EQUIVALENT COHOMOLOGY IN ALGEBRAIC GEOMETRY
LECTURE FIFTEEN: CHEVALLEY'S FORMULA, LINE BUNDLES, DUALITY

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1. Chevalley's formula

In the equivariant setting, Chevalley's formula computes the product of a divisor class $y(s_\alpha)$, for α a simple root, and a general class $y(v)$, for $v \in W$. This will involve classes $y(w)$, for $w = v$, and for $v \leq w$ with $\ell(w) = \ell(v)+1$. In the latter case, such w can be written uniquely in the form $w = v\,s_\beta$, for some positive root β, with $\gamma = v^{-1}(\beta)$ also positive, and $w = s_\gamma\,v$. (It is a general fact — see Section 2 below for more — that $v(R^-) \cap R^+$ consists of $\ell(v)$ roots, and $w(R^-) \cap R^+$ has one more root, which is γ.) Let $n_{\beta\alpha}$ be the coefficient of α when β is written as a sum of positive roots. Set

$$c_\alpha(v,w) = n_{\beta\alpha} = \frac{(\alpha,\alpha)}{(\beta,\beta)}.$$

We have given an explicit formula for $y(s_\alpha)|_v$ in §14.4. We need a basic fact:

Lemma 1.1. $y(v)|_v = \displaystyle\prod_{\beta\in v(R^-)\cap R^+} \beta.$

Proof. Since $p(v)$ is a nonsingular point of $Y(v)$, we know (by §3, Proposition 5.1) that $y(v)|_v$ is the equivariant top Chern class of the normal space $N_{p(v)}$ to $Y(v)$ in X at $p(v)$, i.e., $c^T_{2\ell(v)}(N_{p(v)})$. The tangent space to $Y(v)$ at $p(v)$ has weights $\beta \in v(R^-)\cap R^-$, and the tangent space to X at $p(v)$ has weights $\beta \in v(R^-)$; therefore $N_{p(v)}$ has the complementary weights, $v(R^-)\cap R^+$. □

Proposition 1.2. $y(s_\alpha)\cdot y(v) = \displaystyle\sum_{w=v\,s_\beta} c_\alpha(v,w)\,y(w) + (y(s_\alpha)|_v)\,y(v).$

Proof. We know the terms appearing on the RHS consist of $y(w)$ with $v \leq w$ and $\ell(w) \leq \ell(v)+1$, which are just the terms displayed. Restricting to $p(v)$, we get

$$y(s_\alpha)|_v\,y(v)|_v = 0 + C\,y(v)|_v,$$

where C is the coefficient of $y(v)$ in $y(s_\alpha)\cdot y(v)$. Since $y(v)|_v$ is not zero in Λ, the coefficient C must be equal to $y(s_\alpha)|_v$.

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The other coefficients are "classical", but it is easier to compute them equivariantly (see [Ful-Woo04]). By Poincaré duality, the coefficient of $y(w)$ is

$$\rho_*(y(s_\alpha) \cdot y(v) \cdot x(w))$$

in Λ.

Now $Y(v)$ and $X(w)$ meet transversally in the T-invariant curve E that contains the fixed points $p(v)$ and $p(w)$. This follows readily from the fact the Schubert varieties are nonsingular in codimension one. (This is part of the general fact that Schubert varieties are normal, but an elementary proof is given in [Che94].) Then, since $T_{p(v)}X(v) \subset T_{p(v)}X(w)$ is codimension 1, and it meets $T_{p(v)}Y(v)$ transversally, it follows that $T_{p(v)}X(w) \cap T_{p(v)}Y(v)$ has dimension 1, and similarly for $T_{p(w)}X(w) \cap T_{p(w)}Y(v)$.

It is now easy to find the weight λ of T acting on $T_{p(w)}E$, with the weight $-\lambda$ acting on $T_{p(v)}E$. This is the unique weight that occurs in $T_{p(w)}X(w)$ (so $\lambda \in w(R^-) \cap R^+$), such that $-\lambda$ occurs in $T_{p(v)}Y(v)$ (so $-\lambda \in v(R^-) \cap R^-$, i.e., $\lambda \in v(R^+) \cap R^+$). This is the weight $\gamma = v^{-1}\beta$ described earlier, so $w = s_\gamma\, v = v\, s_\beta$.

