$, we have

$$\varpi_\alpha - v(\varpi_\alpha) = \alpha_1 + \cdots + \alpha_{v(1)-1}$$

for any permutation $v \in S_n$, without needing to find a reduced expression.

Exercise 5.6. Check directly that the two formulas for $y(s_\alpha)|_v$ agree: show that

$$\varpi_\alpha - v(\varpi_\alpha) = \sum_{i:\alpha_i=\alpha} s_{\alpha_1}\cdots s_{\alpha_{i-1}}(\alpha_i)$$

for any simple root α, and any reduced word $(\alpha_1,\ldots,\alpha_d)$ for $v \in W$.[4]

Example 5.7. As noted above, Theorem 5.1 shows that $y(w)|_v$ is nonzero if and only if $p_v \in Y(w)$. This is a special property of the standard torus action on Schubert varieties. In general, for an invariant subvariety Y of a nonsingular variety V, with $[Y]^T \in H_T^*V$, one can have $[Y]^T|_p = 0$ for an isolated fixed point $p \in Y$.

For example, consider $V = \mathbb{P}^4$ with coordinates $x_1,\ldots,x_5$, and a torus T acting by characters $0, \chi_1, -\chi_1, \chi_2, -\chi_2$, where $\chi_1 \neq \chi_2$. Let Y be the hypersurface defined by $x_2x_3 - x_4x_5 = 0$, so $p = [1,0,0,0,0]$ is the singular point of Y. Writing $\zeta = c_1^T(\mathscr{O}(1))$, we have $[Y]^T = 2\zeta$ so $[Y]^T|_p = 0$.

Remark 5.8. As we saw in Chapter 15, Equation (9), Schubert classes in G/P pull back to Schubert classes in G/B. Writing the projection as $\pi\colon G/B \to G/P$, we have $\pi^*y[w] = y(w^{\min})$. This is compatible with restriction to fixed points, and we have

$$y[w]|_{[v]} = y(w^{\min})|_v$$

for any coset representative $v \in [v]$. In particular, Theorem 5.1 includes a formula for restricting G/P Schubert classes.

6. Duality

In Chapter 16, §4, we used an isomorphism $\Phi^w\colon G/B \xrightarrow{\sim} G/B^w$ to relate difference operators with the right W-action on G/B. The particular case where $w = w_\circ$, so $B^{w_\circ} = \dot{w}_\circ B \dot{w}_\circ^{-1} = B^-$, is especially useful for passing between formulas involving $y(w)$ and ones involving $x(w)$. Here we will state several such formulas; their proofs are all immediate from the functoriality of pullbacks.

To set up notation, let $\overline{X} = G/B^-$, with fixed points $\overline{p}_w = \dot{w}B^-$ and Schubert varieties

$$\overline{X}(w) = \overline{B^- \cdot \overline{p}_w} \quad \text{and} \quad \overline{Y}(w) = \overline{B \cdot \overline{p}_w}.$$

Let $\overline{x}(w)$ and $\overline{y}(w)$ be the corresponding Schubert classes in $H_T^*\overline{X}$. The entire discussion for Schubert classes in $\overline{X} = G/B^-$ is parallel to that of $X = G/B$, except that each root is replaced by its negative. For example,

$$\overline{y}(w)|_{\overline{p}_w} = \prod_{\beta \in w(R^-) \cap R^+} (-\beta) = (-1)^{\ell(w)}\, y(w)|_{p_w}.$$

Let $\tau\colon \Lambda \to \Lambda$ be the graded involution which is multiplication by $(-1)^r$ on $\mathrm{Sym}^r M$, so τ is induced by the involution of M taking each root to its negative. Then

$$\overline{y}(w)|_{\overline{p}_v} = \tau(y(w)|_{p_v}) \tag{9}$$

for every $w, v \in W$.

