# EQUIVARIANT COHOMOLOGY IN ALGEBRAIC GEOMETRY
# LECTURE THIRTEEN: TORIC VARIETIES

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## 1

Let $X$ be a complete nonsingular toric variety. In this lecture, we will descibe $H_T^* X$. First we recall some basic notions about toric varieties.

Let $T$ be an $n$-dimensional torus with character group $M$, and let $N = \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ be the dual lattice. Then $X = X(\Sigma)$, for a *complete nonsingular fan* $\Sigma$. That is, $\Sigma$ is a collection of cones $\sigma$ in $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$ such that two cones meet along a face of each; each cone must be generated by part of a basis for $N$ (the nonsingular condition), and the union of the cones is all of $N_{\mathbb{R}}$ (the completeness condition).

The toric variety $X$ is covered by open affines $U_\sigma = \mathrm{Spec}\, \mathbb{C}[\sigma^\vee \cap M]$, where $\sigma^\vee = \{u \,|\, \langle u, v\rangle \geq 0 \text{ for all } v \in \sigma\}$. In fact, $U_\sigma \cong \mathbb{C}^k \times (\mathbb{C}^*)^{n-k}$, where $k = \dim \sigma$, and the $n$-dimensional cones suffice to cover. Also, $U_{\{0\}} = \mathrm{Spec}\, \mathbb{C}[M] = T$, and $U_\sigma \cap U_\tau = U_{\sigma\cap\tau}$. Write $\chi^u \in \mathbb{C}[M]$ for the element corresponding to $u \in M$.

Each cone $\tau$ determines a $T$-invariant subvariety $V(\tau) \subset X$, which is closed and nonsingular, of codimension equal to $\dim \tau$. On open affines, this is given by

$$V(\tau) \cap U_\sigma = \mathrm{Spec}\, \mathbb{C}[\tau^\perp \cap \sigma^\vee \cap M],$$

with the containment in $U_\sigma$ given by

$$\mathbb{C}[\sigma^\vee \cap M] \to \mathbb{C}[\tau^\perp \cap \sigma^\vee \cap M],$$

with $\chi^u \mapsto \chi^u$ if $u \in \tau^\perp$ and $\chi^u \mapsto 0$ otherwise. (This is a homomorphism because $\tau^\perp \cap \sigma^\vee$ is a face of $\sigma^\vee$.) Then $V(\tau)$ is a nonsingular toric variety, for the torus with character group $\tau^\perp \cap M$; it corresponds to a fan in $N/N_\tau$, where $N_\tau$ is the sublattice generated by $\tau$.

The $T$-fixed points of $X$ are $p_\sigma = V(\sigma)$ for $\dim \sigma = n$.

$X$ is projective if and only if there is a lattice polytope $P \subset M_{\mathbb{R}}$, with vertices in $M$, such that $\Sigma$ is the *normal fan* to $P$. That is, to each face $F$ of $P$, the corresponding cone in $\Sigma$ is

$$\sigma_F = \{v \,|\, \langle u', v\rangle \geq \langle u, v\rangle \text{ for all } u' \in P,\, u \in F\}.$$

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This correspondence reverses dimensions: $\dim \sigma_F = \operatorname{codim} F$.

**Example 1.1.** The standard $n$-dimensional simplex corresponds to $\mathbb{P}^n$. An $n$-cube corresponds to $(\mathbb{P}^1)^n$. `Figures...`

For $X$ projective, choose a general vector $v \in N_{\mathbb{R}}$, giving an ordering of the vertices $u_1, \ldots, u_N$ (so that $\langle u_1, v\rangle < \cdots \langle u_N, v\rangle$), and thus an ordering of the $n$-dimensional cones $\sigma_1, \ldots, \sigma_N$. For $1 \le i \le N$, let

$$\tau_i = \bigcap_{\substack{j>i \\ \dim(\sigma_j \cap \sigma_i) = n-1}} \sigma_i \cap \sigma_j,$$

so $\tau_1 = \{0\}$, $\tau_N = \sigma_N$, and $\tau_p \subseteq \tau_q$ implies $p \le q$. (Such an ordering of cones is called a *shelling* of the fan.) This gives a cellular decomposition of $X$, with closures of cells being $V(\tau_1), \ldots, V(\tau_N)$, so

$$[V(\tau_1)], \ldots, [V(\tau_N)]$$

forms a basis for $H^*X$. It follows that $[V(\tau_1)]^T, \ldots, [V(\tau_N)]^T$ form a basis for $H_T^*X$.

