EQUIVARIANT COHOMOLOGY IN ALGEBRAIC GEOMETRY
APPENDIX A: ALGEBRAIC TOPOLOGY

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In this appendix, we collect some basic facts from algebraic topology pertaining to the *fundamental class* of an algebraic variety, and *Gysin push-forward maps* in cohomology. Much of this material can be found in [Ful97, Appendix B], and we often refer there for proofs.

This appendix is in rough form, and will probably change significantly. (Watch the version date.)

1. A brief review of singular (co)homology

Let X be any space, let C_*X be the *complex of singular chains* on X, and let $C^*X = \mathrm{Hom}(C_*X, \mathbb{Z})$ be the *complex of singular cochains*. The **singular homology modules** are defined as

$$H_iX = h_i(C_*X),$$

and the **singular cohomology modules** are

$$H^iX = h^i(C^*X).$$

One sets $H_*X = \bigoplus H_iX$ and $H^*X = \bigoplus H^iX$. We refer to [Spa66] for the details and basic properties of these constructions, summarizing the most relevant facts below.

One sacrifices some geometric intuition in working with cohomology instead of homology, but one gains the advantage of an easily defined ring structure. If $\sigma \in C_kX$ is a singular simplex, let $f_i\sigma \in C_iX$ be the restriction of σ to the *front i-face* of the standard simplex, and let $b_j\sigma$ be the restriction of σ to the *back j-face*. Then one defines the **cup product**

$$H^iX \otimes H^jX \to H^{i+j}X$$

by setting

$$(c \cup d)(\sigma) = c(f_i\sigma)\, d(b_j\sigma),$$

for $c \in C^iX$, $d \in C^jX$, and σ an $(i+j)$-simplex.

Date: April 9, 2007.

This makes H^*X into a skew-commutative graded ring: For $c \in H^iX$ and $d \in H^jX$, one has $c \cup d = (-1)^{ij} d \cup c$. [1]

H_*X becomes a (left) module over H^*X via the **cap product**

$$\cap : H^iX \otimes H_jX \to H_{j-i}X,$$

defined by $c \cap \sigma = c(b_i\sigma)\,(f_{j-i}\sigma)$.

Associated to a continuous map $f : X \to Y$, there are natural *pushforward* and *pullback* maps on homology and cohomology, respectively, denoted $f_* : H_*X \to H_*Y$ and $f^* : H^*Y \to H^*X$. These are related by the *projection formula*, also called "naturality of the cap product":

$$f_*(f^*c \cap \sigma) = c \cap f_*\sigma.$$

If X is triangulated, one also has the *simplicial homology* $H_*^{simp}X$, and a canonical isomorphism $H_*^{simp}X \xrightarrow{\sim} H_*X$. (This shows $H_*^{simp}X$ is independent of the choice of triangulation.)

For $A \subset U \subset X$, with A closed in X and U open, there are natural *excision isomorphisms* $H^i(X,U) \cong H^i(X \smallsetminus A, U \smallsetminus A)$.

If M is a compact, connected, oriented n-manifold, then it has a *fundamental class* $[M] \in H_nM \cong \mathbb{Z}$, characterized by the fact that it maps to a chosen generator of $H_n(M, M \smallsetminus \{x\}) \cong H_n(U, U \smallsetminus \{x\}) \cong \mathbb{Z}$ for all $x \in M$, where U is a ball around x. There is a canonical isomorphism (the "Poincaré isomorphism") $H^iM \xrightarrow{\sim} H_{n-i}M$, given by $c \mapsto c \cap [M]$. This isomorphism, then, makes $H_*M \cong H^*M$ into a ring.

For such M, there is a perfect pairing

$$\int_M : H^*M \times H^*M \to \mathbb{Z}, \tag{1}$$

called the *Poincaré duality pairing.*[2] For $c \in H^iM$ and $d \in H^{n-i}M$, this is given by $(c,d) \mapsto (c \cup d) \cap [M] \in H_0M = \mathbb{Z}$. If $\{x_\alpha\}$ is a homogeneous basis for H^*M, then the *Poincaré dual basis* is the basis $\{y_\alpha\}$ dual for this pairing, so

$$\int_M x_\alpha \cdot y_\beta = \delta_{\alpha\beta}.$$

If V is a closed subset of M, there is also a canonical isomorphism

$$H_iV \cong H^{n-i}(M, M \smallsetminus V). \tag{2}$$

This isomorphism is often called the "Alexander-Lefschetz" isomorphism. (See [Spa66, p. 296–7].)