Write $\Phi = \Phi^{w_\circ}$ for the G-equivariant isomorphism $X \xrightarrow{\sim} \overline{X}$, so $\Phi(gB) = g\dot{w}_\circ B^-$. Since $\Phi(p_{ww_\circ}) = \overline{p}_w$, we see

$$\Phi(X(ww_\circ)) = \overline{Y}(w) \quad \text{and} \quad \Phi(Y(ww_\circ)) = \overline{X}(w).$$

So $\Phi^*\overline{y}(w) = x(ww_\circ)$ and $\Phi^*\overline{x}(w) = y(ww_\circ)$, and we have

$$x(w)|_{p_v} = \overline{y}(ww_\circ)|_{\overline{p}_{vw_\circ}}.$$

Combining this with (9), we obtain

$$x(w)|_{p_v} = \tau(y(ww_\circ)|_{p_{vw_\circ}}). \tag{10}$$

Next consider the automorphism $\tau_\circ = \tau_{w_\circ}\colon X \to X$, coming from the left action of W on G/B as in Chapter 16, §5. The map $\tau_\circ$ is equivariant with respect to the automorphism $\sigma\colon g \mapsto \dot{w}_\circ g \dot{w}_\circ^{-1}$ of G. Restricting σ to the torus $T \subseteq G$, in turn, induces the algebra automorphism $w_\circ\colon \Lambda \to \Lambda$ given by $\lambda \mapsto w_\circ(\lambda)$ for $\lambda \in M$. Since $\tau_\circ$ maps $p_{w_\circ w}$ to p_w, we see $\tau_\circ(X(w_\circ w)) = Y(w)$ and therefore

$$x(w_\circ w)|_{p_{w_\circ v}} = w_\circ \cdot (y(w)|_{p_v}). \tag{11}$$

Like τ, the algebra automorphism $w_\circ$ sends a product of positive roots to a product of negative roots—but in general these are different automorphisms.

Finally, the isomorphism $\Phi \circ \tau_\circ \colon X \to \overline{X}$ is equivariant with respect to the automorphism σ, and takes $Y(w_\circ w w_\circ)$ to $\overline{Y}(w)$, so

$$y(w_\circ w w_\circ)|_{p_{w_\circ v w_\circ}} = w_\circ \cdot \tau(y(w)|_{p_v}). \tag{12}$$

These identities generalize ones we have seen for Schubert polynomials in type A. For instance, Equation (12) here corresponds to Chapter 11, §8, Equation (2).

7. A nonsingularity criterion

For $v \le w$ in W, when is the Schubert variety $X(w)$ nonsingular at the fixed point $p_v \in X(w)$? We will see a criterion in terms of equivariant cohomology, due to Kumar.

We need some information about the tangent cone $C_{p_v}X(w)$. Let

$$V_v = \dot{v}U^-\dot{v}^{-1} \cdot p_v \subseteq X$$

be the T-invariant open affine neighborhood of p_v, and let

$$V(w)_v = X(w) \cap V_v$$

be the corresponding affine neighborhood in $X(w)$. We will write $V(w)_v = \operatorname{Spec} A$, and $\mathfrak{m} \subseteq A$ for the maximal ideal corresponding to $p_v \in V(w)_v$.

LEMMA 7.1. *For each $\beta \in v(R^-)$ such that $s_\beta v \le w$, there is a function $f_\beta \in A$ which is an eigenfunction of weight β for the action of T. (That is, $f_\beta(t^{-1}x) = \beta(t)\, f(x)$ for all $t \in T$ and $x \in V(w)_v$.)*

Furthermore, the f_β generate an $\mathfrak{m}$-primary ideal in A. (That is, $f_\beta(p_v) = 0$ for each β, and p_v is their only common zero.)

From the description of invariant curves we saw in Chapter 15, §4, the roots $\beta \in v(R^-)$ such that $s_\beta v \le w$ are precisely the weights of the T-invariant curves in $X(w)$ through p_v.

We will state the nonsingularity criterion in terms of the equivariant multiplicities defined in Chapter 17.

THEOREM 7.2. *For $v \leq w$, the point p_v is nonsingular in $X(w)$ if and only if*

$$\varepsilon^T_{p_v} X(w) = \prod_{\substack{\beta \in v(R^-) \\ s_\beta v \leq w}} \beta^{-1},$$

where $\varepsilon^T_v X(w)$ is the equivariant multiplicity of $X(w)$ at p_v.