By the transversality of the intersection, we have

$$y(v) \cdot x(w) = [Y(v)]^T \cdot [X(w)]^T = [E]^T.$$

Let $\iota : E \hookrightarrow X$ be the inclusion, and let η be the projection from E to a point. Then

$$\begin{aligned} \rho_*(y(s_\alpha) \cdot y(v) \cdot x(w)) &= \rho_*(y(s_\alpha) \cdot \iota_*(1)) \\ &= \rho_*(\iota_*(\iota^*(y(s_\alpha)))) \\ &= \eta_*(\iota^*(y(s_\alpha))). \end{aligned}$$

Now we use the localization formula to compute this classical push-forward. Note that the restriction of $\iota^*(y(s_\alpha))$ to $p(w)$ is $y(s_\alpha)|_w$, and its restriction to $p(v)$ is $y(s_\alpha)|_v$, so

$$\rho_*(y(s_\alpha) \cdot y(v) \cdot x(w)) = \frac{y(s_\alpha)|_w - y(s_\alpha)|_v}{\gamma}.$$

We know that $y(s_\alpha)|_w = \varpi_\alpha - w(\varpi_\alpha)$, and $y(s_\alpha)|_v = \varpi_\alpha - v(\varpi_\alpha)$, where ϖ_α is the fundamental weight, so

$$\begin{aligned} y(s_\alpha)|_w - y(s_\alpha)|_v &= v(\varpi_\alpha) - w(\varpi_\alpha) \\ &= v(\varpi_\alpha) - s_\gamma v(\varpi_\alpha) \\ &= 2\frac{(v(\varpi_\alpha), \gamma)}{(\gamma, \gamma)}\gamma. \end{aligned}$$

Hence the required coefficient is $2(v(\varpi_\alpha), \gamma)/(\gamma, \gamma) = 2(\varpi_\alpha, \beta)/(\beta, \beta)$, since $v(\beta) = \gamma$ and $(\ ,\ )$ is W-invariant.

The fundamental weight ϖ_α is characterized by the property that $2(\varpi_\alpha, \alpha) = (\alpha, \alpha)$ and $2(\varpi_\alpha, \alpha') = 0$ for any simple root $\alpha' \neq \alpha$; thus $s_{\alpha'}(\varpi_\alpha) =$

$\varpi_\alpha - \delta_{\alpha\alpha'}\alpha'$. Hence $2(\varpi_\alpha, \beta) = 2n_{\beta\alpha}(\varpi_\alpha, \alpha) = n_{\beta\alpha}(\alpha, \alpha)$, giving the coefficient

$$n_{\beta\alpha}\frac{(\alpha,\alpha)}{(\beta,\beta)} = c_\alpha(v,w),$$

as required. $\square$

Exercise 1.3. Prove the formula

$$x(w_0\, s_\alpha)\cdot x(w) = \sum c_\alpha(v,w)\, x(v) + (x(w_0\, s_\alpha)|_w)\, x(w),$$

with the sum over $v \leq w$ with $\ell(v) = \ell(w) + 1$. (Solution: With the notation of the preceding proof, the coefficient of $x(v)$ is $\eta_*\iota^*(x(w_0\, s_\alpha)) = (x(w_0\, s_\alpha)|_w - x(w_0\, s_\alpha)|_v)/\gamma$. Now use Lemma 4.3(2) from Lecture 14 to get $x(w_0\, s_\alpha)|_w - x(w_0\, s_\alpha)|_v = v(\varpi_\alpha) - w(\varpi_\alpha)$; the proof concludes as before.)

Exercise 1.4 (cf. [Ful-Woo04])**.** Prove Chevalley's formula on G/P, for $P = P_J$: For $\alpha \in R^+ \smallsetminus J$, and $v \in W$ a minimal representative for $[v] \in W/W_P$,

$$y[s_\alpha]\cdot y[v] = \sum c_\alpha(v,w) y[w] + (y(s_\alpha)|_v) y[v],$$

the sum over $w \geq v$ with $\ell(w) = \ell(v)+1$ and $[w] \neq [v]$. (Solution: Apply $(\pi_J)^*$. Note that for $w = v\, s_\beta$, $v^{-1}(R^-)\cap R_J^+ = \emptyset$, and $w^{-1}(R^-)\cap R^+ = v^{-1}(R^-)\cup\{v^{-1}(\beta)\}$, so $w^{-1}(R^-)\cap R_J^+ = \emptyset$. Therefore w is a minimal representative for $[w]$.)