[1] The sign conventions in the definition of the cup product vary throughout the literature. For example, ours agree with those of [Spa66] and [Hat02], but are the opposite of those of [Mil-Sta74].

[2] In fact, there are several related (but different) notions which go by the name "Poincaré duality".

For an oriented rank-r real vector bundle $E \xrightarrow{\pi} X$, and A any subspace of X, there is the *Thom isomorphism*

$$H^i(X, X \smallsetminus A) \cong H^{i+r}(E, E \smallsetminus A).$$

In fact, there is a class $\eta \in H^r(E, E \smallsetminus X)$, called the *Thom class*, characterized by the fact that it restricts to the chosen generator of $H^r(\pi^{-1}(p), \pi^{-1}(p) \smallsetminus \{p\})$ for all $p \in X$. The above isomorphism is given by $c \mapsto \pi^*(c) \cup \eta$. (Note: we always identify X with its embedding by the zero section in a vector bundle.)

For a smooth closed submanifold M of a smooth manifold M', there is a neighborhood U of M in M' such that the pair (U, M) is diffeomorphic to (N, M), where N is the normal bundle of the embedding $M \subset M'$.

2. Borel-Moore homology

A better way, at least for our purposes, is to use *Borel-Moore homology*, which we will denote by $\overline{H}_i X$. There are several ways to define these groups; for example, via

- sheaf theory (as was done originally in [Bor-Moo60] and [Bor-Hae61]);
- locally finite chains;
- a one-point compactification X^+, for good spaces X. (Use $H_i(X^+, X^+ \smallsetminus X)$.)

We will use a definition which comes equipped with many nice properties, and which works for any space which can be embedded as a closed subspace of an oriented smooth manifold M:

Definition 2.1. For a space X embedded as a closed subspace in an oriented smooth manifold M, the **Borel-Moore homology groups** are

$$\overline{H}_i X := H^{\dim M - i}(M, M \smallsetminus X). \tag{3}$$

Proposition 2.2. *This definition is independent of the choice of embedding. In fact, given closed embeddings of X into manifolds M and M', there is a canonical isomorphism $H^{\dim M-i}(M, M \smallsetminus X) \cong H^{\dim M'-i}(M', M' \smallsetminus X)$. Moreover, if X is embedded into a third manifold M'', these isomorphisms form a commuting triangle:*

$$\begin{tikzcd}
H^{\dim M-i}(M, M \smallsetminus X) \arrow[rr, "\sim"] \arrow[dr, "\sim"'] & & H^{\dim M'-i}(M', M' \smallsetminus X) \arrow[dl, "\sim"] \\
& H^{\dim M''-i}(M'', M'' \smallsetminus X). &
\end{tikzcd}$$

For the proof, see [Ful97, pp. 216-217].

Note that these Borel-Moore homology groups are not homotopy invariant: For example, our definition says that $\overline{H}_n \mathbb{R}^n = \mathbb{Z}$. Nor are they functorial with respect to arbitrary continuous maps. However, there is functoriality in two important situations.

Proposition 2.3 (Covariance for proper maps). *If $f : X \to Y$ is continuous and proper (i.e., the inverse image of a compact set is compact), then there are maps*

$$f_* : \overline{H}_i X \to \overline{H}_i Y.$$

Proof. Suppose X is embedded as a closed subspace of $\mathbb{R}^n$, and Y is embedded as a closed subspace of $\mathbb{R}^m$. Since f is proper, we can find a map $\varphi : X \to I^n$ such that

$$(f, \varphi) : X \to Y \times I^n \subset \mathbb{R}^m \times \mathbb{R}^n$$

is a closed embedding.[3] (Here $I^n = [a, b]^n$, with $a < 0 < b$.)

$$\begin{array}{rcl}
\overline{H}_i X & = & H^{m+n-i}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^m \times \mathbb{R}^n \smallsetminus X) \\
& \underset{\text{(restrict)}}{\longrightarrow} & H^{n+m-i}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^m \times \mathbb{R}^n \smallsetminus (Y \times I^n)) \\
& \underset{\text{(homotopy)}}{\cong} & H^{m+n-i}(\mathbb{R}^m \times \mathbb{R}^n, \mathbb{R}^m \times \mathbb{R}^n \smallsetminus (Y \times \{0\})) \\
& \underset{\text{(Thom)}}{\cong} & H^{m-i}(\mathbb{R}^m, Y) \\
& = & \overline{H}_i Y.
\end{array}$$