PROOF. One direction is immediate. If $X(w)$ is nonsingular at p_v, the weights on $T_{p_v} X(w)$ coincide with the tangent weights to the T-invariant curves through p_v. (This is a general fact about nonsingular varieties with finitely many invariant curves; see Chapter 7, Proposition 2.3.) Therefore

$$T_{p_v} X(w) = \bigoplus_{\substack{\beta \in v(R^-) \\ s_\beta v \leq w}} \mathfrak{g}_\beta.$$

By an elementary property of equivariant multiplicities, $\varepsilon^T_v X(w)$ is the inverse of the product of tangent weights (Chapter 17, Proposition 4.4(ii)).

Conversely, assume the formula holds. Using the notation of Lemma 7.1, let $A' \subseteq A$ be the subring generated by the functions f_β. Since $\varepsilon^T_v X(w)$ has degree $-\dim X(w) = -\ell(w)$, there are $\ell(w)$ such f_β's. It follows that they form a system of parameters for A at $\mathfrak{m}$. So the subalgebra $A' \cong \mathbb{C}[\{f_\beta \mid \beta \in v(R^-),\ s_\beta v \leq w\}]$ is a polynomial ring, and A is a finitely generated module over A'.

Let $V = V(w)_v = \operatorname{Spec} A$ and $V' = \operatorname{Spec} A'$, and write $\pi \colon V \to V'$ for the corresponding equivariant map of affine varieties. Let $p' \in V'$ be the origin, and note that this is a nondegenerate fixed point, since the tangent weights β are all nonzero. Since the functions f_β are a system of parameters, we have $\pi^{-1}(p') = p_v$. It follows from another property of equivariant mulitplicities (Chapter 17, Proposition 4.4(vi)) that

$$\varepsilon^T_{p_v} V = d \cdot \varepsilon^T_{p'} V',$$

where d is the degree of the finite map π; since equivariant multiplicities are local, we have $\varepsilon^T_v X(w) = \varepsilon^T_{p_v} V$. On the other hand, $p' \in V'$ is nonsingular, with tangent weights β, so as observed above we have

$$\varepsilon^T_{p'} V' = \prod_{\substack{\beta \in v(R^-) \\ s_\beta v \leq w}} \beta^{-1}.$$

It follows that $d = 1$, so $A = A'$ is a polynomial ring, and $V \cong \mathbb{A}^{\ell(w)}$. In particular, p_v is a nonsingular point. □

The criterion may be rephrased in terms of restrictions of Schubert classes.

Corollary 7.3. *For $v \leq w$, the point p_v is nonsingular in $X(w)$ if and only if*

$$x(w)|_v = \prod_{\substack{\beta \in v(R^-) \cap R^- \\ s_\beta v \not\leq w}} \beta.$$

Proof. We have

$$\begin{aligned} x(w)|_v &= c^T_N(T_{p_v} X) \cdot \varepsilon^T_v X(w) \\ &= \left(\prod_{\beta \in v(R^-)} \beta \right) \cdot \varepsilon^T_v X(w), \end{aligned}$$

using another characterization of equivariant multiplicities (Chapter 17, §4, Equation (9)). Dividing both sides by $c^T_N(T_{p_v} X)$, the assertion follows from Theorem 7.2. (For any $\beta \in v(R^-) \cap R^-$, we have $s_\beta v < v \leq w$, so these weights cancel.) □

Using the duality identities from the previous section, it is easy to deduce corresponding nonsingularity criteria for opposite Schubert varieties $Y(w)$. Using the notation of §6, the automorphism $\tau_\circ$ sends $p_{w_\circ v}$ to p_v and $X(w_\circ w)$ to $Y(w)$, so p_v is nonsingular in $Y(w)$ if and only if $p_{w_\circ v}$ is nonsingular in $X(w_\circ w)$. We obtain the following:

Corollary 7.4. *For $v \geq w$, the point p_v is nonsingular in $Y(w)$ if and only if*

$$y(w)|_v = \prod_{\substack{\beta \in v(R^-) \cap R^+ \\ s_\beta v \not\geq w}} \beta.$$

In this case, the tangent space $T_{p_v}Y(w)$ has weights $\beta \in v(R^-)$ such that $s_\beta v \geq w$.

