Let $X$ be a complete nonsingular toric variety. In this lecture, we will describe $H^*_T X$. First we recall some basic notions about toric varieties.

Let $T$ be an $n$-dimensional torus with character group $M$, and let $N = \text{Hom}_\mathbb{Z}(M, \mathbb{Z})$ be the dual lattice. Then $X = X(\Sigma)$, for a complete nonsingular fan $\Sigma$. That is, $\Sigma$ is a collection of cones $\sigma$ in $N_K = N \otimes \mathbb{Z} K$ such that two cones meet along a face of each; each cone must be generated by part of a basis for $N$ (the nonsingular condition), and the union of the cones is all of $N_K$ (the completeness condition).

The toric variety $X$ is covered by open affines $U_\sigma = \text{Spec} \mathbb{C}[\sigma^\vee \cap M]$, where $\sigma^\vee = \{ u | \langle u, v \rangle \geq 0 \text{ for all } v \in \sigma \}$. In fact, $U_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$, where $k = \dim \sigma$, and the $n$-dimensional cones suffice to cover. Also, $U_{\{0\}} = \text{Spec} \mathbb{C}[M] = T$, and $U_\sigma \cap U_\tau = U_{\sigma \cap \tau}$. Write $\chi^u \in \mathbb{C}[M]$ for the element corresponding to $u \in M$.

Each cone $\tau$ determines a $T$-invariant subvariety $V(\tau) \subset X$, which is closed and nonsingular, of codimension equal to $\dim \tau$. On open affines, this is given by

$$V(\tau) \cap U_\sigma = \text{Spec} \mathbb{C}[\tau^\perp \cap \sigma^\vee \cap M],$$

with the containment in $U_\sigma$ given by

$$\mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}[\tau^\perp \cap \sigma^\vee \cap M],$$

with $\chi^u \mapsto \chi^u$ if $u \in \tau^\perp$ and $\chi^u \mapsto 0$ otherwise. (This is a homomorphism because $\tau^\perp \cap \sigma^\vee$ is a face of $\sigma^\vee$.) Then $V(\tau)$ is a nonsingular toric variety, for the torus with character group $\tau^\perp \cap M$; it corresponds to a fan in $N/\mathbb{N}_\tau$, where $\mathbb{N}_\tau$ is the sublattice generated by $\tau$.

The $T$-fixed points of $X$ are $p_\sigma = V(\sigma)$ for $\dim \sigma = n$.

$X$ is projective if and only if there is a lattice polytope $P \subset M_K$, with vertices in $M$, such that $\Sigma$ is the normal fan to $P$. That is, to each face $F$ of $P$, the corresponding cone in $\Sigma$ is

$$\sigma_F = \{ v | \langle u', v \rangle \geq \langle u, v \rangle \text{ for all } u' \in P, u \in F \}.$$
This correspondence reverses dimensions: \( \dim \sigma_F = \text{codim} F \).

**Example 1.1.** The standard \( n \)-dimensional simplex corresponds to \( \mathbb{P}^n \). An \( n \)-cube corresponds to \((\mathbb{P}^1)^n\). Figures...

For \( X \) projective, choose a general vector \( v \in N_{\mathbb{R}} \), giving an ordering of the vertices \( u_1, \ldots, u_N \) (so that \( \langle u_1, v \rangle < \cdots < \langle u_N, v \rangle \)), and thus an ordering of the \( n \)-dimensional cones \( \sigma_1, \ldots, \sigma_N \). For \( 1 \leq i \leq N \), let

\[
\tau_i = \bigcap_{j > i} \sigma_i \cap \sigma_j,
\]

so \( \tau_1 = \{0\} \), \( \tau_N = \sigma_N \), and \( \tau_p \subseteq \tau_q \) implies \( p \leq q \). (Such an ordering of cones is called a *shelling* of the fan.) This gives a cellular decomposition of \( X \), with closures of cells being \( V(\tau_1), \ldots, V(\tau_N) \), so

\[
[V(\tau_1)], \ldots, [V(\tau_N)]
\]

forms a basis for \( H^*X \). It follows that \([V(\tau_1)]^T, \ldots, [V(\tau_N)]^T\) form a basis for \( H^*_T X \).

