CHAPTER 18

Bott-Samelson varieties and Schubert varieties

Schubert varieties in $G/P$ admit explicit equivariant desingularizations by Bott-Samelson varieties. These are certain towers of $\mathbb{P}^1$-bundles, and their cohomology rings are relatively easy to compute.

In this chapter, we use the Bott-Samelson desingularization to obtain a positive formula for restricting a Schubert class to a fixed point. This, in turn, leads to a criterion for a point of a Schubert variety to be nonsingular.

1. Definitions, fixed points, and tangent spaces

Let $G \supset B \supset T$ be as usual: $G$ is a semisimple (or reductive) group, with Borel subgroup $B$ and maximal torus $T$. For each simple root $\alpha$, we have a minimal parabolic subgroup $P_\alpha$, and the corresponding projection of flag varieties is a $\mathbb{P}^1$-bundle, $G/B \to G/P_\alpha$. These spaces occur frequently in this chapter, so we will write

$$X = G/B \quad \text{and} \quad X_\alpha = G/P_\alpha$$

from now on.

For any sequence of simple roots $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d)$, we have a big Bott-Samelson variety $Z(\underline{\alpha}) = Z(\alpha_1, \ldots, \alpha_d)$, defined by

$$Z(\underline{\alpha}) = X \times_{X_{\alpha_1}} X \times_{X_{\alpha_2}} \cdots \times_{X_{\alpha_d}} X.$$ 

Since each projection $X \to X_{\alpha_i}$ is a $\mathbb{P}^1$-bundle, $Z(\underline{\alpha})$ is a tower of $\mathbb{P}^1$-bundles over $X$. In particular, it is a nonsingular projective variety of dimension $\dim X + d$. The group $G$ acts diagonally on $Z(\underline{\alpha})$, equivariantly for each projection $pr_i: Z(\underline{\alpha}) \to X$. (We index these projections from left to right by $0 \leq i \leq d$.)

**Example 1.1.** For $G = SL_n$, so $X = Fl(\mathbb{C}^n)$, a Bott-Samelson variety can be described as a sequence of flags, with the $i$th differing from
the \((i - 1)\)st only in position \(j\), if \(\alpha_i = t_j - t_{j+1}\). That is,

\[
Z(\alpha) = \left\{ (F^{(0)}\bullet, \ldots, F^{(d)}\bullet) \mid E_k^{(i)} = E_k^{(i-1)} \text{ for all } k \neq j, \right. \\
\left. \text{where } \alpha_i = t_j - t_{j+1} \right\}.
\]

When \(n = 3\), these can be represented as configurations of points and lines in \(\mathbb{P}^2\). For instance, suppose \(\alpha = t_1 - t_2\) and \(\beta = t_2 - t_3\). Then a general point of \(Z(\alpha, \beta, \alpha, \beta)\) looks like a quintuple of flags:

\[(\bullet, \bullet, \circ, -\circ, -\circ, \circ).\]

So from left to right, consecutive flags differ by moving the point, then the line, then the point, and finally the line again.

The \(T\)-fixed points of \(Z(\alpha)\) are easily described. An \(\alpha\)-chain (or simply chain) of elements of \(W\) is a sequence

\[\underline{v} = (v_0, v_1, \ldots, v_d)\]

such that for each \(i\), either \(v_i = v_{i-1}\) or \(v_i = v_{i-1} \cdot s_{\alpha_i}\).

**Exercise 1.2.** Show that the \(T\)-fixed points of \(Z(\alpha)\) are the \(2^d \cdot |W|\) points

\[Z(\alpha)^T = \{p_{\underline{v}} = (p_{v_0}, p_{v_1}, \ldots, p_{v_d})\}\]

where each \(\underline{v}\) is an \(\alpha\)-chain.

The (small) Bott-Samelson variety is the fiber \(X(\underline{\alpha}) = p_{r^{-1}}(p_e)\), that is,

\[X(\underline{\alpha}) = \{p_e\} \times_{X_{\alpha_1}} X \times_{X_{\alpha_2}} \cdots \times_{X_{\alpha_d}} X.
\]

The projection \(X(\alpha_1, \ldots, \alpha_d) \to X(\alpha_1, \ldots, \alpha_{d-1})\) is a \(\mathbb{P}^1\)-bundle, so \(X(\underline{\alpha})\) is a nonsingular projective variety of dimension \(d\). Since \(p_e\) is fixed by \(B\), the Bott-Samelson variety \(X(\underline{\alpha})\) comes with an action of \(B\) (but not \(G\), in general).

The Bott-Samelson variety \(X(\underline{\alpha})\) has \(2^d\) \(T\)-fixed points \(p_{\underline{v}'}\) for chains \(\underline{v} = (e, v_1, \ldots, v_d)\). We will index these in two ways: using the chain \(\underline{v}\) and using the subset \(I = \{i_1 < \cdots < i_\ell\} \subseteq \{1, \ldots, d\}\) defined by

\[I = \left\{i \mid v_i = v_{i-1} \cdot s_{\alpha_i} \right\}.
\]
We often use the notation interchangeably, writing $p_{\underline{\alpha}} = p_I$. Sometimes we write $I = I^\alpha$ and $\underline{\nu} = \underline{v}^I$ to indicate the bijection between chains and subsets.

For each subset $I \subseteq \{1, \ldots, d\}$, there is a $B$-invariant subvariety $X(I) \subseteq X(\underline{\alpha})$, defined by

$$X(I) = \{ (x_1, \ldots, x_d) \in X(\underline{\alpha}) \mid x_j = x_{j-1} \text{ for } j \notin I \}.$$ 

In fact, this is canonically isomorphic to another Bott-Samelson variety. Each subset $I = \{i_1 < \cdots < i_\ell\}$ corresponds to a subword $\underline{\alpha}(I) = (\alpha_{i_1}, \ldots, \alpha_{i_\ell})$, and we have

$$X(I) \cong X(\underline{\alpha}(I)).$$

(Use a diagonal embedding of $X^{\ell+1}$ in $X^{d+1}$.) Containment among these subvarieties corresponds to containment of subsets:

$$X(J) \subseteq X(I) \iff J \subseteq I.$$ 

For example, $X(\{1, \ldots, d\}) = X(\underline{\alpha})$, and $X(\emptyset)$ is the point $p_{\emptyset}$.

Each $X(I)$ is the closure of a locally closed set $X(I)\circ$, consisting of the points where $x_i \neq x_{i-1}$ for $i \in I$. In fact, these are cells.

**Lemma 1.3.** We have $X(I)\circ \cong \mathbb{A}^\ell$, where $\ell = \#I$. 

**Proof.** It suffices to consider $I = \{1, \ldots, d\}$. Here one has the $\mathbb{P}^1$-bundle $X(\alpha_1, \ldots, \alpha_d) \to X(\alpha_1, \ldots, \alpha_{d-1})$. The complement of the locus where $x_{d-1} = x_d$ is an $\mathbb{A}^1$-bundle over $X(\alpha_1, \ldots, \alpha_{d-1})$, so the claim follows by induction on $d$. \qed

The subvarieties $X(I)$ therefore determine a cell decomposition of $X(\underline{\alpha})$, and their classes $x(I) = [X(I)]^T$ form a basis for $H^*_T X(\underline{\alpha})$, as $I$ varies over subsets of $\{1, \ldots, d\}$. It also follows that

$$p_I \in X(I) \iff J \subseteq I.$$ 

We will need a description of the tangent spaces.

**Lemma 1.4.** Let $\underline{\nu} = (\nu, \nu_1, \ldots, \nu_d)$ be an $\underline{\alpha}$-chain. The torus weights on $T_{p_{\underline{\alpha}}} X(\underline{\alpha})$ are $-\nu_1(\alpha_1), \ldots, -\nu_d(\alpha_d)$.

More generally, for $K \subseteq I$, with corresponding chains $\underline{\nu}^K$ and $\underline{\nu}^I$, the weights on $T_{p_K} X(I)$ are $-\nu^K_i(\alpha_i)$ for $i \in I$. 