(Applying Equation 11, it suffices to verify that

$$\left\{\beta \in v(R^-) \,|\, s_\beta v \not\geq w\right\} = w_\circ\left(\left\{\gamma \in w_\circ v(R^-) \,|\, s_\gamma w_\circ v \not\leq w_\circ w\right\}\right),$$

which is straightforward, using $w_\circ v \leq w_\circ w$ iff $v \geq w$.)

Combining this with the restriction formula of Theorem 5.1, we arrive at a combinatorial criterion for nonsingularity of $Y(w)$ at p_v.

COROLLARY 7.5. *Fix a reduced word $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d)$ for v, and write $\beta_i = s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i)$. Then p_v is nonsingular in $Y(w)$ if and only if*

$$\sum \beta_{i_1} \cdots \beta_{i_\ell} = \prod_{\substack{\beta \in v(R^-) \cap R^+ \\ s_\beta v \not\geq w}} \beta,$$

where the sum on the left-hand side is over all $I \subseteq \{1, \ldots, d\}$ such that the corresponding subword $\underline{\alpha}(I)$ is a reduced word for w.

EXERCISE 7.6. If $\ell(v) = \ell(w)+1$, show that $p_v \in Y(w)$ is nonsingular. Conclude that Schubert varieties are nonsingular in codimension one. (That is, the singular locus has codimension at least two.)[5]

EXERCISE 7.7. For $G = SL_n$ and $\alpha = t_k - t_{k+1}$, so $s_\alpha = s_k$, show that the (opposite) Schubert variety $Y(s_k) \subseteq SL_n/B$ is singular at w if and only if $\#\{i \leq k \,|\, w(i) > k\} \geq 2$.[6]

EXERCISE 7.8. Use $\mathfrak{S}_{2143} = (x_1 - y_1)(x_1 + x_2 + x_3 - y_1 - y_2 - y_3)$ to determine the singular locus of $Y(2\,1\,4\,3) = \Omega_{2143} \subseteq Fl(\mathbb{C}^4)$.

REMARK 7.9. Using the Bott-Samelson resolution, the additivity property of equivariant multiplicities (Chapter 17, Proposition 4.4(vi)) leads to another formula for $\varepsilon^T_v X(w)$. We have

$$\varepsilon^T_v X(w) = \sum_{\underline{v}} \left(\prod_{i=1}^{\ell} (-v_i(\alpha_i))\right)^{-1}, \tag{13}$$

where $\underline{\alpha} = (\alpha_1, \ldots, \alpha_\ell)$ is a fixed reduced word for w, and the sum is over all $\underline{\alpha}$-chains $\underline{v} = (e, v_1, \ldots, v_\ell)$ such that $v_\ell = v$. (These correspond to the fixed points $p_{\underline{v}} \in X(\underline{\alpha})$ mapping to p_v under the resolution $X(\underline{\alpha}) \to X(w)$, and the corresponding term is $\varepsilon^T_{\underline{v}} X(\underline{\alpha})$.) Clearing denominators, one obtains a formula for $x(w)|_v$ which is different from the one deduced from Billey's formula. In particular, note that the chains indexing terms of the sum need not correspond to *reduced* words for v.

REMARK 7.10. As noted in Remark 5.8, knowing about Schubert varieties in G/B is enough to say something about Schubert varieties in G/P. The projection $\pi\colon G/B \to G/P$ makes $X(w^{\max}) \to X[w]$ and $Y(w^{\min}) \to Y[w]$ into fiber bundles, with nonsingular fiber P/B. So a point $p_{[v]} \in X[w]$ is nonsingular if and only if $p_v \in X(w^{\max})$ is nonsingular, for any coset representative $v \in [v]$; and similarly for $p_{[v]} \in Y[w]$. So Theorem 7.2 and Corollary 7.3 provide nonsingularity criteria for Schubert varieties in G/P.

Notes

Bott and Samelson gave a construction similar to the one indicated in Remark 3.5, and used it to study the cohomology of $G/B = K/S$ [**BoSa55**]. In particular, they prove a non-equivariant version of Corollary 4.4. The algebraic version which is more commonly used in Schubert calculus and representation theory was introduced by Demazure [**De74**] and Hansen [**Han74**], and for this reason the varieties $X(\underline{\alpha})$ are sometimes called *Bott-Samelson-Demazure-Hansen* (or *BSDH*) varieties. The non-equivariant part of the formula for x_i^2 (Exercise 4.3) appears in [**De74**, §4.2].

