EQUIVARIANT COHOMOLOGY IN ALGEBRAIC GEOMETRY
LECTURE TWO: DEFINITIONS AND BASIC PROPERTIES

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For a Lie group G, we are looking for a right principal G-bundle $EG \to BG$, with EG contractible. Such a bundle is universal in the topological setting: if $E \to B$ is any principal G-bundle, then there is a map $B \to BG$, unique up to homotopy, such that E is isomorphic to the pullback of EG. See [Hus75] for the existence of these universal principal bundles; we will not need the general story here.

We will also find principal G-bundles $EG_m \to BG_m$, with $\pi_i(EG_m) = 0$ (and $H^i(EG_m) = 0$) for $0 < i < k(m)$, where $k(m)$ goes to infinity as m grows. For such bundles, we have

$$H^i_G X := H^i(EG \times^G X) = H^i(EG_m \times^G X)$$

for $i < k(m)$. To see this, we need the following proposition:

Proposition 1.1. *If $E \to B$ and $E' \to B'$ are two principal right G-bundles, and $H^i(E) = H^i(E') = 0$ for $0 < i < k$, then there is a canonical isomorphism*

$$H^i(E \times^G X) \cong H^i(E' \times^G X)$$

for $i < k$.

Proof. Let G act diagonally on $E \times E'$, so there is a diagram

$$\begin{tikzcd}
E \times X & E \times E' \times X \arrow[l] \arrow[r] \arrow[d] & E' \times X \arrow[d] \\
E \times^G X \arrow[from=u, d, phantom] & (E \times E') \times^G X \arrow[l] \arrow[r] & E' \times^G X.
\end{tikzcd} \tag{1}$$

Here the vertical maps are G-bundles, the horizontal maps to the left are E-bundles, and the horizontal maps to the right are E'-bundles (all locally trivial). We claim that the maps

$$H^i(E \times^G X) \to H^i((E \times E') \times^G X) \leftarrow H^i(E' \times^G X)$$

are isomorphisms. This is a general fact about fiber bundles; in fact, it follows from the "Leray-Hirsch" lemma below. ☐

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Lemma 1.2. *Let* $E \xrightarrow{\pi} B$ *be a locally trivial fiber bundle, with fiber* F*. Let* R *be a commutative ring (with unit), and consider cohomology with coefficients in* R*. For* $m \geq 0$ *and* $0 \leq i \leq m$*, suppose there are a finite number of elements* $x_{ij} \in H^i(E)$ *whose retrictions to* $H^i(F)$ *form a basis over* R*. Then every element in* $H^m(E)$ *can be uniquely expressed in the form* $\sum_{ij} c_{ij} x_{ij}$ *with* $c_{ij} \in H^{m-i}(B)$*. (Similarly, each element can be written uniquely as* $\sum_{ij} x_{ij} c'_{ij}$*.)*

Corollary 1.3. *If* $H^i F = 0$ *for* $0 < i \leq m$*, then* $H^i B \to H^i E$ *is an isomorphism for* $i \leq m$*.*

Proposition 1.1 is an immediate consequence.

Remark 1.4. One might expect to find Lemma 1.2 in standard topology books or papers that cover spectral sequences, but all references we know include extra hypotheses (e.g., that R is a Dedekind ring, B is simply connected, or other conditions on the spaces). The statement of the Leray-Hirsch theorem given in [Hat02, Thm 4D.1] is very close, and the proof given there adapts easily to prove Lemma 1.2. One can also adapt the discussion in [Spa66, §5.7].

Exercise 1.5. Check compatibility of the isomorphism in Proposition 1.1 with a third principal G-bundle $E'' \to B''$.

Example 1.6. As in Lecture 1, for $G = \mathbb{C}^*$, take $EG_m = \mathbb{C}^m \smallsetminus \{0\}$, so $BG_m = \mathbb{P}^{m-1}$. We have $\Lambda_{\mathbb{C}^*} = \mathbb{Z}[t]$, with $t = c_1(\mathcal{O}(-1))$. We claimed that $t = c_1^G(L)$, where L is the equivariant line bundle $\mathbb{C}$ on a point, with action $g \cdot z = gz$.