Remark 1.5. The fact that $y(s_\alpha)|_w - y(s_\alpha)|_v = v(\varpi_\alpha) - w(\varpi_\alpha)$ is equivalent to the formula $c_1^T(L(-\varpi_\alpha)) = y(s_\alpha) + c$, for some $c \in \Lambda^1$. This formula can be proved by restricting both sides to the curves $E' = X(s_{\alpha'})$, for $\alpha' \in R^+$, which join $p(id)$ to $p(s_{\alpha'})$, and then pushing forward to a point. The right side gives $\delta_{\alpha\alpha'}$; the left gives $(-\varpi_\alpha + s_{\alpha'}(\varpi_\alpha))/\alpha' = \delta_{\alpha\alpha'}$. (See [Ful-Woo04].)

2

Proposition 2.1. *The functions $\psi_v : W \to \Lambda$ (or $W \to Q$) given by $\psi_v(w) = y(v)|_w$ satisfy and are uniquely determined by the following properties:*

1. $\psi_v(w) = 0$ *unless* $v \leq w$.
2. $\psi_v(v) = \prod_{\beta\in v(R^-)\cap R^+} \beta.$
3. $(A_\alpha\psi_v) = \begin{cases} \psi_{v\, s_\alpha} & \text{if } \ell(v\, s_\alpha) < \ell(v); \\ 0 & \text{if } \ell(v\, s_\alpha) > \ell(v). \end{cases}$

Proof. Property (1) holds since $y(v)|_w = 0$ for $p(w) \notin Y(v)$. We have seen (2), and (3) follows from the fact that $D_\alpha y(v)$ is $y(v\, s_\alpha)$ or 0 according as $\ell(v\, s_\alpha) < \ell(v)$ or $\ell(v\, s_\alpha) > \ell(v)$.

To see that these properties characterize ψ_v, note that the function ψ_{w_0} is determined by (1) and (2). Any other $w \in W$ can be written as $w = w_0\, s_{\alpha_1}\cdots s_{\alpha_\ell}$, with $\ell(w) = \ell(w_0) - \ell$, so ψ_w is determined from ψ_{w_0} and Property (3). $\square$

Remark 2.2. These functions ψ_v are denoted ξ^v in the literature stemming from [Kos-Kum86]; cf. [Bil97]. Note that (2) is only needed for $v = w_0$.

Billey has given an explicit formula for these values $(\psi_v)(w)$. To express it, we need a basic fact about roots. If $\alpha_1 \cdots \alpha_\ell$ is a reduced word for $w \in W$, then the $\ell(w)$ roots in $w(R^-) \cap R^+$ get numbered:

$$w(R^-) \cap R^+ = \{\beta_1, \dots, \beta_\ell\},$$

where $\beta_1 = \alpha_1$, and $\beta_i = s_{\alpha_1} s_{\alpha_2} \cdots s_{\alpha_{i-1}}(\alpha_i)$. (See [Bou81, VI §1.6], or [Hum90, p. 14].)

Proposition 2.3 ([Bil97])**.** *For any $v, w \in W$, with reduced word chosen for w, we have*

$$y(v)|_w = \sum \beta_{j_1} \beta_{j_2} \cdots \beta_{j_k}, \tag{1}$$

the sum over all subsets $\{j_1 < \cdots < j_k\}$ of $\{1, \dots, \ell\}$ such that $\alpha_{j_1} \cdots \alpha_{j_k}$ is a reduced word for v.

Billey proves that the right side satisfies the conditions (1), (2), and (3) of Proposition 2.1, with (1) and (2) being clear. Property (3) is also clear if $\alpha_\ell = \alpha$, so the essential point is to prove that the right side of the formula is independent of the choice of a reduced decomposition of w. For this, she uses the *nil-Coxeter algebra*, which is generated over Λ by noncommuting variables u_α, one for each simple root, with relations $u_\alpha^2 = 0$, and $(u_\alpha u_\beta)^m = (u_\beta u_\alpha)^m$ if $(s_\alpha s_\beta)^m = (s_\beta s_\alpha)^m$ in W. This is free over Λ, with basis $\{u_w\}_{w \in W}$, where $u_w = u_{\alpha_1} \cdots u_{\alpha_\ell}$ for any reduced word $\alpha_1 \cdots \alpha_\ell$ for w. Following ideas of Yang-Baxter and Fomin-Kirillov, set