Proof. We will find the weights at any fixed point of the big Bott-Samelson variety. For a chain $\underline{v} = (v_0, v_1, \ldots, v_d)$, consider the point $p = p_{\underline{v}} \in Z(\underline{a})$. The tangent space to $Z(\underline{a})$ at $p$ is the fiber product of vector spaces

$$T_{p_0}X \times_{T_{p[1]}X_{\alpha_1}} T_{p_1}X \times_{T_{p[2]}X_{\alpha_2}} \cdots \times_{T_{p[d]}X_{\alpha_d}} T_{p_d}X,$$

where we have written $p_i = p_{v_i} \in X$ and $p[i] = p[v_i] \in X_{\alpha_i}$ to economize on subscripts. (Note $[v_i] = [v_{i-1}]$ for each $i$, since $\underline{v}$ is an $\underline{a}$-chain.) We have seen descriptions of each of these spaces in Chapter 15. The weights are $v_0(R^-)$, from the first factor, together with weights $-v_i(\alpha_i)$ for $1 \leq i \leq d$, since $g_{-v_i(\alpha_i)}$ is the kernel of $T_{p_i}X \to T_{p[i]}X_{\alpha_i}$.

When $v_0 = e$, the variety $X(\underline{a})$ is the fiber over $p_e$ in the first factor, so the weights $R^- = v_0(R^-)$ are omitted, proving the first claim. The second claim follows from the first, using $X(I) \cong X(\alpha_i, \ldots, \alpha_{i_\ell})$. \qed

### 2. Desingularizations of Schubert varieties

Let $f : X(\underline{a}) \to X$ be the projection onto the last factor; that is, $f$ is the restriction of $p_{d'} : Z(\underline{a}) \to X$. For each $I \subseteq \{1, \ldots, d\}$, with corresponding $\underline{a}$-chain $\underline{v} = (e, v_1, \ldots, v_d)$, we have $f(p_I) = p_{v_d}$. The subset $I$ corresponds to the subword $(\alpha_{i_1}, \ldots, \alpha_{i_\ell})$ of $\underline{a}$, and

$$v_d = s_{\alpha_{i_1}} \cdots s_{\alpha_{i_\ell}}.$$

Since $f$ is proper and $B$-equivariant, $f(X(I))$ contains the Schubert variety $X(v_d) \subseteq X$. However, if $(\alpha_{i_1}, \ldots, \alpha_{i_\ell})$ is not a reduced word for $v_d$, the image of $f$ may be larger.

**Lemma 2.1.** Let $\underline{a} = (\alpha_1, \ldots, \alpha_d)$ be a sequence of simple roots. The set of products $s_{\alpha_{i_1}} \cdots s_{\alpha_{i_\ell}}$ over subwords contains a unique maximum element $w(\underline{a}) \in W$ in Bruhat order, and

$$f(X(\underline{a})) = X(w(\underline{a})).$$

We have $w(\underline{a}) = s_{\alpha_1} \cdots s_{\alpha_d}$ if and only if the word $\underline{a}$ is reduced.
Proof. Since $X(\underline{\alpha})$ is irreducible, the image of the $B$-equivariant morphism $f : X(\underline{\alpha}) \to X$ must be some Schubert variety $X(w)$. It follows that $w = w(\underline{\alpha})$ satisfies the asserted properties. \hfill \Box

In fact, the maximal element $w(\underline{\alpha})$ can be easily computed. Let “*” be the associative product on $W$ defined by

$$w * s_\alpha = \begin{cases} 
ws_\alpha & \text{if } \ell(ws_\alpha) > \ell(w); \\
 w & \text{otherwise}.
\end{cases}$$

This product is called the Demazure product.

**Exercise 2.2.** Show that $w(\underline{\alpha}) = s_{\alpha_1} * \cdots * s_{\alpha_d}$, i.e., it is the Demazure product of reflections from $\underline{\alpha}$. \hfill \Box

**Lemma 2.3.** The map $f : X(\underline{\alpha}) \to X(w)$ is birational if and only if $\alpha$ is a reduced word for $w = w(\underline{\alpha})$.

**Proof.** If $\underline{\alpha}$ is not a reduced word, then $w(\underline{\alpha})$ is the product of reflections for a proper subword, so it has length $\ell(w(\underline{\alpha})) < d$. In this case, $f$ cannot be birational by dimension.

If $\underline{\alpha}$ is reduced, then $w = w(\underline{\alpha}) = s_{\alpha_1} \cdots s_{\alpha_d}$, and $f(p_{\{1,\ldots,d\}}) = p_w$. The map $f : X(\underline{\alpha})^\circ \to X(w)^\circ$ is $B$-equivariant, and therefore also equivariant for the subgroup $U(w) = \tilde{w}U\tilde{w}^{-1} \cap U$. Since the map $u \mapsto u \cdot p_w$ defines an isomorphism $U(w) \tilde{w} \to X(w)^\circ$, it follows that $f : X(\underline{\alpha})^\circ \to X(w)^\circ$ is an isomorphism. \hfill \Box

For a reduced word $\underline{\alpha}$, one can also establish the birationality of $f : X(\underline{\alpha}) \to X(w)$ by examining tangent weights. The tangent space to $X(\underline{\alpha})$ at $p = p_{\{1,\ldots,d\}}$ has weights

$$\alpha_1, s_{\alpha_1}(\alpha_2), \ldots, s_{\alpha_1} \cdots s_{\alpha_{d-1}}(\alpha_d),$$

using Lemma 1.4, for $v_i = s_{\alpha_1} \cdots s_{\alpha_i}$. These are precisely the weights on $T_{p_w} X(w)$ (see Chapter 15, Lemma 2.2).

Given a Schubert variety $X(w) \subseteq G/B$, one obtains a $B$-equivariant desingularization $f : X(\underline{\alpha}) \to X(w)$ by choosing a reduced word for $w$. For a parabolic subgroup $P$, the projection $G/B \to G/P$ maps $X(w^{\min})$ birationally onto $X[w]$, so we obtain desingularizations of these varieties, too.
Corollary 2.4. For a Schubert variety $X[w] \subseteq G/B$, and any reduced word $\underline{w}$ for $w^\min$, one obtains a desingularization $X(\underline{w}) \to X[w]$ by composing $f$ with the projection $G/B \to G/P$. \hfill \Box

These statements have evident analogues for the subvarieties $X(I) \subseteq X(\underline{w})$. If $I$ is a subset, with subword $\underline{w}(I)$, we will write $w(I) = w(\underline{w}(I))$ for the corresponding Demazure product.

Corollary 2.5. Let $I$ be a subset, and let $\underline{v} = (v_1, \ldots, v_d)$ be the corresponding chain. The following are equivalent:

(i) The map $X(I) \to X(w(I))$ is birational.

(ii) $w(I) = v_d$.

(iii) $\ell(v_d) = \# I$.

(iv) The subword $\underline{w}(I)$ is a reduced word for $v_d$. \hfill \Box

Example 2.6. Let $\underline{\alpha} = (\alpha, \alpha)$, for some simple root $\alpha$. Then $X(\underline{\alpha})$ is isomorphic to $\mathbb{P}^1 \times \mathbb{P}^1$. The Demazure product is $s_\alpha * s_\alpha = s_\alpha$, and the map $f: X(\alpha, \alpha) \to X(s_\alpha)$ is identified with the second projection $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1$. The subvarieties $X(I) = X(\underline{v})$ are

\[
\begin{align*}
X(\{1, 2\}) &= X(s_\alpha, e) = X(\underline{\alpha}), \\
X(\{1\}) &= X(s_\alpha, s_\alpha) = \delta(\mathbb{P}^1) \text{ (the diagonal in } \mathbb{P}^1 \times \mathbb{P}^1), \\
X(\{2\}) &= X(e, s_\alpha) = \{p_e\} \times \mathbb{P}^1, \text{ and} \\
X(\emptyset) &= X(e, e) = \{(p_e, p_e)\}.
\end{align*}
\]

While $X(\underline{\alpha})$ always has finitely many fixed points, it often has infinitely many invariant curves—even when $\underline{\alpha}$ is a reduced word.

Exercise 2.7. The following are equivalent, for a sequence of simple roots $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d)$:

(a) $X(\underline{\alpha})$ has finitely many $T$-curves.

(b) The roots $\alpha_1, \ldots, \alpha_d$ are distinct.

(c) $X(\underline{\alpha})$ is a toric variety for the quotient of $T$ whose character lattice has basis $\alpha_1, \ldots, \alpha_d$.

(d) The map $f: X(\underline{\alpha}) \to X(w)$ is an isomorphism.
Another construction of the Bott-Samelson variety $X(\underline{\alpha})$ is sometimes useful.