To see this, recall $c_1^G(L)$ the (ordinary) first Chern class of the line bundle $EG_m \times^G L \to BG_m$. Note that $(z_1, \ldots, z_m) \times z \mapsto [z_1, \ldots, z_m] \times (z_1 z, \ldots, z_m z)$ maps $EG_m \times L$ to the trivial bundle $\mathbb{C}^m_{\mathbb{P}^{m-1}}$, and its image is the tautological subbundle $\mathcal{O}(-1) \subset \mathbb{C}^m_{\mathbb{P}^{m-1}}$. This maps passes to the quotient by $\mathbb{C}^*$, since $(z_1 g, \ldots, z_m g) \times z$ and $(z_1, \ldots, z_m) \times gz$ have the same image, $(z_1 gz, \ldots, z_m gz)$. Thus we get an isomorphism

$$\begin{tikzcd}
EG_m \times^G L \arrow[r, "\sim"] \arrow[d] & \mathcal{O}(-1) \arrow[d] \\
BG_m \arrow[r, "\sim"] & \mathbb{P}^{m-1}.
\end{tikzcd}$$

For $G = (\mathbb{C}^*)^n$, take $EG_m = (\mathbb{C}^m \smallsetminus \{0\})^m$, so $BG_m = (\mathbb{P}^{m-1})^n$. Then $\Lambda_G = \mathbb{Z}[t_1, \ldots, t_n]$, where $t_i = c_1(p_i^* \mathcal{O}(-1))$. Note that $p_i^* \mathcal{O}(-1)$ is isomorphic to the line bundle L_{χ_i}, where χ_i is the character $\chi(z_1, \ldots, z_n) = z_i$.

For $G = GL_n\mathbb{C}$, take $EG_m = M^o_{m,n}$, the set of $m \times n$ matrices of rank n (for $m \geq n$), with G acting on the right by matrix multiplication.

Proposition 2.1. *$\pi_i(M^o_{m,n}) = 0$ for $0 < i \leq 2(m-n)$.*

This follows from some general facts.

Lemma 2.2. *If $Z_r \subset M_{m,n}$ is the set of matrices of rank less than or equal to r, then Z_r is an irreducible subvariety of codimension $(m-r)(n-r)$.*

Exercise 2.3. Prove this.

Lemma 2.4. *If $Z \subset \mathbb{C}^N$ is a Zariski-closed set, of codimension d, then $\pi_i(\mathbb{C}^N \smallsetminus Z) = 0$ for $0 < i \leq 2d-2$. This is always sharp: $\pi_{2d-1}(\mathbb{C}^N \smallsetminus Z) \neq 0$ if Z is nonempty.*

In our case, $M^o_{m,n} = M_{m,n} \smallsetminus Z_{n-1}$, and $\operatorname{codim} Z_{n-1} = m-n+1$, so the proposition follows.

Remark 2.5. Again, we do not know a reference where this sharp bound is proved. See Appendix A.

Note that $BG_m = M^o_{m,n}/G$ is isomorphic to $Gr(n, \mathbb{C}^m)$, by mapping a matrix A to its image $\operatorname{im}(A) \subset \mathbb{C}^m$.

More intrinsically, for $G = GL(V)$, let $EG_m = \operatorname{Hom}^o(V, \mathbb{C}^m)$ be the space of embeddings of V in $\mathbb{C}^m$, with G acting on the right by $(\varphi \cdot g)(v) = \varphi(g \cdot v)$. Then $BG_m = Gr(n, \mathbb{C}^m)$, by $\varphi \mapsto \operatorname{im}(\varphi)$.

Let $E \subset \mathbb{C}^m_{Gr(n,\mathbb{C}^m)}$ be the tautological subbundle of rank n. Then it is a basic fact that $H^*(Gr(n,V))$ is generated by $c_1(E), \dots, c_n(E)$, with relations in degrees $m-n+1, \dots, m$. (We will prove this below; see also [Mil-Sta74].) Therefore

$$\Lambda = \Lambda_{GL(V)} = \Lambda_{GL_n} = \mathbb{Z}[c_1, \dots, c_n].$$

(In topology, one sees this by computing the cohomology of $Gr(n, \mathbb{C}^\infty)$.)

We obtain an equivariant vector bundle on a point from the action of G on V.

Lemma 2.6. *The class c_i is the ith equivariant Chern class of this bundle.*

Proof. As before, we have an isomorphism

$$\begin{tikzcd}
EG_m \times^G V \arrow[r, "\sim"] \arrow[d] & E \arrow[r, hook] \arrow[d] & \mathbb{C}^m_{Gr} \\
BG_m \arrow[r, "\sim"] & Gr(n, \mathbb{C}^m),
\end{tikzcd}$$

where the map is given by $\varphi \times v \mapsto \varphi(v) \in \operatorname{im}(\varphi)$, noting that $\varphi \cdot g \times v$ and $\varphi \times gv$ both map to $\varphi(gv)$. $\square$

Remark 2.7. There is an irreducible representation V_λ of $GL(V) \cong GL_n\mathbb{C}$ for each partition $\lambda = (\lambda_1 \geq \cdots \geq \lambda_n)$. (E.g., for $\lambda = (k, 0, \dots, 0)$, the corresponding representation is $V_\lambda = \operatorname{Sym}^k V$; for $\lambda = (\overbrace{1, \dots, 1}^{k}, 0, \dots, 0)$, it is $V_\lambda = \bigwedge^k V$.) Thus there are classes $c_r^G(V_\lambda \in \mathbb{Z}[c_1, \dots, c_n]$, for $1 \le r \le \dim V_\lambda$.

