EQUIVARIANT COHOMOLOGY IN ALGEBRAIC GEOMETRY
LECTURE ONE: PREVIEW

WILLIAM FULTON
NOTES BY DAVE ANDERSON

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Let G be a Lie group acting on the left on a space X. Around 1960, Borel defined the *equivariant cohomology* H_G^*X as follows [Bor60]. One finds a contractible space EG on which G acts freely (on the right), with quotient $BG = EG/G$. Then form

$$EG \times^G X := EG \times X/(e \cdot g, x) \sim (e, g \cdot x).$$

(In effect, one replaces X by the homotopy-equivalent space $EG \times X$, on which G acts freely, and then forms the quotient. In the modern language of stacks, $EG \times^G X$ represents the (topological) quotient stack $[G\backslash X]$.)

Definition 1.1. The **equivariant cohomology** of X with respect to G is the ordinary (singular) cohomology of $EG \times^G X$:

$$H_G^i X = H^i(EG \times^G X).$$

This definition is independent of the choice of EG, as we will see.

For the special case of a point, we have

$$H_G^*(pt) = H^*(BG),$$

which we will denote by Λ_G or Λ. For any X, the map $X \to pt$ induces a pullback map $\Lambda_G \to H_G^*X$, so the equivariant cohomology of X has the structure of a Λ_G-algebra, at least when $H^i(BG) = 0$ for odd i. In general, this is a richer structure than the usual ring structure of classical cohomology.

Example 1.2. Let $G = \mathbb{C}^*$ (or S^1), and take $EG = \mathbb{C}^\infty \smallsetminus \{0\}$. Then $BG = \mathbb{CP}^\infty$, and $\Lambda_G = \mathbb{Z}[t]$. Here $t = c_1(L)$, with L the tautological line bundle on $\mathbb{CP}^\infty$ (so $L = \mathcal{O}(-1)$!).

Exercise 1.3. For $G = (\mathbb{C}^*)^n$, show that $\Lambda_G = \mathbb{Z}[t_1, \ldots, t_n]$. What is Λ_G for $G = GL_n(\mathbb{C})$?

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Remark 1.4. The term "equivariant cohomology" appeared some time after Borel introduced the notion in his seminar on transformation groups. Originally, it was used by topologists, mainly for finite groups, to answer questions about what kinds of manifolds G can act on, and with what fixed points.

Algebraic geometers were slow to appreciate or use equivariant cohomology — very little was done before 1990 — possibly because the spaces involved are infinite-dimensional and not algebraic. However, they are usually limits of finite-dimensional spaces, as we will see. For example, $\mathbb{C}^\infty = \bigcup_m \mathbb{C}^m$, and $\mathbb{P}^\infty = \bigcup_m \mathbb{P}^m$.

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We always use singular cohomology, with integer coefficients unless otherwise stated, and all spaces will be at least paracompact and Hausdorff. In fact, for us X will be a complex algebraic variety (usually nonsingular, but not necessarily compact), and G will be a linear algebraic group, usually a torus, $GL_n(\mathbb{C})$, or an explicit subgroup of $GL_n(\mathbb{C})$. We will construct finite-dimensional approximations $EG_m \to BG_m$ to $EG \to BG$, all of which will be algebraic manifolds, with

$$H^i_G X = H^i(EG_m \times^G X) \quad \text{for } m \geq m(2i).$$

Two key features of this theory are the existence of *Chern classes* and *fundamental classes*:

(i) If F is an equivariant complex vector bundle on X, then it has **equivariant Chern classes**

$$c_i^G(F) \in H^{2i}_G X,$$

defined as follows. Since F is equivariant, $EG \times^G F$ is a vector bundle on $EG \times^G X$; take $c_i^G(F)$ to be the ith Chern class of this bundle. Equivalently, one can define these using Chern classes of $EG_m \times^G F \to EG_m \times^G X$, for $m \geq m(i)$.

(ii) If $V \subseteq X$ is a G-invariant subvariety of codimension d, there is an **equivariant fundamental class** $[V]^G \in H^{2d}_G X$. In fact, $EG_m \times^G V$ is a subvariety of $EG_m \times^G X$, so we can take its fundamental class in the usual way (see Appendix A). To use this as a definition, one must check these classes are compatible as m varies, independent of the choices of EG and EG_m.

Example 2.1. Let ρ be a complex linear representation of G — that is, an equivariant vector bundle on a point, E_ρ. Thus there are classes $c_i(\rho) = c_i^G(E_\rho) \in H^*_G BG = \Lambda_G$.

Exercise 2.2. For a concrete example, let $G = \mathbb{C}^*$ act on $\mathbb{C}$ by $g \cdot z = g^a z$. This is an equivariant line bundle L_a on a point, and

$$c_1(L_a) = at \in \Lambda_G = \mathbb{Z}[t].$$

(This explains the choice of generator t.)

Example 2.3. Let F be an equivariant vector bundle of rank d, and let s be an equivariant section with $\operatorname{codim}_X Z(s) = d$. Then

$$[Z(s)]^G = c_d^G(F).$$

(Using approximation spaces EG_m, this reduces to the corresponding fact for nonequivariant classes.)