□

Exercise 2.4. Check independence of choices, and naturality: $(g \circ f)_* = g_* \circ f_*$ for

$$X \xrightarrow{f} Y \xrightarrow{g} Z.$$

We also have *contravariance for open inclusions.* Let $U \subset X$ be an open subspace, embed X in an n-manifold M, and let $Y = X \smallsetminus U$. We get restriction maps $\overline{H}_i X \to \overline{H}_i U$ from the long exact cohomology sequence of the triad $(M, M \smallsetminus Y, M \smallsetminus X)$. Indeed, U is a closed subspace of the manifold $M \smallsetminus Y$, so the map is

$$\overline{H}_i X = H^{n-i}(M, M \smallsetminus X) \to H^{n-i}(M \smallsetminus Y, (M \smallsetminus Y) \smallsetminus U) = \overline{H}_i U.$$

Exercise 2.5. Check independence of choice, and naturality for a sequence of open inclusions $U' \subset U \subset X$.

In the case where M, X, and Y are complex varieties, we will use this restriction map to find isomorphisms on top-dimensional homology. Also, this allows restriction of homology classes to small (classical!) open sets.

[3] Take φ to be the composition of $X \hookrightarrow \mathbb{R}^n$ with a homeomorphism $\mathbb{R}^n \to (a, b)^n$, followed by the inclusion $(a, b)^n \subset I^n$. When f is proper, this choice of φ makes (f, φ) a closed embedding. Indeed, $X \hookrightarrow Y \times (a, b)^n$ is always a closed embedding; identify X with its embedding in $(a, b)^n$. Suppose (y, x) is a limit point of $(f, \varphi)(X) \subset Y \times I^n$, so there is a sequence $\{(y_k, x_k)\} \to (y, x)$, with $f(x_k) = y_k$. Then $C = \{y_k\} \cup \{y\}$ is compact, so $f^{-1}C$ is compact and contains $\{x_k\}$, and therefore it must contain x.

Example 2.6. $X = \mathbb{C}^n$. (Or $X =$ ball in $\mathbb{C}^n$.) This is a manifold, so

$$\overline{H}_i X = H^{2n-i}(X, \emptyset) = \begin{cases} \mathbb{Z} & \text{if } i = 2n; \\ 0 & \text{otherwise.} \end{cases}$$

Exercise 2.7. The operations of pushforward and restriction are compatible. Specifically, if $f : X \to Y$ is proper and $U \subset Y$ is open, then the diagram

$$\begin{tikzcd} \overline{H}_i X \arrow[r] \arrow[d] & \overline{H}_i(f^{-1}U) \arrow[d] \\ \overline{H}_i Y \arrow[r] & \overline{H}_i U \end{tikzcd}$$

commutes.

Proposition 2.8. *For $Y \subset X$ closed, with $U = X \smallsetminus Y$, there is a natural long exact sequence*

$$\cdots \to \overline{H}_i Y \to \overline{H}_i X \to \overline{H}_i U \to \overline{H}_{i-1} Y \to \cdots \to \overline{H}_0 U \to 0. \tag{4}$$

This sequence will allow inductions on dimension, when the homologies of two of the three spaces are known.

Proof. When X is embedded as a closed subspace of a manifold M, this is just the long exact cohomology sequence of the triad $(M, M \smallsetminus Y, M \smallsetminus X)$. □

3. Classes of subvarieties

Proposition 3.1. *Let V be a k-dimensional quasi-projective algebraic variety. Then $\overline{H}_i V = 0$ for $i > 2k$, and $\overline{H}_{2k} = \bigoplus \mathbb{Z}$, with one copy of $\mathbb{Z}$ for each k-dimensional irreducible component of V.*

Proof. In general, if X is a disjoint union of spaces X_j, then $\overline{H}_i X = \bigoplus_j \overline{H}_i X_j$. (Exercise.) Now when V is nonsingular, this observation reduces the claim to the case where V is irreducible. Indeed, in a nonsingular variety, connected components coincide with irreducible components. But then V is a connected manifold, so $\overline{H}_i V = H^{2k-i} V$; this is 0 for $i > 2k$ and $\mathbb{Z}$ for $i = 2k$.

For a general (possibly singular and reducible) k-dimensional complex variety V, we can reduce to the nonsingular case using the long exact sequence of Borel-Moore homology. Let $W \subset V$ be the closed set consisting of the singular locus of V, together with all irreducible components of dimension $< k$; thus W is a variety of dimension $< k$. By induction on dimension, then, we may assume the claim holds for W. In particular, $\overline{H}_i W = 0$ for $i > 2k - 2$. Now apply the long exact sequence for $W \subset V$: we have

$$0 = \overline{H}_{2k} W \to \overline{H}_{2k} V \to \overline{H}_{2k}(V \smallsetminus W) \to \overline{H}_{2k-1} W = 0. \tag{5}$$

Thus $\overline{H}_{2k}(V) \cong \overline{H}_{2k}(V \smallsetminus W)$, and since the latter is a k-dimensional nonsingular variety, the claim follows. (The same argument also shows vanishing for $i > 2k$.) □

Remark 3.2. An essential ingredient in the above proof is the fact that the singular locus of a complex variety is a subvariety of *complex* codimension at least 1, and hence of *real* codimension at least 2. Things are somewhat more subtle in the real world, and one must impose additional hypotheses for a similar claim to hold on real varieties.