If \( X \) is not projective, one can always find a refinement \( \Sigma' \) of \( \Sigma \) (by subdividing cones), giving a surjective, birational, \( T \)-equivariant morphism \( \pi : X' \to X \), with \( X' \) projective and nonsingular. Under \( \pi \), \( V(\tau') \) maps to \( V(\tau) \), where \( \tau \) is the smallest cone containing \( \tau' \); this is birational if they have the same dimension. Since \( \pi_* \circ \pi^* = \text{id} \) on \( H^*X \) or \( H^*_T X \), one sees the following:

**Lemma 1.2.** For \( X \) a complete nonsingular toric variety, \( H^*X \) is generated by the classes \([V(\tau)]\) over \( \mathbb{Z} \), and \( H^*_T X \) is generated by \([V(\tau)]^T\) over \( \Lambda \). Also, \( H^*_T X \otimes_{\mathbb{Z}} \mathbb{Z} \to H^*X \).

We will see that \( H^*X \) and \( H^*_T X \) are always free of rank \( N \), the number of \( n \)-dimensional cones.

**Question 1.3.** Is there always a basis of \([V(\tau)]\)’s for \( H^*X \)? If not, an old combinatorial conjecture on shellability is false.

For any cones \( \sigma \) and \( \tau \), if they span a cone \( \gamma \), then \( V(\sigma) \cap V(\tau) = V(\gamma) \); if \( \dim \gamma = \dim \sigma + \dim \tau \), the intersection is transversal, so

\[
[V(\sigma)] \cdot [V(\tau)] = [V(\gamma)] \quad \text{and} \quad [V(\sigma)]^T \cdot [V(\tau)]^T = [V(\gamma)]^T.
\]

If \( \sigma \) and \( \tau \) are contained in a cone of \( \Sigma \), then \( V(\sigma) \cap V(\tau) = \emptyset \), and the corresponding products are zero.

Let \( D_1, \ldots, D_d \) be the \( T \)-invariant divisors, with \( D_i = V(\tau_i) \) for rays \( \tau_i \); let \( v_i \in N \) be the minimal generator of the ray \( \tau_i \). For \( u \in M \), with corresponding rational function \( \chi^u \),

\[
\text{div}(\chi^u) = \sum \langle u, v_i \rangle D_i.
\]
Equivariantly, $\chi^u$ is a rational section of the line bundle $L_u$ corresponding to the character $u$, so

$$u = c^T_i(L_u) = [\text{div}(\chi^u)]^T = \sum \langle u, v_i \rangle [D_i]^T$$

in $H^*_T X$.

Note that $[D_{i_1}] \cdots [D_{i_r}] = [V(\tau)]$ if $v_{i_1}, \ldots, v_{i_r}$ span a cone $\tau$, and the product is 0 otherwise; the same is true for equivariant products.

2

Let $X_1, \ldots, X_d$ be variables, one for each ray.

In $\mathbb{Z}[X] = \mathbb{Z}[X_1, \ldots, X_d]$, we have two ideals:

(i) $I$ is generated by all monomials $X_{i_1} \cdots X_{i_r}$ such that $v_{i_1}, \ldots, v_{i_r}$ do not span a cone of $\Sigma$. It suffices to take minimal such sets, so that any proper subset does span a cone.

The ring $\mathbb{Z}[X]/I$ is called the **Stanley-Reisner ring**: it appears in combinatorics.

(ii) $J$ is generated by all elements $\sum_{i=1}^d \langle u, v_i \rangle X_i$, for $u \in M$. It suffices to let $u$ run through a basis for $M$.

We have

$$(\ast) \quad \mathbb{Z}[X]/(I + J) \to H^*_X,$$

where the map is given by $X_i \mapsto [D_i]$. We have seen that $I$ and $J$ map to 0, so this is well-defined. It is surjective since $[V(\tau)] = [D_{i_1}] \cdots [D_{i_r}]$ if $v_{i_1}, \ldots, v_{i_r}$ span $\tau$. In fact, $(\ast)$ is an isomorphism, as was proved by Jurkiewicz in the projective case, and by Danilov in general [Jur80, Dan78]. We will recover this result.

In $\Lambda[X] = \Lambda[X_1, \ldots, X_d]$, we have two ideals:

(i) $I'$, with the same generators as $I$, i.e., monomials $X_{i_1} \cdots X_{i_r}$ such that $v_{i_1}, \ldots, v_{i_r}$ do not span a cone in $\Sigma$.

(ii) $J'$, with generators $\sum_{i=1}^d \langle u, v_i \rangle X_i - u$, for all $u \in M$ (or a basis of $M$).

