EQUIVARIANT COHOMOLOGY IN ALGEBRAIC GEOMETRY
LECTURE TEN: MORE ON FLAG VARIETIES

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1

A. Molev has just given a simple, efficient, and positive formula for the structure constants $c^{\nu}_{\lambda\mu}$ for multiplication in $H_T^*Gr(k,n)$, without puzzles [Mol07]:

$$c^{\nu}_{\lambda\mu} = \sum_R \sum_T \prod_\alpha (t_{\ell+T(\alpha)-c(\alpha)} - t_{\ell+T(\alpha)-\rho(\alpha)_{T(\alpha)}}). \tag{1}$$

Here $\ell = n-k$, as usual. The rest of the notation is described as follows:

- The outer sum is over all sequences
$$R:\ \mu = \rho^{(0)} \subset \rho^{(1)} \subset \cdots \subset \rho^{(s)} = \nu,$$
where $s = |\nu| - |\mu|$, and $\rho^{(i)}$ is a partition obtained from $\rho^{(i-1)}$ by adding one box. Let r_i be the row of the box added in $\rho^{(i)} \smallsetminus \rho^{(i-1)}$.
- The inner sum is over all "reverse, barred, ν-bounded tableaux T" on the shape λ. This means T is a filling of λ using entries from $\{1,\dots,k\}$, weakly *decreasing* along rows and strictly decreasing down columns. One also chooses s of the entries (or boxes of λ) to be "barred"; these entries must be $r_1, r_2, \dots, r_s$, occurring in this order when the columns of T are read bottom-to-top, left-to-right. Finally, the entries in the jth column of T must be less than or equal to the number of boxes in the jth column of ν (i.e., $T(i,j) \le \nu'_j$).
- The product is over the boxes $\alpha = (i,j)$ of λ containing an unbarred entry of T. Also, $c(\alpha) = j - i$ is the "content" of α, and $\rho(\alpha)$ is the partition $\rho^{(t)}$, where t is the number of barred boxes occurring before α in the column reading order.

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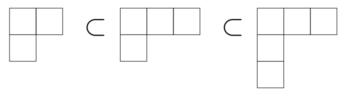
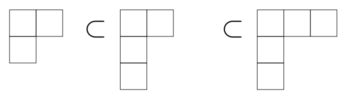
Example 1.1. For $k = \ell = 3$ and $\lambda = \mu = (2,1)$, $\nu = (3,1,1)$, there are two sequences R:

R_1 :  $r_1 = 1,\ r_2 = 3$

R_2 :  $r_1 = 3,\ r_2 = 1$

There is only one tableau for the sequence R_1:

$$\begin{array}{ll} \bar{3} & 1 \\ \bar{1} & \end{array} \qquad t_{3+1-1} - t_{3+1-3} = t_3 - t_1. \quad (\rho = (3,1,1))$$

For R_2, there are two tableaux:

$$\begin{array}{ll} \bar{3} & \bar{1} \\ 1 & \end{array} \qquad t_{3+1+1} - t_{3+1-2} = t_5 - t_2 \quad (\rho = (2,1))$$

$$\begin{array}{ll} \bar{3} & \bar{1} \\ 2 & \end{array} \qquad t_{3+2+1} - t_{3+2-1} = t_6 - t_4. \quad (\rho = (2,1))$$

So the rule says $c^{\nu}_{\lambda\mu} = t_6 - t_4 + t_5 - t_2 + t_3 - t_1$.

Part of the claim is that all terms are positive — i.e., $\rho(\alpha)_{T(\alpha)} > c(\alpha)$. The proof is almost the same as that of the original Molev-Sagan rule [Mol-Sag99] (remarkably, since that rule involved non-positive cancellation), together with a combinatorial argument showing that the "ν-bounded" tableaux pick out the positive (nonzero) terms.

Question 1.2. Is there a bijection between the tableaux T in Molev's rule and the Knutson-Tao puzzles?

Note the independence of k, and the simple dependence on ℓ: Replacing k by $k+h$ and ℓ by $\ell+m$, the coefficient $c^{\nu}_{\lambda\mu}$ for multiplication in $H^*_T Gr(k+h, n+h+m)$ is obtained from that for $H^*_T Gr(k,n)$ by replacing t_i with t_{i+m}. We'll see a generalization of this kind of stability below.