Corollary 3.4 was proved by Willems, using a localization argument similar to the second proof we gave [**Wi04**]. Our geometric argument, using the submanifolds $Y(I)$, appears to be new.

Theorem 5.1 appears as an exercise (without proof) in a book by Andersen, Jantzen, and Soergel [**AJS94**, p. 298]. Billey discovered the formula independently, emphasizing the connection with Schubert calculus [**Bi99**]. Her proof proceeds by decreasing induction on w, with a separate argument that the polynomial is independent of the choice of reduced word. The result is sometimes known as the *AJSB formula*.

Example 5.7 is due to Brion [**Bri00**].

Among simple linear algebraic groups, the automorphisms τ and $w_\circ$ (from §6) are equal precisely in types B_n, C_n, D_{2n}, E_7, E_8, F_4, and G_2; see, e.g., [**Hum81**, §31.6].

Theorem 7.2 is due to Kumar [**Ku96**, Theorem 5.5]. A simplified argument was given by Brion [**Bri97b**, §6.5], and this is essentially the one we use. Lemma 7.1 follows from a result of Polo [**Po94**, Prop. 2.2]; see also Kumar [**Ku02**, Prop. 5.2]. A more detailed study of the tangent cones $C_{p_v}X(w)$ has been carried out by Carrell and Peterson; see, e.g., [**Ca94**].

The formula (13) for $\varepsilon_v^T X(w)$ is due to Rossmann [**Ro89**, (3.8)].

Hints for exercises

[1]Use the subword characterization of Bruhat order, and a greedy algorithm to see that $s_{\alpha_{i_1}} \cdots s_{\alpha_{i_\ell}} \leq s_{\alpha_1} * \cdots * s_{\alpha_d}$ for any subword of $\underline{\alpha}$. See [**KnMi04**, Lemma 3.4].

[2]Consider the point $p = p_{\{1,\ldots,d\}} \in X(\underline{\alpha})$. Using terminology from Chapter 7, §2, the tangent space $T_pX(\underline{\alpha})$ contains parallel weights whenever $\underline{\alpha}$ is a non-reduced word; in this case there are infinitely many T-curves through a neighborhood of p. Whenever the sequence $\underline{\alpha}$ has a repeated root, an instance of the variety considered in Example 2.6 occurs as a subvariety of $X(\underline{\alpha})$, and this has infinitely many T-curves.

To see that $X(\underline{\alpha})$ is toric when all roots are distinct, look at the tangent space to $p_\emptyset$: the characters form part of a basis for M, so there is a dense T-orbit. To see that f is an isomorphism in this case, keep track of fixed points.

[3]Use induction on d. The same argument shows that the analogous map

$$G \times^B P_{\alpha_1} \times^B \cdots \times^B P_{\alpha_d}/B \to Z(\underline{\alpha})$$

is an isomorphism.

[4]Argue inductively as in the second proof of Theorem 5.1. It is obvious for $v = e$. Suppose the equality is known for v, and β is a simple root such that $\ell(vs_\beta) = \ell(v) + 1$. If $\beta \neq \alpha$, the right-hand sides are clearly equal for v and vs_β; since $s_\beta(\varpi_\alpha) = \varpi_\alpha$ for $\beta \neq \alpha$, so are the left-hand sides. If $\beta = \alpha$, then the difference of the right-hand sides is $v(\alpha)$, and the difference of the left-hand sides is $v(\varpi_\alpha) - vs_\alpha(\varpi_\alpha) = v(\alpha)$.

[5]The claim about $p_v \in Y(w)$ being nonsingular follows easily from Billey's formula for $y(w)|_v$. Using B-equivariance, one sees that the nonsingular locus of $Y(w)$ contains the union of Schubert cells $Y(v)^\circ$ for $v \geq w$ and $\ell(v) \leq \ell(w)+1$. (The conclusion also follows from the general fact that Schubert are normal.)

[6]Use the formula for $y(s_\alpha)|_w$ in Chapter 10, Exercise 7.2.