As a first example, we discuss the solution to Exercise (2.4.4). We claim that 
\[ \Lambda_{SL_n \mathbb{C}} = \mathbb{Z}[c_2, \ldots, c_n] = \mathbb{Z}[c_1, c_2, \ldots, c_n]/(c_1) = \Lambda_{GL_n \mathbb{C}}/(c_1). \]
To see this, note that 
\[ M^o_{m,n}/SL_n \mathbb{C} = \{ (L, \varphi) \mid L \subset \mathbb{C}^m, \text{ and } \varphi : \Lambda^n L \cong \mathbb{C} \}, \]
so it is the complement of the zero section of a line bundle on \( Gr(n, \mathbb{C}^m) \):
\[
\begin{array}{c}
M^o_{m,n}/SL_n \mathbb{C} \sim \Lambda^n S \setminus \{0\} \\
\downarrow \\
M^o_{m,n}/GL_n \mathbb{C} \sim Gr(n, \mathbb{C}^m),
\end{array}
\]
where \( S \subset \mathbb{C}^m_{Gr} \) is the tautological subbundle. The claim then follows from the following topological lemma:

**Lemma 1.1.** Let \( E \to X \) be a vector bundle of rank \( e \), such that the map \( a \mapsto a \cdot c_e(E) \) is injective on \( H^\ast X \). Then \( H^\ast(E \setminus \{0\}) = H^\ast(X)/(c_e(E)) \).

In fact, if multiplication by \( c_e(E) \) is injective on \( H^{i+1-2e}X \), then \( H^i(E \setminus \{0\}) = H^i X/(c_e(E) \cdot H^{i-2e}X) \). The proof is immediate from the Gysin sequence of \( E \).

**Example 1.2.** Let \( G = GL(V) \). The \( GL(V) \)-equivariant cohomology ring of \( \mathbb{P}(V) \) is 
\[ H^\ast_{GL(V)} \mathbb{P}(V) = \Lambda[\zeta]/(\zeta^n + c_1 \zeta^{n-1} + \cdots + c_n), \]
where \( n = \dim V \), \( \Lambda = \mathbb{Z}[c_1, \ldots, c_n] \), \( \zeta = c_1^G(\mathcal{O}(1)) \), and \( \mathcal{O}(1) \) is the dual of the tautological subbundle \( \mathcal{O}(-1) \subset V \). To see this, use the approximations \( \text{EG}_m = \text{Hom}^G(V, \mathbb{C}^m) \), with diagrams

\[
\text{Hom}^G(V, \mathbb{C}^m) \cong \mathbb{P}(S)
\]

\[
\text{Hom}^G(V, \mathbb{C}^m)/G \cong \text{Gr}(n, \mathbb{C}^m).
\]

Here \( S \subset \mathbb{C}^m \) is the tautological subbundle on \( \text{Gr}(n, \mathbb{C}^m) \), as before. The isomorphism in the top row is given by \( (\varphi, \ell) \mapsto \varphi(\ell) \subset \text{im}(\varphi) \); similarly, there is an isomorphism

\[
\text{Hom}^G(V, \mathbb{C}^m) \times \mathcal{O}(S) \cong \mathbb{P}(S),
\]

given by \( (\varphi, v) \mapsto \varphi(v) \). Now the computation of \( H^*_G\mathbb{P}(V) \) is reduced to a general fact about projective bundles: For any \( \mathbb{P}(S) \rightarrow Z \), \( \{1, \zeta, \ldots, \zeta^{n-1}\} \) is a basis for \( H^*_\mathbb{P}(S) \) over \( H^*_Z \) (by the Leray-Hirsch lemma), with relation

\[
\zeta^n + c_1(S)\zeta^{n-1} + \cdots + c_n(S) = 0,
\]

since the left-hand side is \( c_n(S \otimes \mathcal{O}(1)) \), and the inclusion \( \mathcal{O} \rightarrow S \otimes \mathcal{O}(1) \) implies this is zero. Finally, \( c_i = c_i(S) \) for large enough \( m \).

**Corollary 1.3.** For \( G = B \), or for \( G = (\mathbb{C}^*)^n \),

\[
H^*_G\mathbb{P}(V) = \Lambda[\zeta]/(\prod_{i=1}^n (\zeta + t_i)),
\]

where \( \Lambda = \mathbb{Z}[t_1, \ldots, t_n] \).

---

Let \( T \) be a torus, so \( T \) is isomorphic to \( (\mathbb{C}^*)^n \), but not by a given isomorphism. Let \( M = \text{Hom}_{\text{alg. gp.}}(T, \mathbb{C}^*) \) be the group of characters (so \( M \cong \mathbb{Z}^n \)), which is naturally dual to \( N = \text{Hom}_{\text{alg. gp.}}(\mathbb{C}^*, T) \), the group of 1-parameter subgroups. For each \( \chi \in M \), there is an associated equivariant line bundle \( L_\chi \) on a point, so \( c_1^T(L_\chi) \in H^2_T(pt) \). We have \( L_{\chi_1\chi_2} = L_{\chi_1} \otimes L_{\chi_2} \), so this map \( M \rightarrow H^2_T(pt) \) is a homomorphism; in fact, it is an isomorphism, as one sees by choosing an isomorphism \( T \cong (\mathbb{C}^*)^n \). Therefore

\[
\text{Sym}^*_\mathbb{Z} M \cong H^*_T(pt) = \Lambda_T,
\]

by a canonical isomorphism.