Exercise 1.3. Prove this fact using puzzles: see what happens when you place a 0 at the beginning of each string, or a 1 at the end of each string.

2

In the last lecture, we saw that under the projection $f : Fl(\mathbb{C}^n) \to Gr(k,n)$, the inverse image of $\Omega_\lambda(F_\bullet)$ is $\Omega_{w(\lambda)}(F_\bullet)$, so $f^*\sigma_\lambda = \sigma_{w(\lambda)}$. (Recall that if $I(\lambda) = \{i_1 < \cdots < i_k\}$ and $J(\lambda) = \{j_1 < \cdots < j_\ell\}$, then $w(\lambda) = j_1 \cdots j_\ell\, i_1 \cdots i_k$..) Replacing k with $k+h$ and ℓ with $\ell + m$ takes $w(\lambda)$ to

$$1\,2 \cdots m\ (j_1 + m) \cdots (j_\ell + m)\ (i_1 + m) \cdots (i_k + m)\ (n+m+1) \cdots (n+m+h).$$

Note that the last h entries are irrelevant, since they are larger than all the preceding entries. In general, the embedding $S_n \hookrightarrow S_{m+n}$ (which lets S_n act on the last n letters in an alphabet of size $m+n$) takes w to $1^m \times w$, where

$$1^m \times w = 1\,2\,\cdots\,m\,(w_1+m)\,\cdots\,(w_n+m).$$

Molev's stability generalizes as follows. For $u, v, w \in S_n$, we have $\sigma_u\,\sigma_v = \sum c_{uv}^w\,\sigma_w$ in $H_T^* Fl(\mathbb{C}^n)$, with $c_{uv}^w \in \Lambda_T = \mathbb{Z}[t_1, \dots, t_n]$.

Proposition 2.1. *$c_{1^m\times u, 1^m\times v}^{1^m\times w}$ is obtained from c_{uv}^w by mapping t_i to t_{i+m}.*

We need an algebraic lemma:

Lemma 2.2 ([Buch-Rim04], Cor. 4)**.** *For $v \in S_{m+n}$, we have*

$$\begin{aligned}&\mathfrak{S}_v(z_1,\dots,z_m,x_1,\dots,x_n|z_1,\dots,z_m,y_1,\dots,y_n)\\ &= \begin{cases} \mathfrak{S}_w(x_1,\dots,x_n|y_1,\dots,y_n) & \textit{if } v = 1^m\times w \textit{ for some } w\in S_n;\\ 0 & \textit{otherwise.}\end{cases}\end{aligned}$$

The proposition follows, since we have

$$\mathfrak{S}_{1^m\times u}(x|t)\cdot\mathfrak{S}_{1^m\times v}(x|t) = \sum c_{1^m\times u,1^m\times v}^w\,\mathfrak{S}_w(x|t).$$

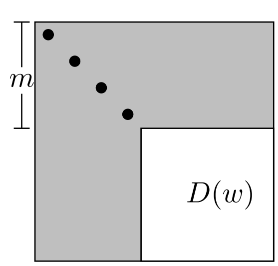
Set $x_i = t_i$ for $1 \le i \le m$ in $x = (x_1, \dots, x_{m+n})$, and apply the lemma.

The lemma can be proved geometrically:

Proof. Recall from last lecture that $\mathfrak{S}_w$ is characterized by the fact that $\mathfrak{S}_w(x|y) = [\Omega_w(\varphi)]$, for $\varphi : E \to F$ a general map of flagged vector bundles. Take general line bundles $L_1, \dots, L_m$, with $z_i = c_1(L_i)$, and let $H_i = L_1 \oplus \cdots \oplus L_i$. Then we have a map $id \times \varphi$ of flagged vector bundles $H_m \oplus E \to H_m \oplus F$, as in the following diagram:

$$\begin{tikzcd} H_1\oplus 0 \arrow[r, phantom, "\subset"] & \cdots \arrow[r, phantom, "\subset"] & H_m\oplus 0 \arrow[r, phantom, "\subset"] & H_m\oplus E_1 \arrow[r, phantom, "\subset"] & \cdots \arrow[r, phantom, "\subset"] & H_m\oplus E_n \arrow[d, "id\times\varphi"'] \\ H_1\oplus 0 & \cdots \arrow[l] & H_m\oplus 0 \arrow[l] & H_m\oplus F_1 \arrow[l] & \cdots \arrow[l] & H_m\oplus F_n. \arrow[l] \end{tikzcd}$$