Exercise 2.4. Let $G = (\mathbb{C}^*)^n$ act on $\mathbb{C}^n$ in the natural way, and compute $[x_i = 0]^G \in H_G^2\mathbb{C}^n$.

Exercise 2.5. Let V and W be G-invariant subvarieties of X. If V and W are disjoint, then $[V]^G \cdot [W]^G = 0$. If G is connected, and V and W intersect properly, with $V \cdot W = \sum m_i Z_i$, then $[V]^G \cdot [W]^G = \sum m_i [Z_i]^G$.

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Much of what we do will involve fiber bundles

$$\begin{array}{c} EG \times^G X \\ \downarrow \\ BG \end{array} \quad \text{and} \quad \begin{array}{c} EG_m \times^G X \\ \downarrow \\ BG_m \end{array},$$

with fiber X. One can think of equivariant geometry as "spread-out geometry". These bundles are spread-out versions of X, in the same spirit as the passages from vector space to vector bundle (with $BGL_n(\mathbb{C})$), projective space $\mathbb{P}^n$ to projective bundle $\mathbb{P}(E)$, Grassmannian to Grassmann bundle, flag manifold to flag bundle, etc. — all familiar constructions in algebraic geometry.

In particular, restricting to a fiber gives a map

$$H_G^* X \to H^* X, \tag{1}$$

which one expects to be surjective (especially after looking at examples). Note that this comes from the fiber square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & EG \times^G X \arrow[d] \\ pt \arrow[r] & BG. \end{tikzcd}$$

If $I_G \subset \Lambda_G$ is the ideal $\bigoplus_{i>0} H^i(BG)$, we see from the diagram that there is a map

$$H_G^* X \otimes_{\Lambda_G} (\Lambda_G / I_G) \to H^* X, \tag{1'}$$

and we expect this to be an isomorphism.

There are several ways to give hypotheses which make these expectations true; the notion is often expressed by saying that X is "equivariantly formal", at least if one takes cohomology with rational coefficients. For all our examples, though, it will be easy to see directly, with integer coefficients.

Remark 3.1. The above "expectations" are certainly not always true. For a simple example, let $X = G$. As we will see later, $H^i_G(G) \cong H^i(G\backslash G) = H^i(pt)$, so the map $H^i_G(G) \to H^i(G)$ is usually not surjective.

Example 3.2. Let X be a homogeneous space (e.g., $\mathbb{P}^n$, $Gr(k,n)$, $Fl(n)$, G/P). Then H^*X has basis of *Schubert classes* $[\Omega_w]$, where $\Omega_w \subset X$ are *Schubert varieties*; these are indexed by $w \in W$, where W is the corresponding Weyl group or cosets. The Schubert varieties are invariant for an action of a maximal torus T (or for a Borel subgroup B), so there are also *equivariant Schubert classes* $\sigma_w = [\Omega_w]^T \in H^*_T X$, mapping to $[\Omega_w] \in H^*X$. (Note: the equivariant classes depend on a choice of flag; the classical ones don't.) These classes σ_w will be a basis for $H^*_T X$ over $\Lambda_T \cong \mathbb{Z}[t_1, \dots, t_n]$ (for $T \cong (\mathbb{C}*)^n$).

The structure of $H^*_T X$ is the subject of *equivariant Schubert calculus*: for $u, v, w \in W$,

$$\sigma_u \cdot \sigma_v = \sum_w c^w_{uv} \sigma_w,$$

with $c^w_{uv} \in \Lambda_T$ homogeneous of degree $\ell(u) + \ell(v) - \ell(w)$. (Here $\ell(w)$ is the codimension of Ω_w in X.) When $\ell(u) + \ell(v) = \ell(w)$, the classical numbers $c^w_{uv} \in \mathbb{Z}$ are nonnegative for geometric reasons (the Kleiman-Bertini theorem). The equivariant coefficients satisfy a positivity condition, too, which we will describe later (see [Gra01]).

For the Grassmannian $Gr(k, \mathbb{C}^n)$, the classical numbers are the *Littlewood-Richardson coefficients*, and combinatorial formulas for these exist; in fact, there is also a combinatorial formula for the equivariant coefficients [Knu-Tao03], which we will discuss later. For the two-step flag manifold $Fl(k_1, k_2); \mathbb{C}^n)$, a combinatorial formula for classical coefficients has recently been proved [Cos07], and there are conjectural formulas for the classical and equivariant numbers by Knutson and Buch. There are also rules for classical Schubert calculus on certain G/P in types other than A: Pragacz showed that a rule of Stembridge computes the structure constants of the Lagrangian Grassmannian [Pra91], and Thomas and Yong recently gave a type-uniform rule for all cominiscule flag varieties [Tho-Yong06].

