EQUIVARIANT COHOMOLOGY IN ALGEBRAIC GEOMETRY
LECTURE THREE: MORE BASICS, FIRST EXAMPLES

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1

As a first example, we discuss the solution to Exercise (2.4.4). We claim that

$$\Lambda_{SL_n\mathbb{C}} = \mathbb{Z}[c_2, \dots, c_n] = \mathbb{Z}[c_1, c_2, \dots, c_n]/(c_1) = \Lambda_{GL_n\mathbb{C}}/(c_1).$$

To see this, note that

$$M^o_{m,n}/SL_n\mathbb{C} = \{(L, \varphi) \,|\, L \subset \mathbb{C}^m, \text{ and } \varphi : \textstyle\bigwedge^n L \cong \mathbb{C}\},$$

so it is the complement of the zero section of a line bundle on $Gr(n, \mathbb{C}^m)$:

$$\begin{array}{ccc} M^o_{m,n}/SL_n\mathbb{C} & \xrightarrow{\sim} & \bigwedge^n S \smallsetminus \{0\} \\ \downarrow & & \downarrow \\ M^o_{m,n}/GL_n\mathbb{C} & \xrightarrow{\sim} & Gr(n, \mathbb{C}^m), \end{array}$$

where $S \subset \mathbb{C}^m_{Gr}$ is the tautological subbundle. The claim then follows from the following topological lemma:

Lemma 1.1. *Let $E \to X$ be a vector bundle of rank e, such that the map $a \mapsto a \cdot c_e(E)$ is injective on H^*X. Then $H^*(E \smallsetminus \{0\}) = H^*(X)/(c_e(E))$.*

In fact, if multiplication by $c_e(E)$ is injective on $H^{i+1-2e}X$, then $H^i(E \smallsetminus \{0\}) = H^iX/(c_e(E) \cdot H^{i-2e}X)$. The proof is immediate from the Gysin sequence of E.

Example 1.2. Let $G = GL(V)$. The $GL(V)$-equivariant cohomology ring of $\mathbb{P}(V)$ is

$$H^*_{GL(V)}\mathbb{P}(V) = \Lambda[\zeta]/(\zeta^n + c_1\zeta^{n-1} + \cdots + c_n),$$

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where $n = \dim V$, $\Lambda = \mathbb{Z}[c_1, \dots, c_n]$, $\zeta = c_1^G(\mathcal{O}(1))$, and $\mathcal{O}(1)$ is the dual of the tautological subbundle $\mathcal{O}(-1) \subset V_{\mathbb{P}(V)}$. To see this, use the approximations $EG_m = \operatorname{Hom}^o(V, \mathbb{C}^m)$, with diagrams

$$\begin{tikzcd}
\operatorname{Hom}^o(V, \mathbb{C}^m) \times^G \mathbb{P}(V) \arrow[r, phantom, "\cong"] \arrow[d] & \mathbb{P}(S) \arrow[d] \\
\operatorname{Hom}^o(V, \mathbb{C}^m)/G \arrow[r, phantom, "\cong"] & Gr(n, \mathbb{C}^m).
\end{tikzcd}$$

Here $S \subset \mathbb{C}^m$ is the tautological subbundle on $Gr(n, \mathbb{C}^m)$, as before. The isomorphism in the top row is given by $(\varphi, \ell) \mapsto \varphi(\ell) \subset \operatorname{im}(\varphi)$; similarly, there is an isomorphism

$$\operatorname{Hom}^o(V, \mathbb{C}^m) \times^G \mathcal{O}_{\mathbb{P}(V)}(-1) \cong \mathcal{O}_{\mathbb{P}(S)}(-1)$$

given by $(\varphi, v) \mapsto \varphi(v)$. Now the computation of $H_G^*\mathbb{P}(V)$ is reduced to a general fact about projective bundles: For any $\mathbb{P}(S) \to Z$, $\{1, \zeta, \dots, \zeta^{n-1}\}$ is a basis for $H^*\mathbb{P}(S)$ over H^*Z (by the Leray-Hirsch lemma), with relation

$$\zeta^n + c_1(S)\zeta^{n-1} + \cdots + c_n(S) = 0,$$

since the left-hand side is $c_n(S \otimes \mathcal{O}(1))$, and the inclusion $\mathcal{O} \hookrightarrow S \otimes \mathcal{O}(1)$ implies this is zero. Finally, $c_i = c_i(S)$ for large enough m.