The locus $\Omega_v(id\times\varphi)$ is empty unless $v = 1^m \times w$, since $v(i) \ne i$ for $i \le m$ would force $\mathrm{rk}(H_m \to H_m) < m$. For $v = 1^m \times w$, the locus is the same as $\Omega_w(\varphi)$, as can be seen from the diagram $D(1^m \times w)$:



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□

This stability corresponds to the embedding $\iota : Fl(n) \hookrightarrow Fl(m+n)$ which sends $L_1 \subset \cdots \subset L_n$ to $\mathbb{C}^1 \subset \cdots \subset \mathbb{C}^m \subset \mathbb{C}^m \oplus L_1 \subset \cdots \subset \mathbb{C}^m \oplus L_n$. We have

$$\iota^*\sigma_v = \begin{cases} \sigma_w & \text{if } v = 1^m \times w; \\ 0 & \text{otherwise,} \end{cases}$$

$$\iota^*x_i = \begin{cases} x_{i-m} & \text{if } i > m; \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\iota^*t_i = \begin{cases} t_{i-m} & \text{if } i > m; \\ 0 & \text{otherwise.} \end{cases}$$

The other obvious embedding puts the "fixed parts" last: $\jmath : Fl(n) \to Fl(n+m)$ sends $L_\bullet$ to $L_1 \subset \cdots \subset L_n \subset L_n \oplus \mathbb{C} \subset \cdots \subset L_n \oplus \mathbb{C}^m = \mathbb{C}^{n+m}$. The corresponding inclusion $S_n \subset S_{n+m}$ is the usual one, with $v \mapsto v$. We have

$$\jmath^*\sigma_v = \begin{cases} \sigma_v & \text{if } v \in S_n \subset S_{n+m}; \\ 0 & \text{otherwise,} \end{cases}$$

$$\jmath^*x_i = \begin{cases} x_i & \text{if } i \leq m; \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\jmath^*t_i = \begin{cases} t_i & \text{if } i \leq m; \\ 0 & \text{otherwise.} \end{cases}$$

An important property of Schubert polynomials, visible from the second stability above, is that $\mathfrak{S}_w(x|y)$ is independent of n, for $w \in S_n$. Also, they multiply with the same structure constants as the Schubert classes σ_w; more precisely, for $u, v \in S_n$ we have

$$\mathfrak{S}_u(x|y) \cdot \mathfrak{S}_v(x|y) = \sum c_{uv}^w(y)\mathfrak{S}_w(x|y), \tag{2}$$

where the sum is over $w \in S_{2n-1}$. In fact, it suffices to consider w which are less than $(2n-1)\,(2n-3)\,\cdots\,3\,1\,2\,4\,\cdots\,(2n-2)$ in *Bruhat order* (to be defined below), and such that $w(n) < w(n+1) < \cdots$. The first condition must be satisfied, since all the monomials which appear on the LHS divide $(x_1^{n-1} \cdots x_{n-1})^2$. To see the second condition holds, recall that $\mathfrak{S}_w$ is symmetric in x_k and x_{k+1} iff $w(k) < w(k+1)$; since x_k does not appear $\mathfrak{S}_u$ or $\mathfrak{S}_v$ for $k \geq n$, the LHS is certainly symmetric in x_k and x_{k+1} for all $k \geq n$.

By the simple stability property, (2) specializes to the corresponding identity in $H_T^*Fl(N)$ for any $N \geq n$, discarding those $\mathfrak{S}_w$ with $w \notin S_N$.

Remark 2.3. If one uses an algebraic proof to see $s_{\lambda'}(x|t) = \mathfrak{S}_{w(\lambda)}(x|t)$, then the degeneracy locus formula for flags implies the formulas of Kempf-Laksov and Thom-Porteous.