The total Chern class can be expressed as

$$c^G(V_\lambda) = \prod_{T \in SSYT(\lambda)} \left(1 + \sum_{i \in T} t_i\right),$$

where the product is over all semistandard Young tableaux with shape λ, and c_i is identified with the ith elementary symmetric polynomial in $t_1, \dots, t_n$ (so c_i is the Chern class $c_i^T(V_\lambda)$, for $T = (\mathbb{C}^*)^n \subset G$).[1]

In fact, $c_r^G(V_\lambda)$ can be written as a *positive* linear combination of *Schur polynomials* s_μ; these are given by

$$s_\mu = \sum_{T \in SSYT(\lambda)} \prod_{i \in T} t_i = \det(c_{\mu'_i + j - i}),$$

where μ' is the partition conjugate to μ. The proof uses the Hard Lefschetz Theorem [Ful-Laz83].

Lascoux gave formulas for $c_r^G(\bigwedge^2 V)$ and $c_r^G(\mathrm{Sym}^2 V)$ [Lascoux]. Beyond this, however, few explicit general formulas for these polynomials are known.

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For any subgroup $G \subset GL_n\mathbb{C}$, we can use the same approximation spaces $EG_m = M^o_{m,n}$, so we have all we need for linear algebraic groups, as well as for compact Lie groups such as $U(n)$.

Remark 3.1. For $G = PGL_n\mathbb{C}$, on the other hand, $H^*_G(pt)$ is only partly understood. Recent work of Vezzosi and Vistoli has led to a presentation of the ring $H^*(BPGL_3)$, as well as a description of the additive structure of $H^*(BPGL_p)$, for p prime. See [Vis05] for these results, and a summary of what else is known.

For $G = (\mathbb{C}^*)^n$, we have seen two choices for EG_m: we can take

$$(\mathbb{C}^m \smallsetminus \{0\})^n = \{A \in M_{m,n} \,|\, \text{no column of } A \text{ is } 0\},$$

or the smaller space $M^o_{m,n}$ consisting of those matrices with independent columns; the two choices give the same answer for $H^*_G X$. Note that using $EG_m = M^o_{m,n}$, we get

$$BG_m = M^o_{m,n}/(\mathbb{C}^*)^n = \left\{ \begin{array}{l} L \subset \mathbb{C}^m \text{ of dimension } n, \text{with a decomposition} \\ L = L_1 \oplus \cdots \oplus L_n \end{array} \right\}.$$

This space could be called the *split Grassmannian*, $Gr^{\text{split}}(n, \mathbb{C}^m)$. It comes equipped with tautological line bundles $L_1, \dots, L_n$, and $t_i = c_1(L_i)$.

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[1] See [Ful97] or Appendix (to be written) for basic facts about Young tableaux and Schur polynomials.

For $G = B = B^+$, the upper-triangular matrices in $GL_n\mathbb{C}$, we have

$$\begin{aligned} M^o_{m,n}/G &= \{L \subset \mathbb{C}^m \text{ with a filtration } L_1 \subset L_2 \subset \cdots \subset L_n = L\} \\ &= Fl(1, 2, \ldots, n; \mathbb{C}^m), \end{aligned}$$

by mapping a matrix A to $(L_1 \subset \cdots \subset L_n)$, with L_i the span of the first i columns of A. There is a tautological sequence of bundles $S_1 \subset \cdots S_n \subset \mathbb{C}^m_{Fl}$, and the cohomology ring $H^*(Fl(1, \ldots, n; \mathbb{C}^m)$ is generated by $t_1, \ldots, t_n$, where $t_i = c_1(S_i/S_{i-1})$. As in with the Grassmannian, this has relations in degrees $m - n + 1, \ldots, m$, so $\Lambda_B = \mathbb{Z}[t_1, \ldots, t_n]$. Thus $H^*_B(pt) = H^*_T(pt)$. (This is a general fact, but here we see it explicitly.)

Exercise 3.2. For $t_i = c_1(S_i/S_{i-1})$, show that $t_i = c_1(L_{\chi_i})$, where $\chi_i : B \to \mathbb{C}^*$ is the character which picks out the ith coordinate on the diagonal (extending χ_i from T to B).