Despite many attempts, no other families of homogeneous spaces even have conjectured combinatorial formulas for the classical or equivariant structure constants. For example, if $X = Fl(n)$, there are integers (or polynomials) c^w_{uv} for each triple of permutations $u, v, w \in S_n$ — but it has been an open problem for a long time to find a combinatorial description of them. However, there are explicit presentations for $H^*Fl(n)$ and $H^*_T Fl(n)$, as well as "Giambelli" formulas for the classes σ_w, which we will describe.

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There is a second general expectation for the behavior of equivariant cohomology with respect to fixed points. Let X^G denote the fixed point set, so the inclusion $X^G \hookrightarrow X$ determines a map $H^*_G X \to H^*_G X^G$. Now $H^*_G X^G = H^*(BG \times X^G) = \Lambda_G \otimes H^* X^G$, when this Künneth formula holds (e.g., if X^G is finite). One expects this map to be an embedding:

$$H^*_G X \hookrightarrow H^*_G X^G. \tag{2}$$

When X^G is finite, $H^*_G X^G = \bigoplus \Lambda_G$, with one copy of Λ_G for each fixed point. In fact, (2) should be an isomorphism after localizing at the quotient field of Λ_G, and one should be able to describe the image explicitly. (Some hypotheses are certainly needed here: for example, X can't be replaced by $X \smallsetminus X^G$.) In the case of a torus, this says giving a class in $H^*_G X$ is the same as giving certain polynomials at each fixed point. The idea here goes back to [Cha-Skj74]; more recently and more generally, see [Gor-Kot-Mac98].

Using the maps (1′) and (2), one can hope to study Schubert calculus by computing the images of σ_w's. The fact that equivariant Schubert calculus involves nonzero polynomial structure constants can help: one sometimes sees non-trivial identities of polynomials which reduce to an uninformative "$0 = 0$" in classical cohomology.

We will concentrate on homogeneous varieties and toric varieties, reviewing some basic facts about these as necessary. Recently, other spaces have been studied, including some Hilbert schemes [Li-Qin-Wang04], [Eva05], [Nie06].

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Example 5.1. In 1982, Lascoux and Schützenberger defined and began studying *Schubert polynomials* $\mathfrak{S}_w(x_1, \ldots, x_n) \in \mathbb{Z}[x_1, \ldots, x_n]$, for $w \in S_n$, which represent the Schubert classes $[\Omega_w] \in H^* Fl(n)$. They also defined *double Schubert polynomials* $\mathfrak{S}_w(x_1, \ldots, x_n; y_1, \ldots, y_n)$. These specialize to the single Schubert polynomials under $y_i \mapsto 0$, and have many wonderful properties — for example, they multiply exactly as Schubert classes do (if n is sufficiently large):

$$\mathfrak{S}_u \cdot \mathfrak{S}_v = \sum_w c^w_{uv} \mathfrak{S}_w.$$

We will have more to say about these polynomials later, but for now, we mention one way they arise naturally from equivariant geometry, which shows how they could have been discovered much earlier. For a permutation $w \in S_n$, form the matrix A_w with 1's in the $w(i)$th column of the ith row, and 0's elsewhere. For example, if $w = 2\,3\,1$, we have

$$A_{231} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

For any $n \times n$ matrix A, let $A^{(p,q)}$ denote the upper-left $p \times q$ submatrix. Now let

$$\Omega_w = \{A \in M_{n,n} \mid \mathrm{rk}(A^{(p,q)}) \le \mathrm{rk}(A_w^{(p,q)}) \text{ for all } 1 \le p, q \le n\}.$$

This is an irreducible subvariety of codimension $\ell(w)$, and is invariant for the action of $T = (\mathbb{C}^*)^n \times (\mathbb{C}^*)^n$, given by $(u, v) \cdot A = u\,A\,v^{-1}$. (In fact, it is also invariant for a similar action of Borel groups.) Thus there is a class

$$[\Omega_w]^T \in H_T^*(M_{n,n}) = \mathbb{Z}[x_1, \ldots, x_n, y_1, \ldots, y_n],$$

where the x's and y's are the usual classes for $\mathbb{C}^*$, which were denoted by t earlier. (Note that $M_{n,n}$ is contractible.)

Claim . This class is equal to the Schubert polynomial of Lascoux and Schützenberger: $[\Omega_w]^T = \mathfrak{S}_w(x_1, \ldots, x_n; y_1, \ldots, y_n)$.

For example, Ω_{231} is defined by $X_{11} = X_{21} = 0$, so we have $\mathfrak{S}_{231} = (x_1 - y_1)(x_2 - y_1)$. In fact, the locus $X_{ij} = 0$ has equivariant class $x_i - y_j$.

This fact was discovered by Fehér and Rimányi [Feh-Rim03] and Knutson and Miller [Knu-Mil05]. Some of the modern story of equivariant cohomology in algebraic geometry has its origins in work of Fehér and Rimányi; see [Feh-Rim02] and references therein.

Exercise 5.2. Compute $\mathfrak{S}_w$ for other $w \in S_n$, with $\ell(w)$ and n small. For instance, try all $w \in S_3$, or those of length at most 2 in S_4. Can you find a relation between $\mathfrak{S}_w$ and $\mathfrak{S}_{w^{-1}}$?

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