**Proposition 2.8.** For a word $\underline{\alpha} = (\alpha_1, \ldots, \alpha_d)$, there is an isomorphism

$$P_{\alpha_1} \times^B P_{\alpha_2} \times^B \cdots \times^B P_{\alpha_d} / B \to X(\underline{\alpha}),$$

given by $[p_1, \ldots, p_d] \mapsto (eB, p_1B, p_1p_2B, \ldots, p_1 \cdots p_dB)$. This is $B$-equivariant, where $B$ acts via left multiplication on $P_{\alpha_1}$. The subvarieties $X(I) \subseteq X(\underline{\alpha})$ are identified with

$$X(I) = \{[p_1, \ldots, p_d] | p_iB = eB \text{ for } i \notin I\},$$

and the point $p_I$ corresponds to $[\varepsilon_1, \ldots, \varepsilon_d]$, where $\varepsilon_i = \dot{e}$ for $i \in I$, and $\varepsilon_j = \dot{s}_{\alpha_i}$ for $j \notin I$.

**Exercise 2.9.** Prove the proposition.

**Remark 2.10.** Bott-Samelson varieties appear in the geometric construction of divided difference operators described in Chapter 16, §1. Let $\underline{\alpha}$ be a reduced word for $w$. The big Bott-Samelson variety $Z(\underline{\alpha})$ maps birationally to the double Schubert variety $Z(w)$ maps birationally to the double Schubert variety

$$Z(w) = G \cdot (p_e, p_w) \subseteq X \times X$$

via the projection $pr_0 \times pr_d$. Using Chapter 16, Proposition 1.2, the operator $D_{w^{-1}}$ on $H^*_X X$ is identified with $pr_d \cdot pr_0^*$.

On the other hand, these projections factor as iterated $\mathbb{P}^1$-bundles, and the diagram

\[
\begin{array}{ccc}
Z(\alpha_1) & \xrightarrow{\cdots} & Z(\alpha_d) \\
\downarrow & & \downarrow \\
X & \xrightarrow{\cdots} & X
\end{array}
\]

shows that $D_{w^{-1}} = D_{\alpha_d} \circ \cdots \circ D_{\alpha_1}$ is independent of the choice of reduced word. One can also see this by restricting the diagram

\[
\begin{array}{ccc}
Z(\underline{\alpha}) & \xrightarrow{\cdots} & Z(\underline{\alpha}) \\
\downarrow & & \downarrow \\
X & \xrightarrow{\cdots} & X
\end{array}
\]

(Use the description of weights on tangent spaces.)

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to the fiber $pr_0^{-1}(p_e)$, obtaining

$$
\begin{array}{c}
X(\underline{\alpha}) \\
p_e
\end{array}
\xleftarrow{f}
\begin{array}{c}
X(w).
\end{array}
$$

Since $f$ is birational, we have

$$
D_{w^{-1}}(x(e)) = pr_* pr^*_0(x(e)) = f_* [X(\underline{\alpha})]^T = [X(w)]^T = x(w).
$$

### 3. Poincaré duality and restriction to fixed points

We have seen that the classes $x(I) = [X(I)]^T$ form a $\Lambda$-module basis for $H^*_TX(\underline{\alpha})$. Next we will study their restrictions to fixed points, and determine the Poincaré dual basis.

Lemma 1.4 leads directly to a description of weights at the fixed points of $X(I) \subseteq X(\underline{\alpha})$. Suppose $K \subseteq I$, so $p_K \in X(I)$, and let $\overline{v^K} \, \overline{v^I}$ be the corresponding chains. The weights on $T_{p_K}X(I)$ are $-\overline{v^K_i}(\alpha_i)$ for $i \in I$. This, in turn, gives a formula for restricting the classes $x(I) = [X(I)]^T$. For any $x \in H^*_TX(\underline{\alpha})$, its restriction to the fixed point $p_I$ is denoted $x|_I$.

**Corollary 3.1.** We have

$$
x(I)|_K = \begin{cases}
\prod_{j \not\in I} v^K_j(-\alpha_j) & \text{if } K \subseteq I; \\
0 & \text{otherwise},
\end{cases}
$$

Let $\{y(I)\}$ be the Poincaré dual basis to $\{x(I)\}$, meaning that $\rho_*(x(I) \cdot y(J)) = \delta_{I,J}$ in $\Lambda$, where $\rho : X(\underline{\alpha}) \to \text{pt}$ is the projection. As we saw in Chapter 4, §6, such a basis always exists. It is natural to look for invariant subvarieties $Y(I)$ representing these Poincaré dual classes. However, no such algebraic subvarieties exist!
Example 3.2. Consider the variety $X(\alpha, \alpha) \cong \mathbb{P}^1 \times \mathbb{P}^1$ from Example 2.6. The basis $\{x(I)\}$ consists of the equivariant classes of

$$
x(\emptyset) = [(p_e, p_e)]^T, \quad x(\{1\}) = [\delta(\mathbb{P}^1)]^T, \quad x(\{2\}) = [(p_e) \times \mathbb{P}^1]^T, \quad \text{and} \quad x(\{1, 2\}) = [\mathbb{P}^1 \times \mathbb{P}^1]^T.
$$

Even non-equivariantly, the Poincaré dual basis cannot be represented by algebraic subvarieties: the class $y(\{2\})$ must have zero intersection with the diagonal class $x(\{1\})$, and no algebraic curve in $\mathbb{P}^1 \times \mathbb{P}^1$ can do this.

Another way of phrasing the conclusion of Example 3.2 is this: we seek a curve $Y(\{2\}) \subseteq \mathbb{P}^1 \times \mathbb{P}^1$ which consists of pairs $(L, L')$ of lines in $\mathbb{C}^2$ such that $L \neq L'$—but the complement of the diagonal is affine, so it contains no complete curves. In fact, this observation indicates a solution. Using the standard Hermitian metric on $\mathbb{C}^2$, we may consider pairs of perpendicular lines $(L, L')$; in terms of a coordinate $z$ on $\mathbb{P}^1$, this is the set of pairs $(z, -1/z)$. This set is a non-algebraic submanifold $Y(\{2\}) \subseteq \mathbb{P}^1 \times \mathbb{P}^1$, which we orient by projecting onto the first factor. (Projection onto the second factor would give the opposite orientation, as the coordinate description shows.) Fixing the metric amounts to reducing $GL_2$ to the maximal compact subgroup $U(2)$, and identifying $\mathbb{P}^1 = GL_2/B$ with $U(2)/(T \cap U(2))$.

The general situation is similar: we construct (non-algebraic) submanifolds $Y(I) \subseteq X(\alpha)$ whose classes represent the Poincaré dual classes $y(I)$. Let $K \subseteq G$ be a maximal compact subgroup, with maximal compact torus $S = T \cap K$, so we have a diffeomorphism $K/S \cong G/B$, and the Weyl group $W = N_K(S)/S$ acts on the right. For $I \subseteq \{1, \ldots, d\}$, we define

$$
Y(I) = \{(e, x_1, \ldots, x_d) \in X(\alpha) \mid x_i = x_{i-1} \cdot s_{\alpha_i} \text{ for } i \in I\}.
$$

This is a $C^\infty$ submanifold, of real codimension $2 \cdot \# I$ in $X(\alpha)$, invariant for the action of the compact torus $S$. Containment among these
submanifolds reverses containment of subsets:

\[ Y(K) \subseteq Y(I) \quad \text{iff} \quad p_K \in Y(I) \quad \text{iff} \quad K \supseteq I. \]

**Lemma 3.3.** Giving each \( Y(I) \) an appropriate orientation (to be specified in the proof), the classes \( y(I) = [Y(I)]^S \) form the Poincaré dual basis to \( x(I) \).

For \( K \supseteq I \), with corresponding \( \alpha \)-chains \( v^K \) and \( v^I \), the normal space to \( Y(I) \subseteq X(\alpha) \) at the fixed point \( p_K \) has characters \( -v^K_i(\alpha_i) \), for \( i \in I \).