If $X$ is not projective, one can always find a refinement $\Sigma'$ of $\Sigma$ (by subdividing cones), giving a surjective, birational, $T$-equivariant morphism $\pi : X' \to X$, with $X'$ projective and nonsingular. Under $\pi$, $V(\tau')$ maps to $V(\tau)$, where $\tau$ is the smallest cone containing $\tau'$; this is birational if they have the same dimension. Since $\pi_* \circ \pi^* = id$ on $H^*X$ or $H_T^*X$, one sees the following:

**Lemma 1.2.** *For $X$ a complete nonsingular toric variety, $H^*X$ is generated by the classes $[V(\tau)]$ over $\mathbb{Z}$, and $H_T^*X$ is generated by $[V(\tau)]^T$ over $\Lambda$. Also, $H_T^*X \otimes_\Lambda \mathbb{Z} \xrightarrow{\sim} H^*X$.*

We will see that $H^*X$ and $H_T^*X$ are always free of rank $N$, the number of $n$-dimensional cones.

**Question 1.3.** Is there always a basis of $[V(\tau)]$'s for $H^*X$? If not, an old combinatorial conjecture on shellability is false.

For any cones $\sigma$ and $\tau$, if they span a cone $\gamma$, then $V(\sigma) \cap V(\tau) = V(\gamma)$; if $\dim \gamma = \dim \sigma + \dim \tau$, the intersection is transversal, so

$$[V(\sigma)] \cdot [V(\tau)] = [V(\gamma)] \quad \text{and} \quad [V(\sigma)]^T \cdot [V(\tau)]^T = [V(\gamma)]^T.$$

If $\sigma$ and $\tau$ are contained in a cone of $\Sigma$, then $V(\sigma) \cap V(\tau) = \emptyset$, and the corresponding products are zero.

Let $D_1, \ldots, D_d$ be the $T$-invariant divisors, with $D_i = V(\tau_i)$ for rays $\tau_i$; let $v_i \in N$ be the minimal generator of the ray $\tau_i$. For $u \in M$, with corresponding rational function $\chi^u$,

$$\operatorname{div}(\chi^u) = \sum \langle u, v_i\rangle D_i.$$

Equivariantly, $\chi^u$ is a rational section of the line bundle $L_u$ corresponding to the character $u$, so

$$u = c_1^T(L_u) = [\mathrm{div}(\chi^u)]^T = \sum \langle u, v_i \rangle [D_i]^T$$

in $H_T^* X$.

Note that $[D_{i_1}] \cdots [D_{i_r}] = [V(\tau)]$ if $v_{i_1}, \ldots, v_{i_r}$ span a cone $\tau$, and the product is 0 otherwise; the same is true for equivariant products.

## 2

Let $X_1, \ldots, X_d$ be variables, one for each ray.

In $\mathbb{Z}[X] = \mathbb{Z}[X_1, \ldots, X_d]$, we have two ideals:

(i) $I$ is generated by all monomials $X_{i_1} \cdots X_{i_r}$ such that $v_{i_1}, \ldots, v_{i_r}$ do not span a cone of $\Sigma$. It suffices to take minimal such sets, so that any proper subset does span a cone.

The ring $\mathbb{Z}[X]/I$ is called the **Stanley-Reisner ring**; it appears in combinatorics.

(ii) $J$ is generated by all elements $\sum_{i=1}^d \langle u, v_i \rangle X_i$, for $u \in M$. It suffices to let $u$ run through a basis for $M$.

We have

$$(*) \qquad \mathbb{Z}[X]/(I + J) \to H^* X,$$

where the map is given by $X_i \mapsto [D_i]$. We have seen that $I$ and $J$ map to 0, so this is well-defined. It is surjective since $[V(\tau)] = [D_{i_1}] \cdots [D_{i_r}]$ if $v_{i_1}, \ldots, v_{i_r}$ span $\tau$. In fact, $(*)$ is an isomorphism, as was proved by Jurkiewicz in the projective case, and by Danilov in general [Jur80, Dan78]. We will recover this result.

In $\Lambda[X] = \Lambda[X_1, \ldots, X_d]$, we have two ideals:

(i) $I'$, with the same generators as $I$, i.e., monomials $X_{i_1} \cdots X_{i_r}$ such that $v_{i_1}, \ldots, v_{i_r}$ do not span a cone in $\Sigma$.

(ii) $J'$, with generators $\sum \langle u, v_i \rangle X_i - u$, for all $u$ in $M$ (or a basis of $M$).

We have

$$(*_T) \qquad \Lambda[X]/(I' + J') \to H_T^* X,$$

by $X_i \mapsto [D_i]^T$. Again, we have seen that $I'$ and $J'$ map to 0. Similarly, this map is surjective. We will prove that $(*_T)$ is also an isomorphism.