$$\mathfrak{R}_w = (1 + \beta_1 u_{\alpha_1})(1 + \beta_2 u_{\alpha_2}) \cdots (1 + \beta_\ell u_{\alpha_\ell}),$$

with $\beta_1, \dots, \beta_\ell$ defined as above. Billey [Bil97] and Stembridge [Ste93] show that $\mathfrak{R}_w$ is independent of the choice of reduced word. The coefficient of u_v in $\mathfrak{R}_w$ is exactly the RHS of (1). Note that if $\ell(w\, s_\alpha) < \ell(w)$, then $\mathfrak{R}_w = \mathfrak{R}_{w\, s_\alpha} \cdot (1 - w(\alpha) u_\alpha)$, and if $\ell(w\, s_\alpha) > \ell(w)$, then $\mathfrak{R}_{w\, s_\alpha} = \mathfrak{R}_w \cdot (1 + w(\alpha) u_\alpha)$. Property (3) is equivalent to these identities.

Remark 2.4. Willems [Wil04] proves this formula by using the Bott tower, which is an iteration of the construction of the correspondences we used to calculate the operators D_v.

Remark 2.5. This gives another expression for $y(s_\alpha)|_w$, as the sum of those β_i for which $\alpha_i = \alpha$. It is not immediately obvious that this is equal to the formula $y(s_\alpha)|_w = \varpi_\alpha - w(\varpi_\alpha)$ we found earlier. The latter formula may be simpler to use. For example, with the usual numberings in type A_n, for $\alpha = \alpha_1$ the first simple root, and $w \in W = S_{n+1}$, we have

$$\varpi_\alpha - w(\varpi_\alpha) = \alpha_1 + \cdots + \alpha_{w(1)-1},$$

which does not require finding a reduced expression for w.

Exercise 2.6. Check that the two formulas agree; i.e., for $\alpha_1 \cdots \alpha_\ell$ a reduced word for w,

$$\varpi_\alpha - w(\varpi_\alpha) = \sum_{\alpha_i = \alpha} s_{\alpha_1} \cdots s_{\alpha_{i-1}}(\alpha).$$

(Hint/solution: use induction on $\ell(w)$. It is obvious for $w = id$. Suppose equality is known for w, and $\ell(ws_\beta) = \ell(w)+1$. If $\beta \neq \alpha$, then the RHS's are clearly equal for w and ws_β, and since $w(\varpi_\alpha) = ws_\beta(\varpi_\alpha)$, so are the LHS's. If $\beta = \alpha$, then the difference of the RHS's is $w(\alpha)$, while the difference of the LHS's is $w(\varpi_\alpha) - ws_\alpha(\varpi_\alpha) = w(\alpha)$.)

Remark 2.7. Kostant and Kumar construct the functions ξ^v in $F(W,Q)$ as duals to elements x_v in the *twisted* group algebra $Q[W]$ (with multiplication given by $qw \cdot q'w' = qw(q')ww'$); for a reduced decomposition $s_{\alpha_1} \cdots s_{\alpha_\ell}$ for v, $x_v = x_{s_{\alpha_1}} \cdots x_{s_{\alpha_\ell}}$, where

$$x_{s_\alpha} = \frac{1}{\alpha} s_\alpha - \frac{1}{\alpha}.$$

These are independent of choice. Such calculations can also be used to show that Billey's formula is independent of choice (see [Kum02, §11.1.10]).

Remark 2.8. Billey's formula shows that $y(v)|_w$ is nonzero if and only if $v \leq w$, i.e., $p(w) \in Y(v)$. In general, however, it is not true that $[Y]^T|_p \neq 0$ for an isolated T-fixed point p in a singular T-variety Y in a smooth T-variety X, as the following example shows.

Example 2.9 (Cf. [Bri00])**.** Let $T = (\mathbb{C}^*)^2$ act on $X = \mathbb{P}^4$ by

$$(z_1, z_2) \cdot [x_1 : \cdots : x_5] = [x_1 : z_1\, x_2 : z_1^{-1}\, x_3 : z_2\, x_4 : z_2^{-1}\, x_5].$$

Let $Y \subset X$ be the hypersurface defined by the equation $X_2\, X_3 - X_4\, X_5 = 0$, and let $p = [1 : 0 : \cdots : 0]$ be the singular point of Y. Then $[Y]^T|_p = 0$. Indeed, $Y = \text{Zeroes}(s)$, where s is a T-invariant section of $\mathcal{O}(2)$. If $\zeta = c_1^T(\mathcal{O}(1))$, then $[Y]^T = 2\zeta$, and

$$H_T^* \mathbb{P}^4 = \Lambda[\zeta]/(\zeta(\zeta + t_1)(\zeta - t_1)(\zeta + t_2)(\zeta - t_2)),$$

so $\zeta \mapsto (0, -t_1, t_1, -t_2, t_2)$ in $H_T^*(\mathbb{P}^4)^T$.