3

Recall that we write p_v for the flag $p_v = \langle e_{v(n)} \rangle \subset \langle e_{v(n)}, e_{v(n-1)} \rangle \subset \cdots$ (which is in the Schubert variety $\Omega_v(F_\bullet)$). For $w \in S_n$, let $\sigma_w|_v$ be the image of σ_w under the restriction map to $H_T^*(p_v) = \Lambda$.

Proposition 3.1. $\sigma_w|_v = \mathfrak{S}_w(t_{v(1)}, \ldots, t_{v(n)}|t_1, \ldots, t_n)$.

Proof. Restricting to p_v, the tautological quotient bundle Q_p becomes

$$\mathbb{C}^n / \langle e_{v(n)}, \ldots, e_{v(n+1-p)} \rangle = \langle e_{v(1)}, \ldots, e_{v(p)} \rangle,$$

so $x_i \mapsto t_{v(i)}$. □

Example 3.2. We have $\sigma_{s_k}|_v = \sum_{i=1}^n (t_{v(i)} - t_i)$. Note that if $u \neq v$, there is at least one k such that $\sigma_{s_k}|_u \neq \sigma_{s_k}|_v$: for example, the minimal k such that $u(k) \neq v(k)$ works.

As usual, $\sigma_w|_v = 0$ unless $p_v \in \Omega_w$, i.e., $\Omega_v \subset \Omega_w$. This is one characterization of the **Bruhat order** on S_n. There are many others: One writes $w \leq v$ if, equivalently,

(i) $\Omega_v \subset \Omega_w$.
(ii) $r_w(q,p) \geq r_v(q,p)$ for all p and q.
(iii) $\{w_1, \ldots, w_k\} \leq \{v_1, \ldots, v_k\}$ for all k, where the order on subsets is by sorting the elements, and comparing termwise.
(iv) There is a chain $w = w^{(0)} \to w^{(1)} \to \cdots \to w^{(s)} = v$, where each step is of the form $u \to u \cdot t$, with $t = (i,j)$ the transposition exchanging entries in positions i and j, and $\ell(u \cdot t) = \ell(u) + 1$. That is, $u_i < u_j$, and u_k does not lie between u_i and u_j for all $i < k < j$.

(v) There is an expression $v = s_{i_1} \cdots s_{i_\ell}$ with $\ell = \ell(v)$ such that w is given by a subsequence of length $\ell(w)$.

Write $u \xrightarrow{k} v$ if $v = u \cdot t$ as in (iv) with $t = (i,j)$ and $i \leq k < j$.

Proposition 3.3. *We have*

$$\begin{aligned} \sigma_w|_w &= \mathfrak{S}_w(t_{w(1)}, \ldots, t_{w(n)}|t_1, \ldots, t_n) \\ &= \prod_{\substack{i<j \\ w(i)>w(j)}} (t_{w(i)} - t_{w(j)}). \end{aligned}$$

There are algebraic proofs (cf. [Buch-Rim04]). Geometrically, it is similar to the Grassmann case: Look at the neighborhood U_w of p_w, and compute the tangent space to Ω_w^o as in the last lecture; the weights on the normal space to Ω_w at p_w will be the weights of $T_{p_w}U_w$ not in $T_{p_w}\Omega_w^o$.

Example 3.4. For $w = 4\,1\,6\,3\,2\,7\,5$, the normal space to Ω_w is given by the $*$'s:

$$\begin{pmatrix} & & & 1 & & & \\ 1 & & & * & & & \\ 0 & & & 0 & & 1 & \\ 0 & & 1 & * & & * & \\ 0 & 1 & * & * & & * & \\ 0 & 0 & 0 & 0 & & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & * & * \end{pmatrix}.$$

The corresponding weights are $t_3 - t_2$, $t_4 - t_1$, $t_4 - t_3$, $t_4 - t_2$, $t_6 - t_3$, $t_6 - t_2$, $t_6 - t_5$, and $t_7 - t_5$.

