EQUIVARIANT COHOMOLOGY IN ALGEBRAIC GEOMETRY
LECTURE ELEVEN: POSITIVITY

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1

We will need a general fact relating intersection products and the diagonal. The setup is as follows: Let $X \xrightarrow{\rho} S$ be a (locally trivial) fiber bundle, where X and S are oriented smooth manifolds, and the fibers are compact n-dimensional manifolds. (In our application, S will be an approximation space for BG.) Then we have a Gysin map $\rho_* : H^i X \to H^{i-n} S$. Assume $H^* X$ is a free module over $H^* S$, and let $\{x_i\}$ be a basis of homogeneous elements for $H^* X$ over $H^* S$. (Note that this does not depend on whether we regard $H^* X$ as a left or right $H^* S$-module.) Let $\{y_i\}$ be the (right) dual basis, so $\deg(y_i) = n - \deg(x_i)$, and

$$\langle x_i, y_j \rangle := \rho_*(x_i \cdot y_j) = \delta_{ij}.$$

Write

$$x_i \cdot x_j = \sum_k c_{ij}^k x_k.$$

Consider the diagonal embedding $\delta : X \hookrightarrow X \times_S X$, with projections $p_1, p_2 : X \times_S X \to X$, and write

$$\delta_*(y_k) = \sum_{i,j} (y_i \times y_j)\, d_{ij}^k,$$

where by definition, $y \times z = p_1^* y \cdot p_2^* z$. (So the classes $y_i \times y_j$ form a basis for $H^*(X \times_S X)$ over $H^* S$.)

Proposition 1.1. $c_{ij}^k = (-1)^{\deg(y_i)\deg(x_j)} d_{ij}^k$.

Question 1.2. Is there a reference for this fact, even in the case where S is a point? (It was certainly known to Lefschetz.)

Question 1.3. Is there a choice of conventions which removes the sign?

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The sign in Proposition 1.1 can become a mess, depending on what conventions one uses for orientations. Here we take $X \times_S X$ to be oriented so that

$$\begin{tikzcd} X \times_S X \arrow[r, "p_2"] \arrow[d, "p_1"'] \arrow[dr, "q"] & X \arrow[d, "\rho"] \\ X \arrow[r, "\rho"'] & S \end{tikzcd}$$

is an oriented square — so $p_{1*}p_2^* = \rho^*\rho_*$. Note that $\rho = q \circ \delta$, $p_a \circ \delta = \mathrm{id}_X$, and $q = \rho \circ p_a$ for $a = 1, 2$.

Lemma 1.4. $q_*(p_1^*(u) \cdot p_2^*(v)) = (-1)^{n(\deg(v)-n)}\rho_*(u) \cdot \rho_*(v)$.

Proof. Using the projection formula,

$$\begin{aligned} LHS = \rho_* p_{1*}(p_1^*(u) \cdot p_2^*(v)) &= \rho_*(u \cdot p_{1*}p_2^*(v)) \\ &= \rho_*(u \cdot \rho^*\rho_*(v)) \\ &= (-1)^{n(\deg(v)-n)}\rho_*(u) \cdot \rho_*(v). \end{aligned}$$

□

The last step uses the fact that the projection formula depends on order (see Appendix A, Remark 5.4):

Exercise 1.5. $\rho_*(x \cdot \rho^* y) = (-1)^{n \deg(y)}\rho_*(x) \cdot y$.

Proof of Proposition 1.1. Compute:

$$\begin{aligned} c_{ij}^k = \sum_\ell c_{ij}^\ell \rho_*(x_\ell y_k) &= \sum \rho_*(c_{ij}^\ell x_\ell \cdot y_k) \\ &= \rho_*(x_i \cdot x_j \cdot y_k) \\ &= q_*\delta_*(\delta^*(p_1^* x_i \cdot p_2^* x_j) \cdot y_k) \\ &= q_*(p_1^* x_i \cdot p_2^* x_j \cdot \delta_* y_k) \\ &= \sum_{a,b} q_*(p_1^* x_i \cdot p_2^* x_j \cdot p_1^* y_a \cdot p_2^* y_b \cdot d_{ab}^k) \\ &= \sum_{a,b} (-1)^{\deg(x_j)\deg(y_a)} q_*(p_1^*(x_i \cdot y_a) \cdot p_2^*(x_j \cdot y_b) \cdot d_{ab}^k) \\ &= (-1)^{\deg(x_j)\deg(y_i)} q_*(p_1^*(x_i \cdot y_i) \cdot p_2^*(x_j \cdot y_j)) \cdot d_{ij}^k, \end{aligned}$$