**Proof.** To compute the tangent spaces of \( Y(I) \), and to orient it, we work from the left, using induction on \( d \). For \( d = 1 \), we have \( Y(\{1\}) = \{s_d B\} \) (a point), and \( Y(\emptyset) = X(\alpha) = \mathbb{P}^1 \), so these are already oriented. Proceeding inductively, consider the projection \( X(\alpha_1, \ldots, \alpha_d) \to X(\alpha_1, \ldots, \alpha_{d-1}) \). If \( d \in I \), this induces an isomorphism \( Y(I) \to Y(I \setminus \{d\}) \). Otherwise, if \( d \notin I \), it induces a \( \mathbb{P}^1 \)-bundle, so there is a fiber square

\[
\begin{array}{ccc}
Y(I) & \leftarrow & X(\alpha_1, \ldots, \alpha_d) \\
\downarrow & & \downarrow \\
Y(\overline{I}) & \leftarrow & X(\alpha_1, \ldots, \alpha_{d-1}),
\end{array}
\]

where we have written \( \overline{I} = I \) as a subset of \( \{1, \ldots, d - 1\} \). By the inductive assumption, we have an orientation of \( Y(\overline{I}) \). The canonical orientation of the \( \mathbb{P}^1 \) fiber then induces an orientation of \( Y(I) \).

This construction also identifies the tangent spaces: assume \( d \notin I \), and for \( K \supseteq I \), write \( p = p_K \) and \( \overline{p} \) for the image of this point in \( Y(\overline{I}) \). The kernel of

\[ T_p Y(I) \to T_{\overline{p}} Y(\overline{I}) \]

is \( g_\beta \), where \( \beta = -v^K_d(\alpha_d) \).

It follows that \( Y(I) \) meets \( X(I) \) transversally in the point \( p_I \). Indeed, we have weight decompositions of the tangent spaces as

\[ T_{p_i} X(I) = \bigoplus_{i \notin I} g_{-v^I_i(\alpha_i)} \]
and

$$T_{p_I}Y(I) = \bigoplus_{i \in I} g_{-v_i^I(\alpha_i)}.$$ 

So these are complementary subspaces of $T_{p_I}X(\alpha)$. By considering fixed points, we see $X(I) \cap Y(J) = \emptyset$ unless $J \subseteq I$, and it follows that the classes $x(I)$ and $y(J)$ form Poincaré dual bases. □

This description of tangent spaces proves a formula for restricting the classes $y(I)$.

**Corollary 3.4.** We have

$$y(I)|_K = \begin{cases} 
\prod_{i \in I} v_i^K(-\alpha_i) & \text{if } K \supseteq I; \\
0 & \text{otherwise.}
\end{cases}$$

A more algebraic proof of Corollary 3.4 uses the localization formula. The dual classes $y(I)$ are uniquely determined by

$$\sum_{p_K \in X(J)} \frac{y(I)|_K}{c_{top}^T(T_{p_K}X(J))} = \delta_{I,J},$$

for every subset $J \subseteq \{1, \ldots, d\}$. We know $p_K \in X(J)$ iff $K \subseteq J$, and in this case $c_{top}^T(T_{p_K}X(J)) = \prod_{j \in J}(-v_j^K(\alpha_j))$. To prove the claimed formula for $y(I)|_K$, it remains to establish the identity

$$\sum_{K: I \subseteq K \subseteq J} \frac{1}{\prod_{j \in J \setminus I}(-v_j^K(\alpha_j))} = \delta_{I,J}.$$  

This is clear if $I = J$, or if $I \notin J$. When $I \subseteq J$, the terms cancel in pairs, as follows. Suppose $j$ is the largest index in $J \setminus I$; then for each $K \ni j$, there is $K' = K \cup \{j\}$, and the corresponding terms cancel. (Indeed, $s_{\alpha_j}(\alpha_j) = -\alpha_j$, so $v_j^{K'}(\alpha_j) = -v_j^K(\alpha)$ and the other factors in the product are equal.)

**Remark 3.5.** The identification $X = G/B = K/S$ leads to a third description of the Bott-Samelson varieties. Each $K_\alpha = K \cap P_{\alpha}$ is a maximal compact subgroup of the minimal parabolic $P_{\alpha}$, and the evident map

$$K_{\alpha_1} \times S K_{\alpha_2} \times S \cdots \times S K_{\alpha_d}/S \to P_{\alpha_1} \times B K_{\alpha_2} \times B \cdots \times B K_{\alpha_d}/B$$
is a diffeomorphism. The submanifolds \( Y(I) \subseteq X(\alpha) \) are easy to identify from this point of view:

\[
Y(I) = \{ [k_1, \ldots, k_d] | k_i S = s_{\alpha_i} S \text{ for } i \in I \}.
\]

For the corresponding projection \( f : X(\alpha) \to X \), one sees

\[
f(Y(\{1, \ldots, k\})) = s_{\alpha_1} \cdots s_{\alpha_k} \cdot X(s_{\alpha_{k+1}} \cdots s_{\alpha_d})
\]

and

\[
f(Y(\{k + 1, \ldots, d\})) = X(s_{\alpha_1} \cdots s_{\alpha_k}) \cdot s_{\alpha_{k+1}} \cdots s_{\alpha_d},
\]

where \( w \cdot X(v) \) and \( X(v) \cdot w \) denote the translations of Schubert varieties by the left and right \( W \)-actions.

### 4. A presentation for the cohomology ring

Multiplication in the basis \( y(I) \) is particularly easy. To simplify the notation, we will write \( p_i = p_{\{i\}} \), \( p_{ij} = p_{\{i,j\}} \), \( y_i = y(\{i\}) \), and \( y_{ij} = y(\{i,j\}) \).

If \( I \cap J = \emptyset \), then \( Y(I) \) and \( Y(J) \) meet transversally in \( Y(I \cup J) \), so

\[
y(I) \cdot y(J) = y(I \cup J) \quad \text{if } I \cap J = \emptyset.
\]

In particular, \( y_i \cdot y_j = y_{ij} \) if \( i \neq j \), and \( y(I) = y_{i_1} \cdots y_{i_\ell} \) if \( I = \{i_1, \ldots, i_\ell\} \).

To determine the structure of \( H^*_T X(\alpha) \), it suffices to give a formula for \( y_i^2 \).

**Proposition 4.1.** We have

\[
y_i^2 = \sum_{j < i} (-\langle \alpha_i, \alpha_j^\vee \rangle) y_{ij} + \alpha_i y_i,
\]

where \( \langle \alpha, \beta^\vee \rangle \) is the pairing between roots and coroots.

**Proof.** By considering degrees and support, we have

\[
y_i^2 = \sum_{j \neq i} c_{ij} y_{ij} + \lambda_i y_i,
\]

for some \( c_{ij} \in \mathbb{Z} \) and \( \lambda_i \in M \). (Since \( p_j \notin Y(\{i\}) \) for \( j \neq i \), we have \( y_i | p_j = 0 \), so the classes \( y_j \) do not appear. Similarly, \( p_\emptyset \notin Y(\{i\}) \), so there is no “constant” term of degree 2 in \( \Lambda \).) So we must determine these coefficients.
Using the restriction formula from Corollary 3.4, we have

\[ y_i |_{p_i} = -v'_i(\alpha_i) = \alpha_i, \]

where the chain corresponding to \( \{i\} \) is \( v' = (e, \ldots, e, s_{\alpha_i}, \ldots, s_{\alpha_i}) \). Since \( p_i \notin Y(\{i, j\}) \) for \( j \neq i \), restricting Equation (5) to this point gives

\[ (\alpha_i)^2 = \lambda_i \alpha_i, \]

and it follows that \( \lambda_i = \alpha_i \).

Similarly, we have

\[ y_i |_{p_{ij}} = \begin{cases} \alpha_i & \text{if } i < j; \\ s_{\alpha_i}(\alpha_i) & \text{if } i > j. \end{cases} \]

(When \( i < j \), the chain \( v' \) corresponding to \( \{i, j\} \) has \( v'_i = s_{\alpha_i}, \) so \( y_i |_{p_{ij}} = -s_{\alpha_i}(\alpha_i) = \alpha_i \). For \( i > j \), the chain has \( v'_i = s_{\alpha_i} s_{\alpha_i}, \) so \( y_i |_{p_{ij}} = -s_{\alpha_i} s_{\alpha_i}(\alpha_i) = s_{\alpha_i}(\alpha_i) \).) Likewise,

\[ y_{ij} |_{p_{ij}} = \begin{cases} \alpha_i s_{\alpha_j}(\alpha_j) & \text{if } i < j; \\ \alpha_j s_{\alpha_i}(\alpha_i) & \text{if } i > j. \end{cases} \]

(For \( i < j \), we have \( v'_j = s_{\alpha_i} s_{\alpha_i}, \) and \( v'_i = s_{\alpha_i} \) as noted before, so Corollary 3.4 gives \( y_{ij} |_{p_{ij}} = \alpha_i \cdot s_{\alpha_i}(\alpha_j) \). If \( i > j \), swap the roles of \( i \) and \( j \).)