We have

$$(\ast_T) \quad \Lambda[X]/(I' + J') \to H^*_T X,$$

by $X_i \mapsto [D_i]^T$. Again, we have seen that $I'$ and $J'$ map to 0. Similarly, this map is surjective. We will prove that $(\ast_T)$ is also an isomorphism.

All this will follow from the construction of a complex often used in toric geometry (see for example Danilov, Lunts, etc.). (refs) For each cone $\tau$, let $v_{i_1}, \ldots, v_{i_k}$ be its generators, and set

$$\mathbb{Z}[\tau] := \mathbb{Z}[X_{i_1}, \ldots, X_{i_k}] = \mathbb{Z}[X]/(X_j | v_j \notin \tau).$$

Consider this as a $\mathbb{Z}$-module, and also as a $\mathbb{Z}[X]/I$-module. Set

$$C_k = \bigoplus_{\dim \tau = k} \mathbb{Z}[\tau].$$
For a face $\gamma$ of $\tau$, there is a canonical surjection $\mathbb{Z}[\tau] \to \mathbb{Z}[\gamma]$. Define $d : C_k \to C_{k-1}$ by taking $\mathbb{Z}[\tau]$ to the sum of those $\mathbb{Z}[\gamma]$ for facets $\gamma$ of $\tau$: Let $v_{i_1}, \ldots, v_{i_k}$ be the generators of $\tau$, with $i_1 < \cdots < i_k$, and let $\gamma$ be generated by $v_{i_1}, \ldots, \hat{v}_{i_p}, \ldots, v_{i_k}$; then $d_k$ is $(-1)^p$ times the canonical surjection $\mathbb{Z}[\tau] \to \mathbb{Z}[\gamma]$.

**Lemma 2.1.** This gives an exact sequence of $\mathbb{Z}[X]/I$-modules

$$0 \to \mathbb{Z}[X]/I \to C_n \xrightarrow{d_n} C_{n-1} \to \cdots \xrightarrow{d_1} C_0 \to 0.$$  

**Proof.** The map $d_k$ is a homomorphism of graded modules over $\mathbb{Z}[X]$, decomposing into a direct sum with one piece for each monomial $X_1^{m_1} \cdots X_d^{m_d}$. All components vanish unless the set of $v_i$ with $m_i > 0$ span a cone $\lambda$ in $\Sigma$. Each $C_k$ contributes a copy of $\mathbb{Z}$ for each $\tau$ that contains $\lambda$. The resulting complex is the one computing the reduced homology of a simplicial sphere $N/N_\Lambda$. □

**Lemma 2.2.** The canonical homomorphism

$$\mathbb{Z}[X]/I \to \Lambda[X]/(I' + J')$$

is an isomorphism.

**Proof.** Let $u_1, \ldots, u_n$ be a basis for $M$. The elements $Z(u_j) = \sum_i (u_j, v_i) X_i - u_j$ form a regular sequence in $\Lambda[X]$ (since $\Lambda = \mathbb{Z}[u_1, \ldots, u_n]$), with quotient $\Lambda[X]/J'$.

In particular, $\Lambda \to \Lambda[X] \to \mathbb{Z}[X]/I$ takes $u \in M$ to $\sum (u, v_i) X_i$. Therefore the exact sequence (1) is an exact sequence of $\Lambda$-modules.

**Proposition 2.3.** $\mathbb{Z}[X]/I \cong \Lambda[X]/(I' + J')$ is free over $\Lambda$ of rank $N$, the number of $n$-dimensional cones.

**Proof.** For a cone $\tau$ spanned by $v_{i_1}, \ldots, v_{i_k}$, choose $v(k+1), \ldots, v(n)$ to complete a basis of $N$. Let $u_1, \ldots, u_n$ be the dual basis of $M$. Then $Z[\tau] \cong \Lambda/(u_{k+1}, \ldots, u_n)$ as a $\Lambda$-module, so the projective dimension of $C_k$ is $\text{pd}_\Lambda C_k = n - k$. It follows by induction that $\text{pd}_\Lambda (\ker(C_k \to C_{k-1})) \leq n - k$. Therefore $\text{pd}_\Lambda \mathbb{Z}[X]/I = 0$. By the (easier) graded version of the Quillen-Suslin theorem, $\mathbb{Z}[X]/I$ is free. Now consider the beginning of (1):

$$0 \to \mathbb{Z}[X]/I \to C_n \to C_{n-1}.$$  

$C_n$ is free over $\Lambda$ on $N$ generators, since $\Lambda \cong \mathbb{Z}[\sigma]$ for $n$-dimensional cones $\sigma$. $C_{n-1}$ is a torsion $\Lambda$-module. Thus $\mathbb{Z}[X]/I$ is free on $N$ generators. □