where the last step uses Lemma 1.4 to see the only nonzero terms are for $a = i$ and $b = j$. Since q has relative dimension $2n$, there is no sign, and this is

$$\begin{aligned} &= (-1)^{\deg(x_j)\deg(y_i)} \rho_*(x_i \cdot y_i)\rho_*(x_j \cdot y_j) \cdot d_{ij}^k \\ &= (-1)^{\deg(x_j)\deg(y_i)} d_{ij}^k, \end{aligned}$$

as asserted. □

Exercise 1.6. $\delta_*(1) = \sum_i y_i \times x_i$.

(What is a reference for this?)

2

We now describe Graham's positivity theorem. Let $X = G/P$, where G is a complex semisimple group and P is parabolic subgroup, and let $T \subset B \subset G$ be a maximal torus and a Borel subgroup. Then $H_T^* X$ has a basis of classes $\sigma_u = [\Omega_u]^T$, for $u \in W^P$. Here Ω_u is a B-invariant subvariety, and W^P is the set of cosets in the Weyl group of G for the subgroup generated by reflections corresponding to roots in P.

Theorem 2.1 ([Gra01])**.** *Write*

$$\sigma_u \cdot \sigma_v = \sum_w = c_{uv}^w \sigma_w.$$

Then $c_{uv}^w \in \mathbb{Z}_{\geq 0}[\chi_1, \dots, \chi_m]$, where the χ_i are the weights of the action of T on $\mathfrak{n}^{opp}$. Here B^{opp} is the opposite Borel subgroup, $B^{opp} = T \cdot N^{opp}$ is its Levi decomposition (so N^{opp} is unipotent), and $\mathfrak{n}^{opp} = \mathrm{Lie}(N^{opp})$.

For example, if B is upper-triangular matrices in $G = GL_n$, then B^{opp} is lower-triangular matrices, and $\mathfrak{n}^{opp}$ is strictly lower-triangular matrices. The weights χ_i are $t_j - t_i$, for $j > i$.

The difficulty in the theorem is that G does not act on $EG \times^G X$.

Proof. There is an equivariant Poincaré duality, which follows from the classical case (and is similar to the version we have seen for $H_T^* Fl(n)$):

$$\langle \sigma_u, \tau_v \rangle = \delta_{uv},$$

where $\tau_v = \widetilde{\sigma}_{w_0 v} = [\widetilde{\Omega}_{w_0 v}]^T$. (Here $\widetilde{\Omega}_{w_0 v}$ is a B^{opp}-invariant subvariety, the orbit closure for the T-fixed point corresponding to $w_0 v$.) Now

$$\delta_*(\tau_w) = \sum_{u,v} c_{uv}^w \tau_u \times \tau_v$$

by Proposition 1.1, where $\delta : X \hookrightarrow X \times X$ is the diagonal embedding.

Let $\mathbf{N} = N^{opp} \times N^{opp}$, and consider the action of $\mathbf{B} = T \cdot \mathbf{N}$ on $X \times X$. (Here $T \subset \mathbf{B}$ is the "diagonal" torus.) There are a finite number of $\mathbf{N}$ orbits (and also of $\mathbf{B}$ orbits), whose closures have classes $\tau_u \times \tau_v$. The subvariety $V = \delta(\widetilde{\Omega}_{w_0 v}) \subset X \times X$ is T-invariant. The theorem then follows from Lemma 2.2 below. □

Lemma 2.2 (cf. [Gra01, Theorem 3.2])**.** *Let $B = T \cdot N$ be a solvable group, with maximal torus T and unipotent radical N, acting on a variety X with a finite number of N-orbits.*[1] *For a T-invariant subvariety V, we have*

$$[V]^T = \sum c_i [W_i]^T$$

[1]As observed by Brion, these are also B-orbits; see [Gra01, Lemma 3.3.].

for B-invariant subvarieties W_i, with $c_i \in \mathbb{Z}_{\geq 0}[\chi_1, \dots, \chi_m]$, where χ_i is positive on $\mathfrak{n} = \mathrm{Lie}(N)$.

See [Gra01] for the proof. In the application, note that the characters of T which are positive on $\mathrm{Lie}(\mathbf{N})$ are those positive on $\mathfrak{n}^{opp}$.