By substituting \( \lambda_i = \alpha_i \) and restricting (5) to \( p_{ij} \), we obtain

\[ \alpha_i^2 = c_{ij} \alpha_j s_{\alpha_j}(\alpha_i) + \alpha_i^2, \]

for \( i < j \), so \( c_{ij} = 0 \) in this case. Doing the same for \( i > j \), we obtain

\[ s_{\alpha_j}(\alpha_i)^2 = c_{ij} \alpha_j s_{\alpha_j}(\alpha_i) + \alpha_i s_{\alpha_j}(\alpha_i), \]

so \( s_{\alpha_j}(\alpha_i) = c_{ij} \alpha_j + \alpha_i \). Since \( s_{\alpha_j}(\alpha_i) = \alpha_i - \langle \alpha_i, \alpha_j^\vee \rangle \alpha_j \), the claim follows. \( \square \)

As a consequence, we obtain a presentation for equivariant cohomology.
§4. A presentation for the cohomology ring

Corollary 4.2. The map \( \eta_i \mapsto y_i \) defines an isomorphism

\[
H^*_T X(\underline{\alpha}) = \Lambda[\eta_1, \ldots, \eta_d]/\left( \eta_i^2 + \sum_{j<i} \langle \alpha_i, \alpha_j \rangle \eta_i \eta_j - \alpha_i \eta_i \right)_{1 \leq i \leq d}
\]

Similar formulas determine multiplication in the \( x(I) \) basis for \( H^*_T X(\underline{\alpha}) \).

Exercise 4.3. Writing \( \beta_i = s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i) \), show that

\[
x_i^2 = \sum_{j<i} \left( -\langle \beta_i, \beta_j \rangle \right) x_{ij} - \beta_i x_i,
\]

where \( x_i = x(\{1, \ldots, d\} \setminus \{i\}) \) and \( x_{ij} = x(\{1, \ldots, d\} \setminus \{i, j\}) \).

The equivariant cohomology of \( G/B \) embeds in that of a Bott-Samelson variety. Let \( (\alpha_1, \ldots, \alpha_N) \) be a reduced word for the longest element \( w_o \), so \( f : X(\underline{\alpha}) \to G/B \) is birational. From the projection formula, the composition \( f^* \circ f^* \) is the identity.

Corollary 4.4. Let

\[
R = \Lambda[ f^* y(s_\alpha) : \alpha \in \Delta ] \subseteq H^*_T X(\underline{\alpha})
\]

be the subalgebra generated by pullbacks of divisor classes. The pullback \( f^* \) identifies \( H^*_T (G/B) \) with the subalgebra of \( H^*_T X(\underline{\alpha}) \) consisting of elements \( x \) such that some integral multiple \( c \cdot x \) lies in \( R \).

Proof. Using rational coefficients, we have seen that \( H^*_T (G/B; \mathbb{Q}) \) is generated over \( \Lambda_\mathbb{Q} = H^*_T (\text{pt}; \mathbb{Q}) \) by the divisor classes \( y(s_\alpha) \). (This follows from the Borel presentation given in Chapter 15, Corollary 6.6. It also follows from Chevalley’s formula, which we will see in Chapter 19, §1.) Using the splitting \( f^* \circ f^* \) and the fact that both \( H^*_T (G/B) \) and \( H^*_T X(\underline{\alpha}) \) are free \( \Lambda \)-modules, it follows that

\[
H^*_T (G/B) = H^*_T (X(\underline{\alpha})) \cap H^*_T (G/B; \mathbb{Q})
\]

as submodules of \( H^*_T (X(\underline{\alpha}); \mathbb{Q}) \). \( \square \)
5. A restriction formula for Schubert varieties

A remarkable formula for the restrictions \( y(w)|_v \) was discovered by Andersen-Jantzen-Soergel, and in a different context, by Billey.

**Theorem 5.1 (Andersen-Jantzen-Soergel, Billey).** Fix a reduced word \((\alpha_1, \ldots, \alpha_d)\) for \( v \in W \). For any \( w \in W \),

\[
y(w)|_v = \sum \beta_i \cdots \beta_{i_\ell},
\]

the sum over all subsets \( I = \{i_1 < \cdots < i_\ell\} \subseteq \{1, \ldots, d\} \) such that \( \underline{\alpha}(I) = (\alpha_{i_1}, \ldots, \alpha_{i_\ell}) \) is a reduced word for \( w \).

Here \( \beta_i = s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i) \), as in Chapter 15, Lemma 1.6. By one of the many characterizations of Bruhat order there exists a subsequence \((\alpha_{i_1}, \ldots, \alpha_{i_\ell})\) as in the theorem if and only if \( w \leq v \), i.e., whenever \( p_v \in Y(w) \).

Considered as a formula for \( y(w)|_v \), one appealing feature is that the right-hand side is positive: the roots \( \beta_i \) which appear are all in \( R^+ \), and it follows that \( y(w)|_v \) is nonzero whenever \( v \geq w \). Another remarkable consequence of the formula is that the polynomial on the right-hand side is independent of the choice of reduced word.

We will give two proofs of this theorem: one based on the geometry of Bott-Samelson varieties, and another using induction and some algebra. We need an easy lemma.

**Lemma 5.2.** For any word \( \underline{\alpha} = (\alpha_1, \ldots, \alpha_d) \) and any \( w \in W \), the pullback for \( f : X(\underline{\alpha}) \to X \) is given by

\[
f^* y(w) = \sum y(I),
\]

the sum over all subsets \( I \) such that \( \#I = \ell(w) \) and the corresponding \( \underline{\alpha} \)-chain \( \underline{v} \) has \( v_d = w \).

**Proof.** Let \( \langle a, b \rangle \) denote the usual pairing in cohomology, given by pushforward of \( a \cdot b \) to a point. By the projection formula, we have \( \langle f^* y(w), x(I) \rangle = \langle y(w), f_* x(I) \rangle \). Since \( f_* x(I) = x(v_d) \) when \( X(I) \to X(v_d) \) is birational, and \( f_* x(I) = 0 \) otherwise, the lemma follows from Corollary 2.5. \( \square \)
Remark 5.3. Applying the lemma to divisor classes, we have 
\[ f^{*}y(s_{\alpha}) = \sum y_{i}, \] 
the sum over \( 1 \leq i \leq d \) such that \( \alpha_{i} = \alpha \). Combining this with Proposition 4.1 gives a method for computing in \( H^{*}_{d}(G/B) \).

First proof of Theorem 5.1. Let \( f : X(\underline{\alpha}) \rightarrow X \) be the projection, and let \( \underline{v} = (v_{1}, \ldots, v_{d}) \) be the \( \underline{\alpha} \)-chain associated to \( I = \{1, \ldots, d\} \), so \( v_{i} = s_{\alpha_{1}} \cdots s_{\alpha_{i}} \) and in particular \( v = v_{d} \). Then \( f(p_{\underline{v}}) = p_{v} \), so \( y(w)|_{v} = (f^{*}y(w))|_{p_{\underline{v}}} \). By Lemma 5.2, this is \( \sum y(K)|_{I} \), the sum over all \( K \) such that \( \#K = \ell(w) \) and the corresponding \( \underline{\alpha} \)-chain \( \underline{v}^{K} \) has \( v_{d}^{K} = w \). On the other hand, by Corollary 3.4, we have \( y(K)|_{I} = \prod_{i \in K}(-v_{i}(\alpha_{i})) \). Since \( -v_{i}(\alpha_{i}) = \beta_{i}, \) the theorem is proved.