**Exercise 2.4.** The Hilbert series

$$\sum_{m=0}^\infty \text{rk}_\mathbb{Z}(\mathbb{Z}[X]/I)_m t^m$$

is equal to

$$\sum_{i=0}^n \frac{(-1)^{n-i} a_i}{(1-t)^i}.$$
(We will not need this, however.)

Consider the diagram

\[ 0 \longrightarrow \mathbb{Z}[X]/I \longrightarrow C_n \xrightarrow{d_n} C_{n-1} \]

where \( \varphi \) takes \( \mathbb{Z}[\sigma] \) to \( H^*_T(p_\sigma) = \Lambda \) as follows: if \( v_1, \ldots, v_n \) span \( \sigma \), let \( u_1, \ldots, u_n \) be the dual basis in \( M \), and let \( \varphi \) be the isomorphism \( \mathbb{Z}[\sigma] = \mathbb{Z}[X_1, \ldots, X_n] \to \Lambda \) given by \( X_i \mapsto u_i \). The left vertical map is the composition \( \mathbb{Z}[X]/I \to \Lambda[X]/(I' + J') \to H^*_T X \), taking \( X_i \) to \( D_i \).

Exercise 2.5. Show that this diagram commutes. (The restriction to \( H^*_T(U_\sigma) \) factors through \( H^*_T(U_\sigma) \), and \( U_\sigma \cong \mathbb{C}^n \), with \( T \) acting by weights \( u_1, \ldots, u_n \). If \( v_i \in \sigma \), with \( i = i_j \), then \( [D_i]^T \) restricts to \( u_j \)—indeed, \( D_i \) restricts to the \( j \)th coordinate hyperplane in \( U_\sigma = \mathbb{C}^n \), so its equivariant class restricts to \( c_1^T(L_{u_j}) = u_j \). If \( v_i \notin \sigma \), then \( [D_i]^T \to 0 \).)

We have seen that the left vertical map is surjective; it follows that it is an isomorphism, proving \( (*)_T \). Tensoring over \( \Lambda \) with \( \mathbb{Z} \), and noting \( \Lambda/M\Lambda = \mathbb{Z} \), we have

\[ (\Lambda[X]/(I' + J')) \otimes_\mathbb{Z} \mathbb{Z} = \mathbb{Z}[X]/(I + J), \]

and \( H^*_T X \otimes_\Lambda \mathbb{Z} \cong H^* X \), so \( (*) \) follows.

Also, we have the following descriptions:

\[ H^*_T X = \mathbb{Z}[X]/I = \ker(d_n) = \{(f_\sigma), f_\sigma \in \mathbb{Z}[\sigma] \cong \Lambda \mid f_\sigma|_\tau = f_{\sigma'}|_\tau \text{ if } \tau \text{ is a facet of } \sigma \text{ and } \sigma'\} \]

where “piecewise polynomial” means continuous functions on \( N_\mathbb{R} \) defined by a polynomial in \( \Lambda \) on each maximal cone \( \sigma \) [Bri97]. This is the GKM theorem for toric varieties (with \( \mathbb{Z} \) coefficients).

Example 2.6. \( H^*_T X = \{ \text{piecewise linear functions} \} = \text{Div}_T M \).

Remark 2.7. If the fan \( \Sigma \) is only simplicial (so the generators of each cone form part of basis for \( N_\mathbb{R} \), but not necessarily for \( N \)), then all the statements here remain true if \( \mathbb{Z} \) is replaced by \( \mathbb{Q} \). (There may also be some multiplicities in products: \( V(\sigma) \cdot V(\tau) = m \cdot V(\gamma) \).)

The ring of piecewise polynomial functions on the support \( |\Sigma| \) can be defined for any fan \( \Sigma \), so it is natural to ask what geometric significance this has, for an arbitrary toric variety \( X = X(\Sigma) \). The answer was given by S. Payne: It is the equivariant operational Chow cohomology, \( A^*_T X \).

There are also descriptions of (ordinary and equivariant) intersection homology groups for singular toric varieties.
§13 TORIC VARIETIES

REFERENCES