Remark 2.10. There is a long history of investigating when a T-fixed point $p(v)$ is a singular point on a Schubert variety $Y(w)$, for $w \leq v$ (or on $X(w)$, $v \leq w$). It is a necessary condition that the restriction $y(w)|_v$ be a product of $\ell(w)$ roots (the weights of T on the normal space to $Y(w)$ at $p(v)$). Formulas for these $y(w)|_v$ can be useful for this study; see [Bri98] and [Bil-Lak00].

3. Line bundles

Recall the homomorphism $b : \Lambda \to H_T^* X$ of graded rings, for $X = G/B$, determined by taking a character λ to $c_1^T(L(\lambda))$. We want to compute $D_\alpha b(P)$ for $P \in \Lambda$.

Recall the classical divided difference operators on Λ. For a simple root α, $\partial_\alpha : \Lambda \to \Lambda$ is defined by

$$\partial_\alpha(P) = \frac{s_\alpha(P) - P}{\alpha},$$

where $s_\alpha(P)$ is defined by the action of W on $\Lambda = \mathrm{Sym}^\bullet M$ coming from the action on M described in Section 14.1.

The following proposition, in terms of operators and functions, appears in [Ara86]:

Proposition 3.1. *For all simple roots α, $D_\alpha b(P) = b(\partial_\alpha P)$.*

Since b is an injection, the classical fact (see [Ber-Gel-Gel73], [Dem74]) that the ∂_α satisfy the usual relations follows:

- $\partial_{\alpha_1} \circ \cdots \circ \partial_{\alpha_\ell} = 0$ if $\ell(s_{\alpha_1} \cdots s_{\alpha_\ell}) < \ell$;
- $\partial_v = \partial_{\alpha_1} \circ \cdots \circ \partial_{\alpha_\ell}$ is independent of the choice of reduced word $\alpha_1 \cdots \alpha_\ell$ for v; and
- $\partial_u \circ \partial_v = \partial_{uv}$ if $\ell(uv) = \ell(u) + \ell(v)$, and is 0 otherwise.

Proof. To show that the left square in the diagram

$$\begin{tikzcd}
\Lambda \arrow[r, "b"] \arrow[d, "\partial_\alpha"'] & H_T^* X \arrow[r, hook] \arrow[d, "D_\alpha"] & F(W,Q) \arrow[d, "A_\alpha"] \\
\Lambda \arrow[r, "b"] & H_T^* X \arrow[r, hook] & F(W,Q)
\end{tikzcd}$$

commutes, it suffices to show that the outer rectangle commutes (see Proposition 14.4.1). Let $b' : \Lambda \to F(W,Q)$ be the composition of the horizontal maps.

Consider first the case $P = \lambda \in M$. We saw in section ? that the corresponding function $b'(\lambda) \in F(W,Q)$ takes w to $w(\lambda)$, and A_α takes this to the constant function

$$\begin{aligned}
w \quad \mapsto \quad & \frac{ws_\alpha(\lambda) - w(\lambda)}{w(\alpha)} \\
= \quad & \frac{w(s_\alpha(\lambda) - \lambda)}{w(\alpha)} \\
= \quad & -\frac{2(\lambda,\alpha)}{(\alpha,\alpha)} \frac{w(\alpha)}{w(\alpha)} \\
= \quad & -\frac{2(\lambda,\alpha)}{(\alpha,\alpha)}.
\end{aligned}$$

Now $\partial_\alpha(\lambda) = \frac{s_\alpha(\lambda)-\lambda}{\alpha} = -\frac{2(\lambda,\alpha)}{(\alpha,\alpha)}$ in $\mathbb{Z} = \Lambda^0$, and b' preserves integers, so the claim is true in this case.