We therefore have a **fundamental class** $[V] \in \overline{H}_{2k}V \cong \mathbb{Z}$, for any irreducible k-dimensional variety V. If V is a closed subvariety of a nonsingular variety X, we also get a fundamental class corresponding to V in $H^{2d}(X)$, where d is the codimension of V. This comes from

$$\overline{H}_{2k}V = H^{2d}(X, X \smallsetminus V) \to H^{2d}(X).$$

The element $\eta_V \in H^{2d}(X, X \smallsetminus V)$ corresponding to $[V] \in \overline{H}_{2k}V$ is called the **refined class** of V in X.

Remark 3.3. One should expect a class representing V in $H^{2d}(X)$ to be "supported on V," and hence to come from $H^{2d}(X, X \smallsetminus V)$. We need a canonical representative, though, and the fundamental class of Borel-Moore homology gives us one.

Remark 3.4. Suppose V and W are irreducible subvarieties of X, of respective codimensions d and e. Then there is a class $[V] \cdot [W] \in H^{2d+2e}(X)$. In fact, using refined classes, we have

$$\eta_V \cdot \eta_W \in H^{2d+2e}(X, X \smallsetminus (V \cap W)),$$

so this product is supported on the intersection of V and W. If V and W intersect properly (i.e., $\operatorname{codim}(V \cap W) = d + e$), then by Proposition 3.1,

$$H^{2d+2e}(X, X \smallsetminus (V \cap W)) = \overline{H}_{top}(V \cap W) = \bigoplus \mathbb{Z},$$

with one copy of $\mathbb{Z}$ for each irreducible component of $V \cap W$. Thus $\eta_V \cdot \eta_W$ assigns an *intersection number* to each irreducible component of $V \cap W$, and we have an *intersection cycle* $V \cdot W$. In fact, these numbers agree with those defined algebraically in intersection theory.

We now turn to the behavior of fundamental classes under morphisms.

Proposition 3.5. *Let X and Y be nonsingular varieties of respective dimensions n and m, and let $f : X \to Y$ be a proper morphism. Let $V \subset X$ be a closed subvariety, and let $W = f(V) \subset Y$. Then*

$$\begin{array}{rccc} f_* : & H^i(X) & \longrightarrow & H^{2m-2n+i}(Y) \\ & \| & & \| \\ & \overline{H}_{2n-i}(X) & \longrightarrow & \overline{H}_{2n-i}(Y) \end{array}$$

maps

$$[V] \mapsto \begin{cases} 0 & \textit{if} \;\; \dim W < \dim V; \\ d[W] & \textit{if} \;\; \dim W = \dim V, \end{cases}$$

where d is the degree of V over W. (By definition, this is the degree of the field extension $[\mathbb{C}(V) : \mathbb{C}(W)]$.)

Proof. The first case ($\dim W < \dim V$) is clear from Proposition 3.1. In the second case, there is an open set $U \subset W$ such that $f^{-1}U \cap V \to U$ is a d-sheeted covering. Taking U to be sufficiently small, we may assume it is a ball, and we have $f^{-1}U \cap V = U_1 \coprod U_2 \coprod \cdots \coprod U_d$, with each U_i mapping isomorphically to U.

Then we have a commuative diagram

$$\begin{tikzcd} \bigoplus_{i=1}^{d} \overline{H}_{2k}(U_i)=\overline{H}_{2k}(f^{-1}U \cap V) \arrow[d] \arrow[dr, phantom, ""] & \overline{H}_{2k}V \arrow[l] \arrow[r] \arrow[d] & \overline{H}_{2k}X \arrow[d] \\ \overline{H}_{2k}U & \overline{H}_{2k}W \arrow[l, "\sim"'] \arrow[r] & \overline{H}_{2k}Y, \end{tikzcd}$$

and the proposition follows. □

The next proposition is about "compatibility of pullbacks."