Corollary 1.3. *For $G = B$, or for $G = (\mathbb{C}^*)^n$,*

$$H_G^*\mathbb{P}(V) = \Lambda[\zeta]/(\textstyle\prod_{i=1}^n(\zeta + t_i)),$$

where $\Lambda = \mathbb{Z}[t_1, \dots, t_n]$.

2

Let T be a torus, so T is isomorphic to $(\mathbb{C}^*)^n$, but not by a given isomorphism. Let $M = \operatorname{Hom}_{\text{alg. gp.}}(T, \mathbb{C}^*)$ be the group of characters (so $M \cong \mathbb{Z}^n$), which is naturally dual to $N = \operatorname{Hom}_{\text{alg. gp.}}(\mathbb{C}^*, T)$, the group of 1-parameter subgroups. For each $\chi \in M$, there is an associated equivariant line bundle L_χ on a point, so $c_1^T(L_\chi) \in H_T^2(pt)$. We have $L_{\chi_1\chi_2} = L_{\chi_1} \otimes L_{\chi_2}$, so this map $M \to H_T^2(pt)$ is a homomorphism; in fact, it is an isomorphism, as one sees by choosing an isomorphism $T \cong (\mathbb{C}^*)^n$. Therefore

$$\operatorname{Sym}_{\mathbb{Z}}^* M \xrightarrow{\sim} H_T^*(pt) = \Lambda_T,$$

by a canonical isomorphism.

This isomorphism is *natural*, in following sense. A homomorphism $\varphi : T \to T'$ corresponds to a homomorphism $M' \to M$ (and dually, to $N \to N'$). Since $L_{\chi'}$ pulls back to L_χ, where $\chi = \varphi^*\chi'$, the diagram

$$\begin{tikzcd}
\operatorname{Sym}^* M' \arrow[r] \arrow[d] & \operatorname{Sym}^* M \arrow[d] \\
\Lambda_{T'} \arrow[r] & \Lambda_T
\end{tikzcd}$$

commutes.

Example 2.1. Let $T = (\mathbb{C}^*)^n/(\mathbb{C}^*)$, $\Lambda = \mathrm{Sym}^* M$, for $M = \{(a_1, \dots, a_n) \in \mathbb{Z}^n \mid \sum a_i = 0\}$. So

$$\Lambda = \mathbb{Z}[t_1 - t_2, \dots, t_{n-1} - t_n] = \mathbb{Z}[t_2 - t_1, \dots, t_n - t_{n-1}]$$

in $\mathbb{Z}[t_1, \dots, t_n] = \Lambda_{(\mathbb{C}^*)^n}$, among (infinitely many) other ways of writing this.

Example 2.2. Define a homomorphism $(\mathbb{C}^*)^n \to (\mathbb{C}^*)^r$ by

$$(z_1, \dots, z_n) \mapsto \left(\textstyle\prod_{i=1}^n z_i^{a_{i1}}, \dots, \prod_{i=1}^n z_i^{a_{ir}}\right),$$

for some $n \times r$ matrix A (corresponding to a map $\mathbb{Z}^r \to \mathbb{Z}^n$)). The corresponding map $\mathbb{Z}[s_1, \dots, s_r] \to \mathbb{Z}[t_1, \dots, t_n]$ takes s_j to $\sum_{i=1}^n a_{ij} t_i$.

Example 2.3. For the map $\mathbb{C}^* \to \mathbb{C}^*$ given by $z \mapsto z^{-a}$, corresponding equivariant maps $\mathbb{C}^m \smallsetminus \{0\} \to \mathbb{C}^m \smallsetminus \{0\}$ are given by

$$(x_1, \dots, x_m) \mapsto \|x\|^{-a} (\bar{x}_1^a, \dots, \bar{x}_m^a),$$

where $\|x\| = \sum |x_i|^2$. The induced map $f : \mathbb{P}^{m-1} \to \mathbb{P}^{m-1}$ on BG_m has $f^* \mathcal{O}(1) = \mathcal{O}(-a)$.