Since all maps are additive, it suffices by induction on the degree of P to prove that the validity of the formula for $P_1 = \lambda \in M$ and P_2 implies it for

$P_1 \cdot P_2$. Now

$$\partial_\alpha(P_1P_2) = \partial_\alpha(P_1)\,P_2 + s_\alpha(P_1)\,\partial_\alpha(P_2),$$

which follows from the formula $s_\alpha(P_1P_2) - P_1P_2 = (s_\alpha(P_1) - P_1)P_2 + s_\alpha(P_1)(s_\alpha(P_2) - P_2)$; and

$$A_\alpha(\psi_1\psi_2) = A_\alpha(\psi_1)\,\psi_2 + s_\alpha(\psi_1)\,A_\alpha(\psi_2),$$

which follows from the formula $(\psi_1\psi_2)(ws_\alpha) - (\psi_1\psi_2)(w) = (\psi_1(ws_\alpha) - \psi_1(w))\psi_2(w) + \psi_1(ws_\alpha)(\psi_2(ws_\alpha) - \psi_2(w))$. (Here W acts on $F(W,Q)$ by $(v\psi)(w) = \psi(wv)$.)

The conclusion follows from the two displayed formulas, once we verify that b' respects the action of W. Since b' is a ring homomorphism, it suffices to verify this for $\lambda \in M$. Then $(v \cdot b'(\lambda))(w) = b'(\lambda)(wv) = (wv)(\lambda) = w(v(\lambda)) = b'(v(\lambda))(w)$, so $v \cdot b'(\lambda) = b'(v(\lambda))$ as required. □

Remark 3.2. It follows that the operators D_α also satisfy a "Leibniz-type" formula: for $x_1, x_2 \in H_T^*X$,

$$D_\alpha(x_1x_2) = D_\alpha(x_1)x_2 + s_\alpha(x_1)D_\alpha(x_2),$$

where the action of W on H_T^*X is the one described in the following corollary.

Corollary 3.3. *There is a unique left action of W on H_T^*X, preserving its grading and Λ-algebra structure, and satisfying*

$$s_\alpha \cdot x = D_\alpha(x \cdot y(s_\alpha)) - D_\alpha(x)y(s_\alpha)$$

for all $x \in H_T^*X$ and simple roots α. This action satisfies and is determined by the formula

$$(v \cdot x)|_w = x|_{wv}$$

for all $v, w \in W$ and $x \in H_T^*X$.

Proof. This is the induced action of W on H_T^*X as a subalgebra of $F(W,\Lambda)$. If x' is the image of x in $F(W,\Lambda)$, and y' is the image of $y(s_\alpha)$, then

$$A_\alpha(x'y') = A_\alpha(x')y' + s_\alpha(x')A_\alpha(y').$$

Since $A_\alpha(y') = D_\alpha(y(s_\alpha)) = 1$, this reads

$$s_\alpha(x') = A_\alpha(x'y') - A_\alpha(x')y'.$$

This implies the first displayed formula of the corollary, and proves that $H_T^*X \hookrightarrow F(W,\Lambda)$ id preserved by the action of W. □

4. Uniqueness of structure constants

Proposition 4.1. *The coefficients p_{uv}^w in the formula $y(u)\cdot y(v) = \sum p_{uv}^w y(w)$ satisfy and are uniquely determined by three properties:*

(1) $p_{uu}^u = \displaystyle\prod_{\beta \in u(R^-) \cap R^+} \beta$;

(2) $y(s_\alpha)|_u - y(s_\alpha)|_v)\, p^u_{uv} = \sum_{v^+} c_\alpha(v, v^+)\, p^u_{u\, v^+}$; *and*

(3) $(y(s_\alpha)|_w - y(s_\alpha)|_u) p^w_{uv} = \sum_{u^+} c_\alpha(u, u^+)\, p^w_{u^+ v} - \sum_{w^-} c_\alpha(w^-, w)\, p^{w^-}_{uv}$,

the sums over u^+ *with* $\ell(u^+) = \ell(u) + 1$ *and* w^- *with* $\ell(w^-) = \ell(w) - 1$.

Proof. As in the type A case, (3) follows from associativity of the product $y(s_\alpha) \cdot y(u) \cdot y(v)$, and (2) follows from (3), commutativity, and the fact that $p^w_{uv} = 0$ if $\ell(u) > \ell(w)$ (in fact, unless $u \leq w$ and $v \leq w$).