For the second proof, we use a variation on the functions \( \psi_{\underline{w}} \) which we studied in Chapter 16. These were given by \( \psi_{\underline{w}}(w) = y(w)|_{\underline{w}} \). Here we will use functions \( \varphi_{\underline{w}} : W \rightarrow \Lambda \), defined by
\[ \varphi_{\underline{w}}(w) = y(w)|_{\underline{v}} = \psi_{\underline{w}}(v). \]
Properties of these functions are immediate from the corresponding properties of \( \psi_{\underline{w}} \) (Chapter 16, Proposition 2.5). We only need an inductive formula.

Lemma 5.4. We have
\[ \begin{align*}
(7) \quad \varphi_{\underline{w}}(w) &= \varphi_{v_{s_{\alpha}}}(w) \quad \text{if } \ell(w_{s_{\alpha}}) > \ell(w); \\
(8) \quad \varphi_{\underline{w}}(w) &= \varphi_{v_{s_{\alpha}}}(w) - v(\alpha) \varphi_{v_{s_{\alpha}}}(w_{s_{\alpha}}) \quad \text{if } \ell(w_{s_{\alpha}}) < \ell(w).
\end{align*} \]

Proof. Using the operators \( A_{\alpha} \) from Chapter 16, Proposition 2.5, we have
\[ \psi_{w}(v_{s_{\alpha}}) - \psi_{w}(v) = v(\alpha) (A_{\alpha} \psi_{w})(v) \]
\[ = \begin{cases} 
0 & \text{if } \ell(w_{s_{\alpha}}) > \ell(w); \\
v(\alpha) \psi_{w_{s_{\alpha}}}(v) & \text{if } \ell(w_{s_{\alpha}}) < \ell(w).
\end{cases} \]
This immediately proves (7), as well as (8) with \( \varphi_{\underline{w}}(w_{s_{\alpha}}) \) appearing on the right-hand side in place of \( \varphi_{v_{s_{\alpha}}}(w_{s_{\alpha}}) \). But by (7), we have \( \varphi_{\underline{w}}(w_{s_{\alpha}}) = \varphi_{v_{s_{\alpha}}}(w_{s_{\alpha}}) \) (since \( \ell(w_{s_{\alpha}}) > \ell(w_{s_{\alpha}} \cdot s_{\alpha}) \)).
Using the lemma, if we know the function $\varphi_{v_\alpha}$, for some $\alpha$, then we know $\varphi_v$. For instance, we know

$$
\varphi_e(w) = \begin{cases} 
1 & \text{if } w = e; \\
0 & \text{otherwise}
\end{cases}
$$

(since $p_e \not\in Y(w)$ for $w \neq e$). This determines the rest!

**Second proof of Theorem 5.1.** We use induction on $\ell(v)$. For $\ell(v) = 0$, so $v = e$, this is the case observed above, so the theorem holds. In general, fix a reduced word for $v$ as in the theorem. Set $f_v(w)$ to be the right-hand side of the formula (6), and let $\alpha = \alpha_d$. We assume the formula for $\varphi_{v_\alpha}$ is known, using the reduced word $(\alpha_1, \ldots, \alpha_{d-1})$ for it.

If $\ell(ws_\alpha) > \ell(w)$, then no reduced word for $w$ ends in $\alpha$, and it follows that $f_v(w) = f_{v_\alpha}(w)$. Since $\varphi_v(w) = \varphi_{v_\alpha}(w)$ by Lemma 5.4, the formula holds in this case.

If $\ell(ws_\alpha) < \ell(w)$, then no reduced word for $ws_\alpha$ ends in $\alpha$. Consider subsets $I = \{i_1 < \cdots < i_\ell\}$ corresponding to reduced words for $w$. For those $I$ such that $i_\ell = d$, the sequence $(\alpha_{i_1}, \ldots, \alpha_{i_{\ell-1}})$ is a reduced word for $ws_\alpha$, and $\beta_d = -v(\alpha) = (v_{s_\alpha})(\alpha)$. So the sum of such terms is

$$
\sum_{I \text{ with } i_\ell = d} \beta_{i_1} \cdots \beta_{i_{\ell-1}} \beta_{i_\ell} = -v(\alpha) \varphi_{v_\alpha}(ws_\alpha).
$$

The other terms, where $i_\ell < d$, sum to $\varphi_{v_\alpha}(w)$. Applying Lemma 5.4, the full sum is $\varphi_v(w)$, as required. \hfill \Box

**Example 5.5.** Theorem 5.1 includes a formula for the restrictions of divisor classes $y(s_\alpha)|_v$, as the sum of those $\beta_i$ for which $\alpha_i = \alpha$. On the other hand, we saw $y(s_\alpha) = \omega_\alpha - v(\omega_\alpha)$ in Chapter 16, Lemma 2.6. The latter is often simpler to use in this case. For example, with $G = SL_n$ and $\alpha = t_1 - t_2$, we have

$$
\omega_\alpha - v(\omega_\alpha) = \alpha_1 + \cdots + \alpha_{v(1)-1}
$$

for any permutation $v \in S_n$, without needing to find a reduced expression.
Exercise 5.6. Check directly that the two formulas for $y(s_{\alpha})|_v$ agree: show that
\[ \omega_{\alpha} - v(\omega_{\alpha}) = \sum_{i: \alpha_i = \alpha} s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i) \]
for any simple root $\alpha$, and any reduced word $(\alpha_1, \ldots, \alpha_d)$ for $v \in W$.\(^4\)

Example 5.7. As noted above, Theorem 5.1 shows that $y(w)|_v$ is nonzero if and only if $p_v \in Y(w)$. This is a special property of the standard torus action on Schubert varieties. In general, for an invariant subvariety $Y$ of a nonsingular variety $V$, with $[Y]^T \in H^*_T V$, one can have $[Y]^T|_p = 0$ for an isolated fixed point $p \in Y$.

For example, consider $V = \mathbb{P}^4$ with coordinates $x_1, \ldots, x_5$, and a torus $T$ acting by characters $0, \chi_1, -\chi_1, \chi_2, -\chi_2$, where $\chi_1 \neq \chi_2$. Let $Y$ be the hypersurface defined by $x_2x_3 - x_1x_5 = 0$, so $p = [1, 0, 0, 0, 0]$ is the singular point of $Y$. Writing $\zeta = c^T_1(\mathcal{O}(1))$, we have $[Y]^T = 2\zeta$ so $[Y]^T|_p = 0$.

Remark 5.8. As we saw in Chapter 15, Equation (9), Schubert classes in $G/P$ pull back to Schubert classes in $G/B$. Writing the projection as $\pi: G/B \to G/P$, we have $\pi^* y[w] = y(w^{\text{min}})$. This is compatible with restriction to fixed points, and we have
\[ y[w]|_{[v]} = y(w^{\text{min}})|_v \]
for any coset representative $v \in [v]$. In particular, Theorem 5.1 includes a formula for restricting $G/P$ Schubert classes.

6. Duality

In Chapter 16, §4, we used an isomorphism $\Phi^w: G/B \simto G/B^w$ to relate difference operators with the right $W$-action on $G/B$. The particular case where $w = w_o$, so $B^w = \dot{w}_o B \dot{w}_o^{-1} = B^-$, is especially useful for passing between formulas involving $y(w)$ and ones involving $x(w)$. Here we will state several such formulas; their proofs are all immediate from the functoriality of pullbacks.
To set up notation, let $X = G/B^-$, with fixed points $p_w = \hat{w}B^-$ and Schubert varieties

$$X(w) = B^- \cdot p_w \quad \text{and} \quad Y(w) = B \cdot p_w.$$  

Let $\bar{X}(w)$ and $\bar{Y}(w)$ be the corresponding Schubert classes in $H^*_T \bar{X}$. The entire discussion for Schubert classes in $X = G/B$, except that each root is replaced by its negative. For example,

$$\bar{y}(w)|_{p_v} = \prod_{\beta \in \imath(w(R^-) \cap R^+)} (-\beta) = (-1)^{\ell(w)} y(w)|_{p_w}.$$  

Let $\tau: \Lambda \to \Lambda$ be the graded involution which is multiplication by $(-1)^{\ell}$ on $\text{Sym}^\ell M$, so $\tau$ is induced by the involution of $M$ taking each root to its negative. Then

$$\bar{y}(w)|_{p_v} = \tau(y(w)|_{p_v})$$  

for every $w, v \in W$.