Proposition 3.6. *Let $f : X \to Y$ be a morphism of nonsingular varieties, with $V \subset Y$ irreducible of codimension d. Assume there is a classical open neighborhood $U \subset Y$ where $V \cap U = V^o \subset U$ is connected, nonsingular, and defined by (holomorphic) equations $h_1, \ldots, h_d$, such that $W \cap f^{-1}U = W^o \subset f^{-1}U$ is also connected, nonsingular, and defined by the equations $h_1 \circ f, \ldots, h_d \circ f$. Then $f^*\eta_V = \eta_W$.*

Proof. We have

$$\mathbb{Z} = H^{2d}(Y, Y \smallsetminus V) \xrightarrow{f^*} H^{2d}(X, X \smallsetminus W) = \mathbb{Z},$$

so $\eta_V \mapsto c \cdot \eta_W$ for some $c \in \mathbb{Z}$. The assumptions on U and $f^{-1}U$ guarantee that the restriction maps are isomorphisms:

$$\begin{array}{rcl} H^{2d}(Y, Y \smallsetminus V) & \xrightarrow{\sim} & H^{2d}(U, U \smallsetminus V^o) \\ H^{2d}(X, X \smallsetminus W) & \xrightarrow{\sim} & H^{2d}(f^{-1}U, f^{-1}U \smallsetminus W^o). \end{array}$$

Thus we may replace Y with U, and reduce to the situation where Y is a vector bundle over V – in fact, a trivial bundle – and similarly X is a vector bundle over W. The claim is that the Thom class in $H^{2d}(Y, Y \smallsetminus V)$ pulls back to the Thom class in $H^{2d}(X, X \smallsetminus V)$. But the hypotheses mean that X is the pullback of Y, when X and Y are considered as vector bundles over W and V. The assertion thus reduces to naturality of the Thom class of vector bundles. □

Thus for irreducible closed subvarieties V and W of a nonsingular variety X, with $\dim V = k$, $\dim W = l$, and $\dim X = n$, we obtain an intersection class in $\overline{H}_{2n-2k-2l}(V \cap W) = H^{2k+2l}(X, X \smallsetminus (V \cap W))$, corresponding to $[V]\cdot[W]$. Each $(k+l-n)$-dimensional irreducible component Z of $V \cap W$ gives an intersection number $i(Z, V \cap W, X)$, which is the projection of $[V] \cdot [W]$ onto the factor of $\overline{H}_{2k+2l-2n}(V \cap W) = \bigoplus \mathbb{Z}$ corresponding to Z.

Exercise 3.7. If there is an open set of Z on which V and W meet transversally, then $i(Z, V \cap W, X) = 1$. (See [Ful97, p. 222].)

Remark 3.8. Geometric considerations ("reduction to the diagonal") can be used to show that the intersection numbers $i(Z, V \cap W, X)$ are nonnegative whenever X is a manifold; hence any product $[V] \cdot [W]$ can be expressed as a *nonnegative* sum of classes $[Z]$. Is this a formal property of the theory, or does it depend on the geometry? For example, if X is a $\mathbb{Q}$-homology manifold (i.e., a variety having the $\mathbb{Q}$-homology of a manifold), is it true that the corresponding intersection numbers $i(Z, V \cap W, X)$ are positive rational numbers?

Remark 3.9. When X is a manifold, the ring structure on H^*X comes from the diagonal embedding. More specifically, one has the diagonal map $\delta : X \to X \times X$, and for classes $\alpha, \beta \in H^*X$,

$$\alpha \cup \beta = \delta^*(\alpha \times \beta) = \delta^*(p_1^*(\alpha) \cup p_2^*(\beta)). \tag{6}$$

For subvarieties V, W, we get $[V] \cdot [W] = \delta^*[V \times W]$. This technique of "reduction to the diagonal" works because δ is a *regular embedding.*

Proposition 3.10. *If $X = X_s \supset X_{s-1} \supset \cdots \supset X_0 = \emptyset$ are closed algebraic subsets, and $X_i \smallsetminus X_{i-1} = \coprod_j U_{ij}$ with $U_{ij} \cong \mathbb{C}^{n(i,j)}$, then the classes $[\overline{U}_{ij}]$ form a $\mathbb{Z}$-linear basis for $\overline{H}_*X$.*

Proof. Use induction on i, and assume the proposition holds for X_{i-1}. Associated to the inclusion $X_{i-1} \subset X_i$, we have an exact sequence

$$\to \overline{H}_k(X_{i-1}) \to \overline{H}_k(X_i) \to \bigoplus_j \overline{H}_k(U_{ij}) \to \overline{H}_{k-1}(X_{i-1}) \to . \tag{7}$$