The proof of the uniqueness statement is also the same as in the type A case, using Corollary 14.4.6. □

5. Duality

Consider first the variety G/B^-. This has Schubert varieties $\overline{X}(w) = \overline{B^- \overline{p}(w)}$, where $\overline{p}(w) = n_w B^-/B^-$, and $\overline{Y}(w) = \overline{B\, \overline{p}(w)}$, with classes $\overline{x}(w) \in H^{2N-2\ell(w)}(G/B^-)$ and $\overline{y}(w) \in H^{2\ell(w)}(G/B^-)$, for $w \in W$. For clarity, in this section we will write $y(w)|_{p(v)}$, etc., in place of $y(w)|_v$. The entire discussion for G/B goes through for G/B^-, except that each root gets replaced by its negative. For example,

$$\overline{y}(w)|_{\overline{p}(w)} = \prod_{\beta \in w(R^-) \cap R^+} (-\beta) = (-1)^{\ell(w)} y(w)|_{p(w)}.$$

Let $\tau : \Lambda \to \Lambda$ be the graded involution that is multiplication by $(-1)^r$ on $\mathrm{Sym}^r M$; τ is induced by the involution of M that takes each root to its negative.

Proposition 5.1. *We have*

(a) $\overline{y}(w)|_{\overline{p}(v)} = \tau(y(w)|_{p(v)})$;

(b) $\overline{y}(u) \cdot \overline{y}(v) = \sum \tau(p^w_{uv})\, \overline{y}(w)$.

Proof. Part (a), in this and the following propositions, follows from the functoriality of pullbacks. Part (b) follows from what we have just seen, or by applying Proposition 4.1. □

There is a canonical G-equivariant isomorphism $\Phi : G/B \to G/B^-$, defined by

$$\Phi(gB/B) = g n_0 B^-/B^-,$$

where $n_0 \in N(T)$ is any representative of w_0. (This is the isomorphism obtained by identifying each of G/B and G/B^- with the space of Borel subgroups of G, since $gBg^{-1} = g n_0 B^- (g n_0)^{-1}$.) Since $\Phi(p(w w_0)) = \overline{e}(w)$, we see that Φ maps $X(w w_0)$ to $\overline{Y}(w)$, and $Y(w w_0)$ to $\overline{X}(w)$. Thus $\Phi^* \overline{y}(w) = x(w w_0)$, and $\Phi^* \overline{x}(w) = y(w w_0)$, so we have the following:

Proposition 5.2. (a) $x(w)|_{p(v)} = \overline{y}(w w_0)|_{\overline{p}(v w_0)} = \tau(y(w w_0)|_{p(v w_0)})$;

(b) $x(uw_0) \cdot x(vw_0) = \sum \tau(p_{uv}^w)\, x(ww_0)$.

Consider next the mapping $\Psi : G/B \to G/B$ given by left multiplication by a representative n_0 for w_0; i.e., $\Psi(gB/B) = n_0 gB/B$. This map Ψ is equivariant with respect to the homomorphism $\psi : G \to G$, $\psi(g) = n_0 g n_0^{-1}$. Since Ψ maps $p(w)$ to $p(w_0 w)$, $p(w_0 w)$ to $p(w)$, and sends $X(w_0 w)$ to $Y(w)$, we have $\Psi^* y(w) = x(w_0 w)$ and $\Psi^*(x(w)) = y(w_0 w))$. Note that $\psi^* : \Lambda \to \Lambda$ is induced by the map from M to M that takes a weight λ to $w_0(\lambda)$; let

$$\tau_0 = \psi^* \colon \Lambda \to \Lambda$$

be this involution. Note that τ_0 also takes a product of positive weights to a product of negative weights. Applying the homomorphism Ψ^*, we find

Proposition 5.3. (a) *$x(w_0 w)|_{p(w_0 v)} = \tau_0(y(w)|_{p(v)})$;*

(b) *$x(w_0 u) \cdot x(w_0 v) = \sum \tau_0(p_{uv}^w)\, x(w_0 w)$.*

Finally, $\Phi \circ \Psi : G/B \to G/B^-$ is equivariant with respect to ψ, taking $Y(w_0 w w_0)$ to $X(w w_0)$ to $\overline{Y}(w)$, so we have

Proposition 5.4. (a) *$y(w_0 w w_0)|_{p(w_0 v w_0)} = \tau_0 \tau(y(w)|_{p(v)})$;*

(b) *$y(w_0 u w_0) \cdot y(w_0 v w_0) = \sum \tau_0 \tau(p_{uv}^w)\, y(w_0 w w_0)$.*

Note that $\tau_0 \tau = \tau \tau_0$ preserves products of positive roots.

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