Exercise 2.4. For $G = GL_n$, find a map $EG_m \to EG_m$ equivariant with respect to the homomorphism $g \mapsto {}^t g^{-1}$, where $EG_m = M^o_{m,n}$ is $m \times n$ matrices of rank n. (Hint/Solution: Take $A \mapsto \overline{A} \cdot ({}^t A \overline{A})^{-1}$.)

Exercise 2.5. Let T be a torus acting on a vector space V by characters $\chi_1, \dots, \chi_n$. Then

$$H_T^* \mathbb{P}(V) = \Lambda_T[\zeta] / \left(\textstyle\prod_{i=1}^n (\zeta + \chi_i)\right).$$

***Exercise 2.6.** Compute $H_T^* \mathbb{P}^{n-1}$ for the action of $T = (\mathbb{C}^*)^n/\mathbb{C}^*$ on $\mathbb{P}^{n-1}$.

3

Proposition 3.1. *Let G be a Lie group acting on a space X.*

(i) *If G acts freely on X, with $X \to G \backslash X$ a locally trivial fiber bundle, then*

$$H_G^i X = H^i(G \backslash X).$$

(ii) *If G acts trivially on X, then*

$$H_G^i X = H^i(BG \times X).$$

Thus $H_G^* X = \Lambda_G \otimes_{\mathbb{Z}} H^* X$ when $H^k(BG)$ is free and finitely generated for all k.

Proof. For (i), consider the fiber square

$$\begin{array}{ccc} EG_m \times X & \longrightarrow & X \\ \downarrow & & \downarrow \\ EG_m \times^G X & \longrightarrow & G \backslash X, \end{array}$$

whose vertical maps are principal G-bundles, and whose horizontal maps are EG_m-bundles. By the Leray-Hirsch lemma, $H^i(G\backslash X) \xrightarrow{\sim} H^i(EG_m \times^G X)$, for $m > m(i)$.

For (ii), note that $EG_m \times^G X = BG_m \times X$. The second statement follows from the Künneth formula. □

Proposition 3.2. *If $H^iX = 0$ for $0 < i \le m$, then $H^i_G(pt) \to H^i_G X$ is an isomorphism for $i \le m$. Thus if X is contractible, $H^*_G X = \Lambda_G$.*

Proof. In the diagram

$$\begin{tikzcd} EG_m \times X \arrow[r] \arrow[d] & EG_m \arrow[d] \\ EG_m \times^G X \arrow[r] & BG_m, \end{tikzcd}$$

the vertical maps are G-bundles, and the horizontal maps are X-bundles. The statement follows from the Leray-Hirsch lemma, applied to the bottom horizontal map. □

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A space X satisfying the hypotheses of the following proposition is called **(equivariantly) formal** with respect to the ring R.

Proposition 4.1. *Assume H^iX is finitely generated and free over R for $0 \le i \le m$, and suppose there are elements $x_{ij} \in H^i_G X$ that restrict to a basis for H^iX. Then every element of $H^m_G X$ has a unique expression $\sum x_{ij}c_{ij}$, for $c_{ij} \in H^{m-i}BG$. (There is also a unique expression $\sum c'_{ij}x_{ij}$, for possibly different elements $c'_{ij} \in H^{m-i}BG$.)*

If in addition $H^iX = 0$ for $i > m$, and $H^kBG = 0$ for k odd, then $H^*_G X$ is finitely generated and free over Λ_G, with basis $\{x_{ij}\}$. Moreover, the map

$$H^*_G X \otimes_{\Lambda_G} R \to H^*X$$

is an isomorphism. In fact, for any homomorphism $G' \to G$, the map

$$H^*_G X \otimes_{\Lambda_G} \Lambda_{G'} \to H^*_{G'} X$$

is an isomorphism.