When k is odd, $\overline{H}_k(X_{i-1}) = 0$ by induction, and $\overline{H}_k(U_{ij}) = \overline{H}_k(\mathbb{C}^{n(i,j)}) = 0$ by calculation. Therefore $\overline{H}_k X_i = 0$, as well, and we have short exact sequences

$$0 \to \overline{H}_{2k}(X_{i-1}) \to \overline{H}_{2k}(X_i) \to \bigoplus_j \overline{H}_{2k}(U_{ij}) \to 0. \tag{8}$$

Now $[\overline{U}_{ij}]$ maps to $[U_{ij}]$ under the map $\overline{H}_*(X_i) \to \oplus_j \overline{H}_*(U_{ij})$, and the latter classes form a basis, so the former are independent in $\overline{H}_*(X_i)$. The proposition follows. □

Exercise 3.11. Show that $H^*\mathbb{P}^n = \mathbb{Z}[\zeta]/(\zeta^{n+1})$, where ζ corresponds to $[H]$, the class of a hyperplane.

Exercise 3.12. Compute f^* and f_* for the Segre embedding $\mathbb{P}^m \times \mathbb{P}^n \hookrightarrow \mathbb{P}^{nm+n+m}$.

Exercise 3.13. If a connected group G acts continuously on a space X, show that g acts trivially on H^*X for each $g \in G$. For varieties, show $g \cdot [V] = [gV] = [V]$.

Remark 3.14. In fact, more is true: The operations from intersection theory in algebraic geometry make sense in Borel-Moore homology, and are compatible. See [Ful98, §19].

4. Homotopy type of the complement of an affine algebraic set

Proposition 4.1. *If $Z \subset \mathbb{C}^N$ is a Zariski-closed set, of codimension d, then $\pi_i(\mathbb{C}^N \smallsetminus Z) = 0$ for $0 < i \leq 2d-2$. This is always sharp: $\pi_{2d-1}(\mathbb{C}^N \smallsetminus Z) \neq 0$ if Z is nonempty.*

Proof. (D. Speyer.) Identify $\mathbb{C}^n$ with $\mathbb{R}^{2n}$. For a smooth (C^∞) map $f : S^i \to \mathbb{C}^n \smallsetminus Z$, let $\mathscr{S} = \{p \in \text{(real) line between } Z \text{ and } f(S^i)\}$. (This is analogous to a secant variety in algebraic geometry.) Consider the number

$$\begin{aligned} \dim_{\mathbb{R}} \mathscr{S} &= \dim_{\mathbb{R}} Z + \dim_{\mathbb{R}} f(S^i) + \dim_{\mathbb{R}} \mathbb{R} \\ &\leq 2n - 2d + i + 1, \end{aligned}$$

since smoothness of f implies $\dim f(S^i) \leq i$. The condition that this number be less than $2n$ is exactly that $i \leq 2d-2$. For such i, then, $\mathscr{S} \subsetneq \mathbb{C}^n$; therefore we can find a point $p \notin \mathscr{S}$. Since p does not lie on any line joining Z and $f(S^i)$, the set of line segments between p and $f(S^i)$ lies in $\mathbb{C}^n \smallsetminus Z$. Use this to extend f to a map of the ball $\tilde{f} : D^{i+1} \to \mathbb{C}^n \smallsetminus Z$, thus showing that f is null-homotopic.

Since every continous map between smooth manifolds is homotopic to a smooth map (see [Bott-Tu95, 213–214]), the homotopy groups can be computed using only smooth maps. Thus $\pi_i(\mathbb{C}^n \smallsetminus Z) = 0$ for $i \leq 2d-2$.

On the other hand, it follows from this and the Hurewicz isomorphism theorem that $\pi_{2d-1}(\mathbb{C}^n \smallsetminus Z) = H_{2d-1}(\mathbb{C}^n \smallsetminus Z)$, and $H_{2d-2}(\mathbb{C}^n \smallsetminus Z) = 0$. Now by the universal coefficient theorem and the long exact sequence for the pair $(\mathbb{C}^n, \mathbb{C}^n \smallsetminus Z)$, we have

$$\begin{aligned} H_{2d-1}(\mathbb{C}^n \smallsetminus Z)^\vee &\cong H^{2d-1}(\mathbb{C}^n \smallsetminus Z) \\ &\cong H^{2d}(\mathbb{C}^n, \mathbb{C}^n \smallsetminus Z) \\ &= \overline{H}_{2n-2d} Z, \end{aligned}$$

and we know this top Borel-Moore homology group is nonzero. ◻

5. Gysin maps

All maps and manifolds are assumed to be smooth. We need a few preliminary notions about compatibility of orientations of manifolds. Throughout, let X, Y, X', and Y' be oriented manifolds of (real) dimensions n, m, n', and m', respectively.