Write $\Phi = \Phi^{w_0}$ for the $G$-equivariant isomorphism $X \sim \bar{X}$, so $\Phi(gB) = g\hat{w}_0 B^-$. Since $\Phi(p_{ww_0}) = p_w$, we see

$$\Phi(X(ww_0)) = \bar{Y}(w) \quad \text{and} \quad \Phi(Y(ww_0)) = X(w).$$

So $\Phi^* \bar{y}(w) = x(ww_0)$ and $\Phi^* \bar{X}(w) = y(ww_0)$, and we have

$$x(w)|_{p_v} = \bar{y}(ww_0)|_{p_{ww_0}}.$$  

Combining this with (9), we obtain

$$x(w)|_{p_v} = \tau(y(ww_0)|_{p_{ww_0}}).$$

Next consider the automorphism $\tau_0 = \tau_{w_0}: X \to X$, coming from the left action of $W$ on $G/B$ as in Chapter 16, §5. The map $\tau_0$ is equivariant with respect to the automorphism $\sigma: g \mapsto \hat{w}_0 g \hat{w}_0^{-1}$ of $G$. Restricting $\sigma$ to the torus $T \subseteq G$, in turn, induces the algebra automorphism $w_0: \Lambda \to \Lambda$ given by $\lambda \mapsto w_0(\lambda)$ for $\lambda \in M$. Since $\tau_0$ maps $p_{ww_0}$ to $p_w$, we see $\tau_0(X(ww)) = Y(w)$ and therefore

$$x(ww)|_{p_{ww}} = w_0 \cdot (y(w)|_{p_v}).$$
Like $\tau$, the algebra automorphism $w_0$ sends a product of positive roots to a product of negative roots—but in general these are different automorphisms.

Finally, the isomorphism $\Phi \circ \tau : X \to \overline{X}$ is equivariant with respect to the automorphism $\sigma$, and takes $Y(w_0 w w_0)$ to $\overline{Y}(w)$, so

$$y(w_0 w w_0)|_{p_{w_0 w w_0}} = w_0 \cdot \tau(y(w)|_{p_v}).$$

These identities generalize ones we have seen for Schubert polynomials in type A. For instance, Equation (12) here corresponds to Chapter 11, §8, Equation (2).

7. A nonsingularity criterion

For $v \leq w$ in $W$, when is the Schubert variety $X(w)$ nonsingular at the fixed point $p_v \in X(w)$? We will see a criterion in terms of equivariant cohomology, due to Kumar.

We need some information about the tangent cone $C_{p_v}X(w)$. Let

$$V_v = \delta U^{-1} \cdot p_v \subseteq X$$

be the $T$-invariant open affine neighborhood of $p_v$, and let

$$V(w)_v = X(w) \cap V_v$$

be the corresponding affine neighborhood in $X(w)$. We will write $V(w)_v = \text{Spec} A$, and $m \subseteq A$ for the maximal ideal corresponding to $p_v \in V(w)_v$.

**Lemma 7.1.** For each $\beta \in \nu(R^-)$ such that $s_\beta v \leq w$, there is a function $f_\beta \in A$ which is an eigenfunction of weight $\beta$ for the action of $T$. (That is, $f_\beta(t^{-1}x) = \beta(t) f(x)$ for all $t \in T$ and $x \in V(w)_v$.)

Furthermore, the $f_\beta$ generate an $m$-primary ideal in $A$. (That is, $f_\beta(p_v) = 0$ for each $\beta$, and $p_v$ is their only common zero.)

From the description of invariant curves we saw in Chapter 15, §4, the roots $\beta \in \nu(R^-)$ such that $s_\beta v \leq w$ are precisely the weights of the $T$-invariant curves in $X(w)$ through $p_v$.

We will state the nonsingularity criterion in terms of the equivariant multiplicities defined in Chapter 17.
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Theorem 7.2. For \( v \leq w \), the point \( p_v \) is nonsingular in \( X(w) \) if and only if

\[
\varepsilon_{p_v}^T X(w) = \prod_{\beta \in \nu^-(\mathbb{R}^-), \ s_{\beta v} \leq w} \beta^{-1},
\]

where \( \varepsilon_v^T X(w) \) is the equivariant multiplicity of \( X(w) \) at \( p_v \).

Proof. One direction is immediate. If \( X(w) \) is nonsingular at \( p_v \), the weights on \( T_{p_v} X(w) \) coincide with the tangent weights to the \( T \)-invariant curves through \( p_v \). (This is a general fact about nonsingular varieties with finitely many invariant curves; see Chapter 7, Proposition 2.3.) Therefore

\[
T_{p_v} X(w) = \bigoplus_{\beta \in \nu^-(\mathbb{R}^-), \ s_{\beta v} \leq w} \mathcal{g}_{\beta}. \]

By an elementary property of equivariant multiplicities, \( \varepsilon_v^T X(w) \) is the inverse of the product of tangent weights (Chapter 17, Proposition 4.4(ii)).

Conversely, assume the formula holds. Using the notation of Lemma 7.1, let \( A' \subseteq A \) be the subring generated by the functions \( f_{\beta} \). Since \( \varepsilon_v^T X(w) \) has degree \(-\dim X(w) = -\ell(w)\), there are \( \ell(w) \) such \( f_{\beta} \)'s. It follows that they form a system of parameters for \( A \) at \( m \).

So the subalgebra \( A' \cong \mathbb{C}\{f_{\beta} \mid \beta \in \nu^-(\mathbb{R}^-), \ s_{\beta v} \leq w\} \) is a polynomial ring, and \( A \) is a finitely generated module over \( A' \).

Let \( V = V(w)_v = \text{Spec} A \) and \( V' = \text{Spec} A' \), and write \( \pi : V \to V' \) for the corresponding equivariant map of affine varieties. Let \( p' \in V' \) be the origin, and note that this is a nondegenerate fixed point, since the tangent weights \( \beta \) are all nonzero. Since the functions \( f_{\beta} \) are a system of parameters, we have \( \pi^{-1}(p') = p_v \). It follows from another property of equivariant multiplicities (Chapter 17, Proposition 4.4(vi)) that

\[
\varepsilon_{p_v}^T V = d \cdot \varepsilon_{p_v}^T V',
\]

where \( d \) is the degree of the finite map \( \pi; \) since equivariant multiplicities are local, we have \( \varepsilon_v^T X(w) = \varepsilon_{p_v}^T V \). On the other hand, \( p' \in V' \) is
nonsingular, with tangent weights $\beta$, so as observed above we have

$$\varepsilon^T_{p'} V' = \prod_{\substack{\beta \in v(R^-) \cap R^- \setminus \{v\} \leq w}} \beta^{-1}.$$  

It follows that $d = 1$, so $A = A'$ is a polynomial ring, and $V \cong \mathbb{A}^{v(w)}$. In particular, $p_v$ is a nonsingular point. \hfill $\square$

The criterion may be rephrased in terms of restrictions of Schubert classes.

**Corollary 7.3.** For $v \leq w$, the point $p_v$ is nonsingular in $X(w)$ if and only if

$$x(w)|_v = \prod_{\beta \in v(R^-) \cap R^- \setminus \{v\} \leq w} \beta.$$ 

**Proof.** We have

$$x(w)|_v = c^T_N(T_{p_v} X) \cdot \varepsilon^T_{\varepsilon} X(w)$$

$$= \left( \prod_{\beta \in v(R^-) \setminus R^+} \beta \right) \cdot \varepsilon^T_{\varepsilon} X(w),$$

using another characterization of equivariant multiplicities (Chapter 17, §4, Equation (9)). Dividing both sides by $c^T_N(T_{p_v} X)$, the assertion follows from Theorem 7.2. (For any $\beta \in v(R^-) \cap R^-$, we have $s_\beta v < v \leq w$, so these weights cancel.) \hfill $\square$

Using the duality identities from the previous section, it is easy to deduce corresponding nonsingularity criteria for opposite Schubert varieties $Y(w)$. Using the notation of §6, the automorphism $\tau_v$ sends $p_{w,v}$ to $p_v$ and $X(w_v w)$ to $Y(w)$, so $p_v$ is nonsingular in $Y(w)$ if and only if $p_{w,v}$ is nonsingular in $X(w_v w)$. We obtain the following:

**Corollary 7.4.** For $v \geq w$, the point $p_v$ is nonsingular in $Y(w)$ if and only if

$$y(w)|_v = \prod_{\substack{\beta \in v(R^-) \cap R^+ \setminus \{v\} \leq w}} \beta.$$
In this case, the tangent space $T_{p_v} Y(w)$ has weights $\beta \in v(R^-)$ such that $s_\beta v \geq w$.