All this will follow from the construction of a complex often used in toric geometry (see for example Danilov, Lunts, etc.). (`refs`) For each cone $\tau$, let $v_{i_1}, \ldots, v_{i_k}$ be its generators, and set

$$\mathbb{Z}[\tau] := \mathbb{Z}[X_{i_1}, \ldots, X_{i_k}] = \mathbb{Z}[X]/(X_j \mid v_j \notin \tau).$$

Consider this as a $\mathbb{Z}$-module, and also as a $\mathbb{Z}[X]/I$-module. Set

$$C_k = \bigoplus_{\dim \tau = k} \mathbb{Z}[\tau].$$

For a face $\gamma$ of $\tau$, there is a canonical surjection $\mathbb{Z}[\tau] \to \mathbb{Z}[\gamma]$. Define $d : C_k \to C_{k-1}$ by taking $\mathbb{Z}[\tau]$ to the sum of those $\mathbb{Z}[\gamma]$ for facets $\gamma$ of $\tau$: Let $v_{i_1}, \dots, v_{i_k}$ be the generators of $\tau$, with $i_1 < \cdots < i_k$, and let $\gamma$ be generated by $v_{i_1}, \dots, \hat{v}_{i_p}, \dots, v_{i_k}$; then $d_k$ is $(-1)^p$ times the canonical surjection $\mathbb{Z}[\tau] \to \mathbb{Z}[\gamma]$.

**Lemma 2.1.** *This gives an exact sequence of $\mathbb{Z}[X]/I$-modules*

$$0 \to \mathbb{Z}[X]/I \to C_n \xrightarrow{d_n} C_{n-1} \to \cdots \xrightarrow{d_1} C_0 \to 0. \tag{1}$$

*Proof.* The map $d_k$ is a homomorphism of graded modules over $\mathbb{Z}[X]$, decomposing into a direct sum with one piece for each monomial $X_1^{m_1} \cdots X_d^{m_d}$. All components vanish unless the set of $v_i$ with $m_i > 0$ span a cone $\lambda$ in $\Sigma$. Each $C_k$ contributes a copy of $\mathbb{Z}$ for each $\tau$ that contains $\lambda$. The resulting complex is the one computing the reduced homology of a simplicial sphere in $N/N_\lambda$. □

**Lemma 2.2.** *The canonical homomorphism*

$$\mathbb{Z}[X]/I \to \Lambda[X]/(I' + J')$$

*is an isomorphism.*

*Proof.* Let $u_1, \dots, u_n$ be a basis for $M$. The elements $Z(u_j) = \sum_i \langle u_j, v_i \rangle X_i - u_j$ form a regular sequence in $\Lambda[X]$ (since $\Lambda = \mathbb{Z}[u_1, \dots, u_n]$), with quotient $\Lambda[X]/J'$. □

In particular, $\Lambda \to \Lambda[X] \to \mathbb{Z}[X]/I$ takes $u \in M$ to $\sum \langle u, v_i \rangle X_i$. Therefore the exact sequence (1) is an exact sequence of $\Lambda$-modules.

**Proposition 2.3.** $\mathbb{Z}[X]/I \cong \Lambda[X]/(I' + J')$ *is free over $\Lambda$ of rank $N$, the number of $n$-dimensional cones.*

*Proof.* For a cone $\tau$ spanned by $v_{i_1}, \dots, v_{i_k}$, choose $v(k+1), \dots, v(n)$ to complete a basis of $N$. Let $u_1, \dots, u_n$ be the dual basis of $M$. Then $\mathbb{Z}[\tau] \cong \Lambda/(u_{k+1}, \dots, u_n)$ as a $\Lambda$-module, so the projective dimension of $C_k$ is $\operatorname{pd}_\Lambda C_k = n - k$. It follows by induction that $\operatorname{pd}_\Lambda(\ker(C_k \to C_{k-1})) \le n - k$. Therefore $\operatorname{pd}_\Lambda \mathbb{Z}[X]/I = 0$. By the (easier) graded version of the Quillen-Suslin theorem, $\mathbb{Z}[X]/I$ is free. Now consider the beginning of (1):

$$0 \to \mathbb{Z}[X]/I \to C_n \to C_{n-1}.$$

$C_n$ is free over $\Lambda$ on $N$ generators, since $\Lambda \cong \mathbb{Z}[\sigma]$ for $n$-dimensional cones $\sigma$. $C_{n-1}$ is a torsion $\Lambda$-module. Thus $\mathbb{Z}[X]/I$ is free on $N$ generators. □

**Exercise 2.4.** The Hilbert series

$$\sum_{m=0}^{\infty} \operatorname{rk}_{\mathbb{Z}}(\mathbb{Z}[X]/I)_m t^m$$

is equal to

$$\sum_{i=0}^{n} \frac{(-1)^{n-i} a_i}{(1-t)^i}.$$

(We will not need this, however.)