Proposition 3.5 (Equivariant Monk rule)**.** *We have*

$$\sigma_{s_k} \cdot \sigma_w = \sum_{w \xrightarrow{k} w^+} \sigma_{w^+} + (\sigma_{s_k}|_w)\,\sigma_w.$$

Proof. As for the Grassmannian case, the only possible σ_v appearing on the RHS have $v \leq w$ and $\ell(w) - \ell(v) \leq 1$. (One sees this by Poincaré duality, intersecting with $\widetilde{\sigma}_{w_0 v}$.) The sum in the first part of the RHS is the classical Monk rule; see [Ful97] for a proof. The second part is seen by restriction to p_w, using the fact that $\sigma_w|_w \neq 0$ (and $\sigma_{w^+}|_w = 0$). $\square$

Proposition 3.6. *The polynomials c_{uv}^w satisfy and are uniquely determined by the following three properties:*

(i) $c_{ww}^w = \sigma_w|_w = \displaystyle\prod_{\substack{i<j \\ w(i)>w(j)}} (t_{w(i)} - t_{w(j)})$*;*

(ii) $(\sigma_{s_k}|_u - \sigma_{s_k}|_v)\, c_{uv}^u = \displaystyle\sum_{v \xrightarrow{k} v^+} c_{uv^+}^u$*; and*

(iii) $(\sigma_{s_k}|_w - \sigma_{s_k}|_u) c_{uv}^w = \displaystyle\sum_{u \xrightarrow{k} u^+} c_{u^+v}^w - \sum_{w^- \xrightarrow{k} w} c_{uv}^{w^-}.$

Proof. The proof is essentially the same as in the Grassmannian case. To show that (iii) is satisfied, use the Monk rule and associativity:

$$\begin{aligned} \sigma_{s_k} \cdot (\sigma_u \cdot \sigma_v) &= \sum c_{uv}^w \sigma_{s_k} \cdot \sigma_w \\ &= \sum_{w \xrightarrow{k} w^+} c_{uv}^w \sigma_{w^+} + \sum_w c_{uv}^w (\sigma_{s_k}|_w) \sigma_w, \end{aligned}$$

and

$$\begin{aligned}(\sigma_{s_k} \cdot \sigma_u) \cdot \sigma_v &= \sum_{u \xrightarrow{k} u^+} \sigma_{u^+} \cdot \sigma_v + (\sigma_{s_k}|_u)\sigma_u \cdot \sigma_v \\ &= \sum_{u \xrightarrow{k} u^+} c_{u^+v}^w \sigma_w + (\sigma_{s_k}|_u) \sum_w c_{uv}^w \sigma_w.\end{aligned}$$

Equating the coefficients of σ_w on the RHS's gives

$$\sum_{w^- \xrightarrow{k} w} c_{uv}^{w^-} + (\sigma_{s_k}|_w) c_{uv}^w \quad = \quad \sum_{u \xrightarrow{k} u^+} c_{u^+v}^w \sigma_w + (\sigma_{s_k}|_u) c_{uv}^w,$$

which is (iii).

Setting $w = v$ in (iii) gives

(ii′) $(\sigma_{s_k}|_v - \sigma_{s_k}|_u) c_{uv}^v = \sum_{u \xrightarrow{k} u^+} c_{u^+v}^v$,

using the fact that $c_{uv}^{v^-} = 0$, since $v \not\leq v^-$. Using commutativity ($c_{uv}^w = c_{vu}^w$) and interchanging u and v turns (ii′) into (ii).

The uniqueness statement is also almost the same as before. If $u = v = w$, c_{uv}^w is given by (i). If $u = w$, then c_{uv}^w is given by (ii), using induction on $\ell(u) - \ell(v)$: one starts with $v = w_0$ and uses the fact that one can always find a k such that $\sigma_{s_k}|_u \neq \sigma_{s_k}|_v$. Finally, if $u \neq w$, (iii) gives c_{uv}^w by induction on $\ell(w) - \ell(v)$. □

Remark 3.7. Conditions (i), (ii′), and (iii) also characterize c_{uv}^w.

Remark 3.8. These conditions also determine the (unknown!) classical coefficients c_{uv}^w, as well as the Grassmannian coefficients $c_{\lambda\mu}^\nu = c_{w(\lambda)\,w(\mu)}^{w(\nu)}$.

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