Proof. The first statement is simply an application of the Leray-Hirsch lemma to the fiber bundle $EG \times^G X \to BG$ (or $EG_m \times^G X \to BG_m$), which has fiber X.

For the second part, note that $H^kBG = 0$ for k odd implies Λ_G is a commutative ring, so $H^*_G X$ is a Λ_G-algebra, with basis $\{x_{ij}\}$ as above. In the fiber square

$$\begin{tikzcd} EG' \times^{G'} X \arrow[r] \arrow[d] & EG \times^G X \arrow[d] \\ BG' \arrow[r] & BG, \end{tikzcd}$$

the vertical maps are fiber bundles with fiber X, so the basis $\{x_{ij}\}$ for H_G^*X restricts to a basis $\{x'_{ij}\}$ for $H_{G'}^*X$. $\square$

5

Let G be a linear algebraic group acting on nonsingular algebraic varieties X and Y, and let $f : X \to Y$ be a proper, G-equivariant morphism. Then there are **equivariant Gysin maps**

$$f_*^G : H_G^i X \to H_G^{i+2d} Y,$$

where $d = \dim Y - \dim X$. (We will sometimes write f_* for f_*^G.)

These are constructed from the ordinary Gysin maps associated to the proper morphism

$$EG_m \times^G X \to EG_m \times^G Y$$

of nonsingular algebraic varieties, with the difference in dimensions being d. Identifying $H_G^i X$ with $H^i(EG_m \times^G X)$ for sufficiently large m, and similarly for Y, the equivariant Gysin map is defined to be the ordinary Gysin map $H^i(EG_m \times^G X) \to H^{i+2d}(EG_m \times^G Y)$.

These Gysin maps have the following properties:

(i) (Functoriality) If $g : Y \to Z$ is another proper G-equivariant morphism, then so is $g \circ f$, and $(g \circ f)_* = g_* \circ f_*$.

(ii) (Projection formula) For $b \in H_G^* Y$ and $a \in H_G^* X$,

$$f_*^G(f^* b \cdot a) = b \cdot f_*^G a.$$

(iii) (Naturality) Given a fiber square of nonsingular varieties and G-equivariant maps

$$\begin{tikzcd} X' \arrow[r, "g'"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g"] & Y, \end{tikzcd}$$

with f and f' proper, and $\dim Y - \dim X = \dim Y' - \dim X'$, then

$$g^* \circ f_*^G = (f')_*^G \circ (g')^*.$$

(iv) (Embedding) If $f : X \to Y$ is a closed G-equivariant embedding, its normal bundle N becomes a G-equivariant vector bundle on X, and the composition $f^* \circ f_*^G : H_G^i X \to H_G^{i+2d} X$ is multiplication by $c_d^G(N)$.

(v) If $g : X' \to Y$ is also G-equivariant, and $g(X') \cap f(X) = \emptyset$, then $g^* \circ f_*^G = 0$. (This follows from property (iii).)

(vi) If $V \subset X$ is a G-invariant subvariety, and $W = f(V) \subset Y$, then W is a G-invariant subvariety of Y, and

$$f_*^G[V]^G = \begin{cases} \deg(V/W)[W]^G & \text{if } \dim W = \dim V; \\ 0 & \text{if } \dim W < \dim V. \end{cases}$$

These properties follow directly from the corresponding properties for ordinary Gysin maps (see Appendix A), applied to approximation spaces. For example, to prove (vi), apply the non-equivariant Gysin map to $EG_m \times^G V \to EG_m \times^G W$, which has the same degree as $V \to W$.

Note that the ordinary version of (iii) implies that the definition of f_*^G is independent of the choice of EG_m or EG. Indeed, if $E \to E'$ is an equivariant map of two such choices, then applying the non-equivariant version of (iii) to the fiber square

$$\begin{tikzcd} E \times^G X \arrow[r] \arrow[d] & E' \times^G X \arrow[d] \\ E \times^G Y \arrow[r] & E' \times^G Y \end{tikzcd}$$

shows that the pushforwards f_*^G for the two choices agree.