Consider the diagram

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}[X]/I \arrow[r] \arrow[d] & C_n \arrow[r, "d_n"] \arrow[d, "\varphi"] & C_{n-1} \\ & H_T^*X \arrow[r, "\text{restr}"] & H_T^*X^T, & \end{tikzcd}$$

where $\varphi$ takes $\mathbb{Z}[\sigma]$ to $H_T^*(p_\sigma) = \Lambda$ as follows: if $v_{i_1}, \ldots, v_{i_n}$ span $\sigma$, let $u_1, \ldots, u_n$ be the dual basis in $M$, and let $\varphi$ be the isomorphism $\mathbb{Z}[\sigma] = \mathbb{Z}[X_{i_1}, \ldots, X_{i_n}] \to \Lambda$ given by $X_{i_j} \mapsto u_j$. The left vertical map is the composition $\mathbb{Z}[X]/I \to \Lambda[X]/(I' + J') \to H_T^*X$, taking $X_i$ to $[D_i]^T$.

**Exercise 2.5.** Show that this diagram commutes. (The restriction to $H_T^*(p_\sigma)$ factors through $H_T^*(U_\sigma)$, and $U_\sigma \cong \mathbb{C}^n$, with $T$ acting by weights $u_1, \ldots, u_n$. If $v_i \in \sigma$, with $i = i_j$, then $[D_i]^T$ restricts to $u_j$ — indeed, $D_i$ restricts to the $j$th coordinate hyperplane in $U_\sigma = \mathbb{C}^n$, so its equivariant class restricts to $c_1^T(L_{u_j}) = u_j$. If $v_i \notin \sigma$, then $[D_i]^T \mapsto 0$.)

We have seen that the left vertical map is surjective; it follows that it is an isomorphism, proving $(*_T)$. Tensoring over $\Lambda$ with $\mathbb{Z}$, and noting $\Lambda/M\Lambda = \mathbb{Z}$, we have

$$(\Lambda[X]/(I' + J')) \otimes_\Lambda \mathbb{Z} = \mathbb{Z}[X]/(I + J),$$

and $H_T^*X \otimes_\Lambda \mathbb{Z} \xrightarrow{\sim} H^*X$, so $(*)$ follows.

Also, we have the following descriptions:

$$\begin{aligned} H_T^*X &= \mathbb{Z}[X]/I \\ &= \ker(d_n) \\ &= \{(f_\sigma),\ f_\sigma \in \mathbb{Z}[\sigma] \cong \Lambda \mid f_\sigma|_\tau = f_{\sigma'}|_\tau \text{ if } \tau \text{ is a facet of } \sigma \text{ and } \sigma'\} \\ &= \{\text{piecewise polynomial functions on } N_\mathbb{R}\}, \end{aligned}$$

where "piecewise polynomial" means continuous functions on $N_\mathbb{R}$ defined by a polynomial in $\Lambda$ on each maximal cone $\sigma$ [Bri97]. This is the GKM theorem for toric varieties (with $\mathbb{Z}$ coefficients).

**Example 2.6.** $H_T^2X = \{\text{piecewise linear functions}\} = \operatorname{Div}_T M$.

**Remark 2.7.** If the fan $\Sigma$ is only *simplicial* (so the generators of each cone form part of basis for $N_\mathbb{R}$, but not necessarily for $N$), then all the statements here remain true if $\mathbb{Z}$ is replaced by $\mathbb{Q}$. (There may also be some multiplicities in products: $V(\sigma) \cdot V(\tau) = m \cdot V(\gamma)$.)

The ring of piecewise polynomial functions on the support $|\Sigma|$ can be defined for any fan $\Sigma$, so it is natural to ask what geometric significance this has, for an arbitrary toric variety $X = X(\Sigma)$. The answer was given by S. Payne: It is the *equivariant operational Chow cohomology*, $A_T^*X$.

There are also descriptions of (ordinary and equivariant) intersection homology groups for singular toric varieties.

## References

[Bri97] M. Brion, "The structure of the polytope algebra," *Tohoku Math. J. (2)* **49** (1997), no. 1, 1–32.

[Bri-Ver] M. Brion and M. Vergne, "An equivariant Riemann-Roch theorem for complete, simplicial toric varieties," *J. Reine Angew. Math.* **482** (1997), 67–92.

[Dan78] V. Danilov, "The geometry of toric varieties," *Russ. Math. Surveys* **33** (1978), 97–154.

[Ful93] W. Fulton, *Introduction to Toric Varieties*, Princeton 1993.

[Jur80] J. Jurkiewicz, "Chow ring of projective nonsingular torus embedding," *Colloq. Math.* **43** (1980), no. 2, 261–270.