Example 2.3. Consider the action of $B = T \cdot N$ on $\mathbb{P}^1$ given by the map $B \to GL_2$,

$$t \cdot n \mapsto \begin{pmatrix} \chi_1(t) & \varphi(n) \\ 0 & \chi_2(t) \end{pmatrix},$$

where χ_1 and χ_2 are characters, and $\varphi : N \to \mathbb{G}_a = \mathbb{A}^1$ is a homomorphism making this an action. There are two T-fixed points, $p = [1:0]$ and $q = [0:1]$; and one B-fixed point, p. Clearly $[p]^T = [p]^T$, and $[\mathbb{P}^1]^T = 1$. The nontrivial case is $[q]^T = [p]^T + (\chi_1 - \chi_2) \cdot 1$. Indeed, we know $H_T^*\mathbb{P}^1 = \Lambda[\zeta]/(\zeta + \chi_1)(\zeta + \chi_2)$, and $[p]^T = \zeta + \chi_2$ and $[q]^T = \zeta + \chi_1$, so $[q]^T - [p]^T = \chi_1 - \chi_2$. Note that $\chi_1 - \chi_2$ is the weight of T on $\mathbb{A}^1$, so it is positive on $\mathfrak{n}$.

Remark 2.4. It is not necessary that $X = G/P$ in the theorem: all that is needed is a basis $\{\sigma_u\}$ of classes of B-invariant subvarieties, and a Poincaré dual basis $\{\tau_u\}$ of classes of B^{opp}-invariant subvarieties.

3

We conclude with some more facts about double Schubert polynomials. First, there is a duality on $Fl(\mathbb{C}^n)$ (see [BKTY04, §4.1]). Consider a complete flags of bundles on a variety X (e.g., $X = Fl(\mathbb{C}^n)$),

$$F_1 \subset \cdots \subset F_n = V = E_n \to \cdots \to E_1,$$

so there are degeneracy loci $\Omega_w = \Omega_w(F_\bullet \to E_\bullet)$, where

$$\Omega_w(F_\bullet \to E_\bullet) = \{x \in X \mid \mathrm{rk}(F_p \to E_q) \leq r_w(q,p)\}.$$

Let $F'_i = \ker(V \to E_{n-i})$, and let $E'_i = V/F_{n-i}$, so we have

$$F'_1 \subset \cdots \subset F'_n = V = E'_n \to \cdots \to E'_1,$$

with degeneracy loci $\Omega'_w = \Omega_w(F'_\bullet \to E'_\bullet)$.

Exercise 3.1. $\Omega_w = \Omega'_{w_0\, w^{-1}\, w_0}$.

If $x_i = c_1(\ker(E_i \to E_{i-1}))$ and $y_i = c_1(F_i/F_{i-1})$, then the exchanges $E_\bullet \leftrightarrow E'_\bullet$ and $F_\bullet \leftrightarrow F'_\bullet$ exchange x_i and y_{n+1-i}. The exercise implies

$$\begin{aligned} \mathfrak{S}_w(x|y) &\equiv \mathfrak{S}_{w_0\, w^{-1}\, w_0}(y_n, \dots, y_1 | x_n, \dots, x_1) \pmod{I} \\ &= (-1)^{\ell(w)} \mathfrak{S}_{w_0\, w\, w_0}(x_n, \dots, x_1 | y_n, \dots, y_1) \\ &= \mathfrak{S}_{w_0\, w\, w_0}(-x_n, \dots, -x_1 | -y_n, \dots, -y_1), \end{aligned}$$

where I is the ideal generated by $e_i(x) - e_i(y)$, for $1 \leq i \leq n$.

Example 3.2. For $w = s_k$, $\mathfrak{S}_{s_k} = x_1 + \cdots + x_k - (y_1 + \cdots + y_k)$. Since $w_0\, s_k\, w_0 = s_{n-k}$, we see

$$\mathfrak{S}_{n-k}(-x_n, \ldots, -x_1 | -y_n, \ldots, -y_1) = -x_n - \cdots - x_{k+1} + y_n + \cdots + y_{k+1}.$$

For $w \in S_n$, let $w' = w_0\, w\, w_0$. (In one-line notation, this is w read "backwards and opposite": $(w_1 \cdots w_n)' = (n+1-w_n) \cdots (n+1-w_1)$. For example, $(2\,1\,6\,3\,5\,4)' = 3\,2\,4\,1\,6\,5$.