One consequence of (iv) which will be useful later is the following:

Proposition 5.1. *Let $Y \subset X$ be a closed G-invariant subvariety of codimension d, and let $p \in Y$ be a nonsingular fixed point. Then the image of $[Y]^G$ under the restriction map $H_G^*X \to H_G^*(p)$ is $c_d^G(N(p))$, where $N(p)$ is the normal space to Y in X at p.*

When f is the projection of a fiber bundle, in the non-equivariant case and with coefficients in $\mathbb{R}$, the Gysin map f_* can be interpreted as integration over the fiber. The equivariant Gysin map can be interpreted similarly as integration over the fibers of $EG_m \times^G X \to EG_m \times^G Y$.

There are generalizations of properties (iii), (iv), and (v) involving excess normal bundles. See Appendix A, or [Ful-Mac81, §4].

A special case of the Gysin construction comes from the map $X \xrightarrow{p} pt$, when X is compact. The ordinary Gysin map is $p_* : H^{2N}X \to H^0(pt) = \mathbb{Z}$ (where $N = \dim X$), while $p_*(H^jX) = 0$ for $j \neq 2N$. In the equivariant situation, however, the map is

$$p_*^G : H_G^iX \to H_G^{i-2N}(pt),$$

which can be nonzero for $i \geq 2N$. These maps are sometimes written as $a \mapsto \int_X a$.

Using the equivariant pushforward to a point, we have an equivariant version of Poincaré duality:

Proposition 5.2. *Let $\{x_\alpha\}$ be a (homogeneous, right) basis for H_G^*X over Λ_G, and assume X is formal with respect to the coefficient ring R. Then* *there is a unique (homogeneous, left) basis* $\{y_\alpha\}$ *of* H_G^*X *over* Λ_G *such that*

$$p_*(y_\beta \cdot x_\alpha) = \delta_{\beta\alpha}$$

in Λ_G.

This follows from a general fact about fiber bundles; see Appendix A. The bases $\{x_\alpha\}$ and $\{y_\alpha\}$ are sometimes called *(equivariant) Poincaré dual bases.*

For $x, y \in H_G^*X$, set $\langle y, x\rangle = p_*(y \cdot x) \in \Lambda_G$. Then with $\{x_\alpha\}$ and $\{y_\beta\}$ as in Proposition 5.2, for any $y \in H_G^*X$, we have $y = \sum c_\alpha y_\alpha$, where $c_\alpha = \langle y, x_\alpha\rangle$. Indeed, if we write $y = \sum c_\beta y_\beta$, then by the projection formula we have

$$\begin{aligned}\langle y, x_\alpha\rangle &= \sum c_\beta \, \langle y_\beta, x_\alpha\rangle \\ &= \sum c_\beta \, \delta_{\beta\alpha} \\ &= c_\alpha.\end{aligned}$$

Example 5.3. Let $x_\alpha = [V_\alpha]^G$ for G-invariant subvarieties $V_\alpha \subset X$, and suppose there are G-invariant subvarieties W_α such that V_α and W_α intersect transversally in a point, and $V_\alpha \cap W_\beta = \emptyset$ if $\alpha \neq \beta$ and $\operatorname{codim}(V_\alpha) + \operatorname{codim}(W_\beta) \leq \dim X$. Then we can take $y_\alpha = [W_\alpha]^G$ to form a dual basis.

Example 5.4. If $X = \mathbb{P}^{n-1}$, $G = (\mathbb{C}^*)^n$, and

$$V_a = \{[* : \cdots : * : \underbrace{0 : \cdots : 0}_{a}]\} \subset \mathbb{P}^{n-1},$$

then $x_a = [V_a]^G$ is a basis for $H^*\mathbb{P}^{n-1}$, for $0 \leq a \leq n-1$. If we take

$$W_a = \{[0 : \cdots : 0 : \underbrace{* : \cdots : *}_{a+1}]\},$$

then $y_a = [W_a]^G$ is the Poincaré dual basis.

References

[Ful-Mac81] W. Fulton and R. MacPherson, "Categorical framework for the study of singular spaces," *Mem. Amer. Math. Soc.* **31** (1981), no. 243.