Suppose $f : X \to Y$ is a closed embedding, with normal bundle $N = N_{X/Y}$. There is an exact sequence

$$0 \to TX \to TY|_X \to N \to 0,$$

and a canonical isomorphism $\bigwedge^n TX \otimes \bigwedge^{m-n} N \to \bigwedge^m TY|_X$; then N is said to be **compatibly oriented** for f if this isomorphism preserves orientations.

More generally, any proper map $f : X \to Y$ factors through a closed embedding in $Y \times \mathbb{R}^N$. The given orientations on X and Y, together with the standard orientation on $\mathbb{R}^N$, induce a compatible orientation on $N = N_{X/Y\times\mathbb{R}^N}$. (That is, N is compatibly oriented for the sequence

$$0 \to TX \to T(Y \times \mathbb{R}^N)|_X \to N \to 0,$$

as above.)

There is also a notion of compatible orientations for a fiber square of oriented manifolds. Suppose

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

is such a square, with f (and hence f') proper. Factor f through a closed embedding in $Y \times \mathbb{R}^N$, and factor f' through $Y' \times \mathbb{R}^N$ by pullback. Give $N = N_{X/Y\times\mathbb{R}^N}$ and $N' = N_{X'/Y'\times\mathbb{R}^N}$ the orientations compatible for f and f'. Then the square is said to be **compatibly oriented** if this orientation on N' agrees with that induced by $N' \cong (g')^*N$; in other words, $(g')^*N$ is compatible for the sequence

$$0 \to TX' \to T(Y' \times \mathbb{R}^N)|_{X'} \to (g')^*N \to 0.$$

Exercise 5.1. Check these definitions are independent of the choice of factorization of f.

When $f : X \to Y$ is a locally trivial fiber bundle, with smooth fiber F, the sequence

$$0 \to TF \to TX|_F \to f^*TY|_F \to 0$$

induces a compatible orientation on F. In this case, an equivalent way of saying a fiber square is compatibly oriented is to require that the two orientations on F induced by $f : X \to Y$ and $f' : X' \to Y'$ agree.

Remark 5.2. One must be a little careful with this definition. For example, fix an orientation on S^1, and give $S^1 \times S^1$ the induced orientation. Then the fiber for the first projection inherits the original orientation, but for the second projection, the compatible orientation on the fiber is the opposite orientation.

Exercise 5.3. Fix orientations on S^n and S^m, and give $S^n \times S^m$ the induced orientation. Check that compatibility of orientations for the second projection depends on the parity of nm.

Now for a proper map $f : X \to Y$, there are **Gysin maps**

$$f_* : H^i X \to H^{i+d} Y, \tag{9}$$

defined via the pushforward for Borel-Moore homology:

$$H^iX = \overline{H}_{n-i}X \xrightarrow{f_*} \overline{H}_{n-i}Y = H^{i+m-n}Y.$$

These Gysin maps satisfy the following properties:

(i) (Functoriality) For maps $X \xrightarrow{f} Y \xrightarrow{g} Z$, we have

$$(g \circ f)_* = g_* \circ f_*.$$

(ii) (Projection formula) For $x \in H^*X$ and $y \in H^*Y$,

$$f_*(f^*(y) \cdot x) = y \cdot f_*(x).$$

(iii) (Naturality) Suppose $f : X \to Y$ is proper, and

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y \end{tikzcd}$$

is a compatibly oriented fiber square, with $n - m = n' - m'$. Then $f'_* \circ (g')^* = g^* \circ f_*$.

(iv) (Embedding) When $f : X \hookrightarrow Y$ is a closed embedding, f_* factors as

$$H^*X \underset{t}{\xrightarrow{\sim}} H^{*+m-n}(Y, Y \smallsetminus X) \to H^{*+m-n}Y, \tag{10}$$

where t comes from the isomorphism defined by the Thom class of the normal bundle $N = N_{X/Y}$, when N is compatibly oriented for f. Thus the composition $f^* \circ f_* : H^iX \to H^{i+m-n}X$ is (right) multiplication by the Euler class $e(N)$.