Corollary 3.3. *The polynomial* $c_{u'v'}^{w'}$ *is obtained from* c_{uv}^{w} *by interchanging* t_i *and* $-t_{n+1-i}$.

Exercise 3.4. For λ a partition contained in the $k \times \ell$ rectangle, show that $w(\lambda') = w(\lambda)'$, where λ' denotes the conjugate partition (as usual). Thus the Corollary generalizes the relation between $c_{\lambda\mu}^{\nu}$ and $c_{\lambda'\mu'}^{\nu'}$ we saw in Lecture 8.

Remark 3.5. The involution $D : Fl(\mathbb{C}^n) \to Fl((\mathbb{C}^n)^\vee) \cong Fl(\mathbb{C}^n)$ is equivariant for $g \mapsto ({}^t g)^{-1}$, and takes $\Omega_w(F_\bullet)$ to $\Omega_{w'}(\widetilde{F}_\bullet)$, so we see $\sigma_w \mapsto \sigma_{w'}$ by $x_i \mapsto -x_{n+1-i}$ and $t_i \mapsto -t_{n+1-i}$, as above.

More generally, one can consider products of the form

$$\mathfrak{S}_u(x|s) \cdot \mathfrak{S}_v(x|t) = \sum_w c_{uv}^w(s,t)\, \mathfrak{S}_w(x|t),$$

where $c_{uv}^w(s,t)$ is a homogeneous polynomial of degree $\ell(u) + \ell(v) - \ell(w)$ in $s = (s_1, \ldots, s_n)$ and $t = (t_1, \ldots, t_n)$. These specialize to the equivariant coefficients: $c_{uv}^w = c_{uv}^w(t,t)$. For the Grassmannian (so $u = w(\lambda)$, etc.), these are the coefficients studied by Molev and Sagan. They satisfy a vanishing property: $c_{uv}^w(s,t) = 0$ unless $v \leq w$ in Bruhat order. (Note that $c_{uv}^w(s,t)$ need not vanish when $u \not\leq w$!)

Proof. (D. Anderson.) On a variety Y, consider a vector bundle E of rank n, with two general flags of subbundles $S_1 \subset \cdots \subset S_n = E$ and $T_1 \subset \cdots \subset T_n = E$; let $s_i = c_1(S_i/S_{i-1})$ and $t_i = c_1(T_i/T_{i-1})$. Let $X = \mathbf{Fl}(E)$, with tautological quotients $E_X \to Q_{n-1} \to \cdots \to Q_1$, and let $x_i = c_1(\ker(Q_i \to Q_{i-1}))$. Then $\mathfrak{S}_u(x|s)$ is the class of the degeneracy locus $\Omega_u(S_\bullet \to Q_\bullet)$, and $\mathfrak{S}_v(x|t)$ is the class of $\Omega_v(T_\bullet \to Q_\bullet)$.

The classes $\mathfrak{S}_w(x|t)$ form a basis for H^*X over H^*Y, so one can write $\mathfrak{S}_u(x|s) \cdot \mathfrak{S}_v(x|t) = \sum_w c_{uv}^w(s,t)\, \mathfrak{S}_w(x|t)$ modulo the ideal defining H^*X; taking n sufficiently large, there are no relevant relations and this becomes an identity of polynomials.

The class $\mathfrak{S}_u(x|s) \cdot \mathfrak{S}_v(x|t)$ is supported on $\Omega_v(T_\bullet \to Q_\bullet)$, so it comes from a refined class in $H^*(X, X \smallsetminus \Omega_v(T_\bullet \to Q_\bullet))$. Since $H^*(X \smallsetminus \Omega_v(T_\bullet \to Q_\bullet))$ has a basis of classes $[\Omega_w(T_\bullet \to Q_\bullet)]$ for $v \not\leq w$, the vanishing follows from the exact sequence for the pair $(X, X \smallsetminus \Omega_v(T_\bullet \to Q_\bullet)$. $\square$

References

[BKTY04] A. Buch, A. Kresch, H. Tamvakis, and A. Yong, "Schubert polynomials and quiver formulas," *Duke Math. J.* **122** (2004), no. 1, 125–143.

[Gra01] W. Graham, "Positivity in equivariant Schubert calculus," *Duke Math. J.* **109** (2001), 599–614.