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Equivariant cohomology $H^*_G X$ is functorial in both X and G. Specifically, let G act on X and G' act on X', let $\varphi : G \to G'$ be a continuous homomorphism of Lie groups, and let $f : X \to X'$ be continuous and **equivariant** with respect to φ; that is,

$$f(g \cdot x) = \varphi(g) \cdot f(x)$$

for $x \in X$, $g \in G$. Then there is a degree-preserving ring homomorphism

$$H^*_{G'}X' \to H^*_G X, \tag{2}$$

and this is functorial for compositions. In fact, one can find a continuous map $EG \to EG'$, equivariant for the right actions of G and G', so there is a commutative diagram

$$\begin{tikzcd} EG \arrow[r] \arrow[d] & EG' \arrow[d] \\ BG \arrow[r] & BG'. \end{tikzcd}$$

These maps are well-defined up to homotopy (see [Hus75]). Thus we get an induced map

$$EG \times^G X \to EG' \times^{G'} X',$$

and the map of (2) is the cohomology pullback for this.

More generally, suppose $E \to B$ is a right principal G-bundle and $E' \to B'$ is a right principal G'-bundle, such that $\pi_i(E) = \pi_i(E') = 0$ for $0 < i < k$, and suppose we have an equivariant map $E \to E'$.

Claim . In this situation, the corresponding map

$$H^i(E' \times^{G'} X') \to H^i(E \times^G X) \tag{3}$$

is the same as the map in (2), for $i < k$.

This shows that one can use approximation spaces to see the functorial maps in (2).

To prove the claim, form products to obtain a commutative diagram

$$\begin{tikzcd} E \times X & E \times EG \times X \arrow[l] \arrow[r] & EG \times X \\ E' \times X' \arrow[from=1-1, d] & E \times EG' \times X' \arrow[from=1-2, d] \arrow[l] \arrow[r] & EG' \times X', \arrow[from=1-3, d] \end{tikzcd} \tag{4}$$

and then take quotients to get a commutative diagram

$$\begin{tikzcd} E \times^G X & (E \times EG) \times^G X \arrow[l] \arrow[r] & EG \times^G X \\ E' \times^{G'} X' \arrow[from=1-1, d] & (E' \times EG') \times^{G'} X' \arrow[from=1-2, d] \arrow[l] \arrow[r] & EG' \times^{G'} X'. \arrow[from=1-3, d] \end{tikzcd} \tag{5}$$

(The principal bundle maps between these two diagrams look like the diagram (1).) As in Proposition 1.1, the cohomology maps induced by the horizontal arrows in (5) are canonical isomorphisms, so the claim follows.

For the cases we need, we will construct explicit maps $EG_m \to EG'_m$ on approximation spaces.

Two important special cases are the following:

(i) Given $G \to G'$, an action of G' on X' induces an action of G on X', so we get a map $H^*_{G'}X' \to H^*_G X'$. (In particular, there is a map $\Lambda_{G'} \to \Lambda_G$.)

In practice, it may help to change the group in either direction. The smaller group G should have more fixed points, which may help in calculation; the larger group G' may reveal more structure.

(ii) The map $X \to pt$ gives $H^*_G X$ the structure of a Λ_G-algebra, via the induced map $\Lambda_G = H^*_G(pt) \to H^*_G X$.

Example 4.1. The inclusions $(\mathbb{C}^*)^n \subset B^+ \subset GL_n\mathbb{C}$ give rise to a sequence $\Lambda_{GL_n} \to \Lambda_{B^+} \to \Lambda_{(\mathbb{C}^*)^n}$, i.e.,

$$\mathbb{Z}[c_1, \dots, c_n] \to \mathbb{Z}[t_1, \dots, t_n] \to \mathbb{Z}[t_1, \dots, t_n],$$

where the first map sends c_i to the ith elementary symmetric polynomial $e_i(t_1 \dots, t_n)$, and the second map is the identity.

Exercise 4.2. If $T = (\mathbb{C}^*)^n \subset B = B^+$, and B acts on X, then show that $H^*_B X \to H^*_T X$ is an isomorphism. (In fact, $ET_m \to EB_m$ is a fiber bundle with contractible fibers, and the same is true for $ET_m \times^T X \to EB_m \times^B X$.)

Exercise 4.3. The inclusions of compact subgroups $(S^1)^n \subset (\mathbb{C}^*)^n$ and $U(n) \subset GL_n\mathbb{C}$ give isomorphisms of equivariant cohomology rings.

Exercise 4.4. What is $\Lambda_{SL_n\mathbb{C}}$?

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