(Applying Equation 11, it suffices to verify that
\[ \{ \beta \in v(R^-) \mid s_\beta v \not\preceq w \} = w_o \left( \{ \gamma \in w_o v(R^-) \mid s_\gamma w_o v \not\preceq w_o w \} \right), \]
which is straightforward, using $w_o v \leq w_o w$ iff $v \geq w$.)

Combining this with the restriction formula of Theorem 5.1, we arrive at a combinatorial criterion for nonsingularity of $Y(w)$ at $p_v$.\(^5\)

**Corollary 7.5.** Fix a reduced word $\alpha = (\alpha_1, \ldots, \alpha_d)$ for $v$, and write $\beta_i = s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha_i)$. Then $p_v$ is nonsingular in $Y(w)$ if and only if
\[ \sum \beta_i \cdot \cdots \cdot \beta_{\ell} = \prod_{\beta \in v(R^-) \cap R^+} \beta, \]
where the sum on the left-hand side is over all $I \subseteq \{1, \ldots, d\}$ such that the corresponding subword $\underline{\alpha}(I)$ is a reduced word for $w$.

**Exercise 7.6.** If $\ell(v) = \ell(w) + 1$, show that $p_v \in Y(w)$ is nonsingular. Conclude that Schubert varieties are nonsingular in codimension one. (That is, the singular locus has codimension at least two.)\(^5\)

**Exercise 7.7.** For $G = SL_n$ and $\alpha = t_k - t_{k+1}$, so $s_\alpha = s_k$, show that the (opposite) Schubert variety $Y(s_k) \subseteq SL_n/B$ is singular at $w$ if and only if $\# \{i \leq k \mid w(i) > k \} \geq 2$.\(^6\)

**Exercise 7.8.** Use $\Xi_{2143} = (x_1 - y_1)(x_1 + x_2 + x_3 - y_1 - y_2 - y_3)$ to determine the singular locus of $Y(2143) = \Omega_{2143} \subseteq Fl(C^4)$.

**Remark 7.9.** Using the Bott-Samelson resolution, the additivity property of equivariant multiplicities (Chapter 17, Proposition 4.4(vi)) leads to another formula for $\epsilon_{p_v}^T X(w)$. We have
\[ (13) \quad \epsilon_{p_v}^T X(w) = \sum_{\underline{\alpha}} \left( \prod_{i=1}^{\ell} ( -v_i(\alpha_i) ) \right)^{-1}, \]
where $\underline{\alpha} = (\alpha_1, \ldots, \alpha_\ell)$ is a fixed reduced word for $w$, and the sum is over all $\underline{\alpha}$-chains $\underline{v} = (v, v_1, \ldots, v_\ell)$ such that $v_\ell = v$. (These correspond to the fixed points $p_{\underline{v}} \in X(\underline{\alpha})$ mapping to $p_v$ under the
resolution $X(\alpha) \to X(w)$, and the corresponding term is $\varepsilon_1 X(\alpha)$. Clearing denominators, one obtains a formula for $x(w)|_v$ which is different from the one deduced from Billey’s formula. In particular, note that the chains indexing terms of the sum need not correspond to reduced words for $v$.

**Remark 7.10.** As noted in Remark 5.8, knowing about Schubert varieties in $G/B$ is enough to say something about Schubert varieties in $G/P$. The projection $\pi: G/B \to G/P$ makes $X(w^{\max}) \to X[w]$ and $Y(w^{\min}) \to Y[w]$ into fiber bundles, with nonsingular fiber $P/B$. So a point $p[v] \in X[w]$ is nonsingular if and only if $p_v \in X(w^{\max})$ is nonsingular, for any coset representative $v \in [v]$; and similarly for $p[v] \in Y[w]$. So Theorem 7.2 and Corollary 7.3 provide nonsingularity criteria for Schubert varieties in $G/P$.

**Notes**

Bott and Samelson gave a construction similar to the one indicated in Remark 3.5, and used it to study the cohomology of $G/B = K/S$ [BoSa55]. In particular, they prove a non-equivariant version of Corollary 4.4. The algebraic version which is more commonly used in Schubert calculus and representation theory was introduced by Demazure [De74] and Hansen [Han74], and for this reason the varieties $X(\alpha)$ are sometimes called Bott-Samelson-Demazure-Hansen (or BSDH) varieties. The non-equivariant part of the formula for $x_1^2$ (Exercise 4.3) appears in [De74, §4.2].

Corollary 3.4 was proved by Willems, using a localization argument similar to the second proof we gave [Wi04]. Our geometric argument, using the submanifolds $Y(I)$, appears to be new.

Theorem 5.1 appears as an exercise (without proof) in a book by Andersen, Jantzen, and Soergel [AJS94, p. 298]. Billey discovered the formula independently, emphasizing the connection with Schubert calculus [Bi99]. Her proof proceeds by decreasing induction on $w$, with a separate argument that the polynomial is independent of the choice of reduced word. The result is sometimes known as the AJSB formula.

Example 5.7 is due to Brion [Bri00].

Among simple linear algebraic groups, the automorphisms $\tau$ and $w_o$ (from §6) are equal precisely in types $B_n, C_n, D_{2n}, E_7, E_8, F_4$, and $G_2$; see, e.g., [Hum81, §31.6].
Theorem 7.2 is due to Kumar [Ku96, Theorem 5.5]. A simplified argument was given by Brion [Bri97b, §6.5], and this is essentially the one we use. Lemma 7.1 follows from a result of Polo [Po94, Prop. 2.2]; see also Kumar [Ku02, Prop. 5.2]. A more detailed study of the tangent cones $C_{p_v} X(w)$ has been carried out by Carrell and Peterson; see, e.g., [Ca94].

The formula (13) for $\epsilon_v^T X(w)$ is due to Rossmann [Ro89, (3.8)].

Hints for exercises

1. Use the subword characterization of Bruhat order, and a greedy algorithm to see that $s_{a_1} \cdots s_{a_i} \leq s_{a_1} \cdots s_{a_d}$ for any subword of $\underline{a}$. See [KnMi04, Lemma 3.4].

2. Consider the point $p = p_{\{1, \ldots, d\}} \in X(\underline{a})$. Using terminology from Chapter 7, §2, the tangent space $T_p X(\underline{a})$ contains parallel weights whenever $\underline{a}$ is a non-reduced word; in this case there are infinitely many $T$-curves through a neighborhood of $p$. Whenever the sequence $\underline{a}$ has a repeated root, an instance of the variety considered in Example 2.6 occurs as a subvariety of $X(\underline{a})$, and this has infinitely many $T$-curves.

To see that $X(\underline{a})$ is toric when all roots are distinct, look at the tangent space to $p_{\emptyset}$: the characters form part of a basis for $M$, so there is a dense $T$-orbit. To see that $f$ is an isomorphism in this case, keep track of fixed points.

3. Use induction on $d$. The same argument shows that the analogous map

$$G^d P_{a_1} \times^B \cdots \times^B P_{a_d} / B \to Z(\underline{a})$$

is an isomorphism.

4. Argue inductively as in the second proof of Theorem 5.1. It is obvious for $v = e$. Suppose the equality is known for $v$, and $\beta$ is a simple root such that $\ell(v s_p) = \ell(v) + 1$. If $\beta \neq \alpha$, the right-hand sides are clearly equal for $v$ and $v s_p$; since $s_\beta(\omega_\alpha) = \omega_\alpha$ for $\beta \neq \alpha$, so are the left-hand sides. If $\beta = \alpha$, then the difference of the right-hand sides is $v(\alpha)$, and the difference of the left-hand sides is $v(\omega_\alpha) - v s_\alpha(\omega_\alpha) = v(\alpha)$.

5. The claim about $p_v \in Y(w)$ being nonsingular follows easily from Billey’s formula for $y(w)|_v$. Using $B$-equivariance, one sees that the nonsingular locus of $Y(w)$ contains the union of Schubert cells $Y(v)^v$ for $v \geq w$ and $\ell(v) \leq \ell(w) + 1$. (The conclusion also follows from the general fact that Schubert are normal.)

6. Use the formula for $y(s_\alpha)|_w$ in Chapter 10, Exercise 7.2.