Remark 5.4. In general, order is important in the above formulas. For example, if one exchanges factors, the projection formula (ii) becomes

$$f_*(x \cdot f^*(y)) = (-1)^{bd} f_*(x) \cdot y,$$

where $b = \deg(y)$ and $d = m - n$. Indeed, we have

$$\begin{aligned} f_*(x \cdot f^*(y)) &= (-1)^{\deg(x)\deg(y)} f_*(f^*(y) \cdot x) \\ &= (-1)^{\deg(x)\deg(y)} y \cdot f_*(x) \\ &= (-1)^{\deg(x)\deg(y)} (-1)^{\deg(y)(\deg(x)-d)} f_*(x) \cdot y. \end{aligned}$$

Remark 5.5. When X and Y are complex algebraic varieties, there are Gysin pushforward maps induced by certain "nice" maps in more general situations. For example, assuming all varieties are pure-dimensional (but not necessarily irreducible), and all maps are proper, there are Gysin maps for the following:

(i) Proper maps of nonsingular varieties $X \to Y$, as above.

(ii) Any (proper) map $X \to Y$, where Y is nonsingular. Regardless of whether X is nonsingular, there is a map $H^*X \to \overline{H}_*X$ given by cap product with $[X]$. Composing this with $\overline{H}_*X \to \overline{H}_*Y = H^*Y$ gives the pushforward.

(iii) Regular embeddings $X \hookrightarrow Y$. That is (for general Y) X is locally defined by a regular sequence, so the ideal sheaf I/I^2 is locally free. (This is somewhat harder to construct.)

(iv) Smooth maps $X \to Y$.

(v) Local complete intersection morphisms $X \to Y$. (This follows from (iii) and (iv), since any such map may be factored $X \hookrightarrow Y \times P \to Y$ as a regular embedding followed by a smooth projection.)

6. Poincaré duality for fiber bundles

For our purposes, the right setting in which to express Poincaré duality is the following. Let $p : X \to S$ be a fiber bundle with X and S smooth oriented manifolds, with smooth, compatibly oriented fiber F, as above. For $x, y \in H^*X$, set $\langle y, x\rangle = p_*(y \cdot x) \in H^*S$.

Proposition 6.1. *Assume that there are finitely many (homogeneous) elements $\{x_i\}$ which form a basis for H^*X as a right H^*S-module, and whose restrictions $\bar{x}_i$ form a basis for H^*F over the coefficient ring R. Then there is a unique (homogeneous, left) basis $\{y_i\}$ of H^*X over H^*S such that*

$$\langle y_j, x_i\rangle = \delta_{ji}$$

in H^*S.

Proof. Order the x_i's so that $x_i \in H^{k(i)}X$, with $d = k(1) \geq k(2) \geq \cdots \geq k(m) = 0$, and take $\bar{y}_1, \ldots, \bar{y}_m$ to be the basis for H^*F dual to $\{\bar{x}_i\}$ under ordinary Poincaré duality. We will use induction on r to find lifts $y_1, \ldots, y_r$ such that $\langle y_j, x_i\rangle = \delta_{ji}$ for all i, and for $j \leq r$.

For $r = 1$, $y_1 \in H^0X$ is the unique lift of $\bar{y}_1 \in H^0F$ via $H^0X \xrightarrow{\sim} H^0F$.

Now assume $y_1, \ldots, y_{r-1}$ have been found. Take any lift $y_r' \in H^{n-k(r)}X$ of $\bar{y}_r$, and set

$$y_r = y_r' - \sum_{j=1}^{r-1} a_j y_j,$$

where

$$a_j = \langle y_r', x_j\rangle \in H^{k(j)-k(r)}S.$$

A straightforward check shows that this choice of y_r works.

To see that the y_j's form a basis, it is enough to show they generate. Let M be a free (left) H^*S-module with generators $y_1, \ldots, y_m$, and consider the maps

$$M \hookrightarrow H^*X \to \mathrm{Hom}(H^*X, H^*S),$$

where the first is the obvious inclusion, and the second is given by $y \mapsto \langle y, \cdot\rangle$. The composition is surjective, because $y_i \mapsto x_i^*$. Thus the second map is surjective, but as a map of free modules of the same rank, it must be an isomorphism. □

Exercise 6.2. Compare $x_i \cdot x_j = \sum_k c_{ij}^k x_k$ with $\delta_*(y_k) = \sum_{i,j} a_{ij}^k y_i \times y_j$, where $\delta : X \to X \times_S X$ is the diagonal map, $y_i \times y_j = p_1^*(y_i) \cdot p_2^*(y_j)$, and $a_{ij}^k, c_{ij}^k \in H^*S$. In fact, show that $c_{ij}^k = \pm a_{ij}^k$, and determine the sign.

(Of course, when all degrees are even – as they will be for applications in algebraic geometry – there is no sign.) The geometric meaning of this exercise is that the structure constants of the cohomology ring determine the Gysin map δ_*, and vice versa.

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