This isomorphism is *natural*, in following sense. A homomorphism \( \varphi : T \rightarrow T' \) corresponds to a homomorphism \( M' \rightarrow M \) (and dually, to \( N \rightarrow N' \)). Since \( L_{\chi'} \) pulls back to \( L_\chi \), where \( \chi = \varphi^*\chi' \), the diagram

\[
\text{Sym}^* M' \rightarrow \text{Sym}^* M
\]

\[
\Lambda_{T'} \rightarrow \Lambda_T
\]

commutes.
Example 2.1. Let $T = (\mathbb{C}^*)^n/(\mathbb{C}^*)$, $\Lambda = \text{Sym}^* M$, for $M = \{(a_1, \ldots, a_n) \in \mathbb{Z}^n \mid \sum a_i = 0\}$. So
$$\Lambda = \mathbb{Z}[t_1 - t_2, \ldots, t_{n-1} - t_n] = \mathbb{Z}[t_2 - t_1, \ldots, t_n - t_{n-1}]$$
in $\mathbb{Z}[t_1, \ldots, t_n] = \Lambda (\mathbb{C}^*)^n$, among (infinitely many) other ways of writing this.

Example 2.2. Define a homomorphism $(\mathbb{C}^*)^n \to (\mathbb{C}^*)^r$ by
$$(z_1, \ldots, z_n) \mapsto (\prod_{i=1}^n z_i^{a_{i1}}, \ldots, \prod_{i=1}^n z_i^{a_{ir}}),$$
for some $n \times r$ matrix $A$ (corresponding to a map $\mathbb{Z}^r \to \mathbb{Z}^n$). The corresponding map $\mathbb{Z}[s_1, \ldots, s_r] \to \mathbb{Z}[t_1, \ldots, t_n]$ takes $s_j$ to $\sum_{i=1}^n a_{ij} t_i$.

Example 2.3. For the map $\mathbb{C}^* \to \mathbb{C}^*$ given by $z \mapsto z^{-a}$, corresponding equivariant maps $\mathbb{C}^m \setminus \{0\} \to \mathbb{C}^m \setminus \{0\}$ are given by
$$(x_1, \ldots, x_m) \mapsto \|x\|^{-a}(\bar{x}_1^a, \ldots, \bar{x}_m^a),$$
where $\|x\| = \sum |x_i|^2$. The induced map $f : \mathbb{P}^{m-1} \to \mathbb{P}^{m-1}$ on $BG_m$ has $f^* \mathcal{O}(1) = \mathcal{O}(-a)$.

Exercise 2.4. For $G = GL_n$, find a map $EG_m \to EG_m$ equivariant with respect to the homomorphism $g \mapsto t^g$, where $EG_m = M_{m,n}$ is $m \times n$ matrices of rank $n$. (Hint/Solution: Take $A \mapsto A \cdot (A^{-1} \bar{A})^{-1}$.)

Exercise 2.5. Let $T$ be a torus acting on a vector space $V$ by characters $\chi_1, \ldots, \chi_n$. Then
$$H^*_T \mathbb{P}(V) = \Lambda_T [\zeta]/(\prod_{i=1}^n (\zeta + \chi_i)).$$

*Exercise 2.6. Compute $H^*_T \mathbb{P}^{n-1}$ for the action of $T = (\mathbb{C}^*)^n/\mathbb{C}^*$ on $\mathbb{P}^{n-1}$.

3

Proposition 3.1. Let $G$ be a Lie group acting on a space $X$.

(i) If $G$ acts freely on $X$, with $X \to G \setminus X$ a locally trivial fiber bundle, then
$$H^*_G X = H^i (G \setminus X).$$

(ii) If $G$ acts trivially on $X$, then
$$H^*_G X = H^i (BG \times X).$$
Thus $H^*_G X = \Lambda_G \otimes \mathbb{Z} H^* X$ when $H^k(BG)$ is free and finitely generated for all $k$.

Proof. For (i), consider the fiber square
$$\begin{array}{ccc}
EG_m \times X & \longrightarrow & X \\
\downarrow & & \downarrow \\
EG_m \times G \times X & \longrightarrow & G \setminus X,
\end{array}$$

whose vertical maps are principal $G$-bundles, and whose horizontal maps are $EG_m$-bundles. By the Leray-Hirsch lemma, $H^i(G\backslash X) \sim H^i(EG_m \times^G X)$, for $m > m(i)$.

For (ii), note that $EG_m \times^G X = BG_m \times X$. The second statement follows from the Künneth formula. 

**Proposition 3.2.** If $H^i X = 0$ for $0 < i \leq m$, then $H^i_G(pt) \rightarrow H^i_GX$ is an isomorphism for $i \leq m$. Thus if $X$ is contractible, $H^*_G X = \Lambda_G$.

**Proof.** In the diagram

\[ \begin{array}{ccc}
EG_m \times X & \rightarrow & EG_m \\
\downarrow & & \downarrow \\
EG_m \times^G X & \rightarrow & BG_m,
\end{array} \]

the vertical maps are $G$-bundles, and the horizontal maps are $X$-bundles. The statement follows from the Leray-Hirsch lemma, applied to the bottom horizontal map. \qed

A space $X$ satisfying the hypotheses of the following proposition is called (equivariantly) formal with respect to the ring $R$.

**Proposition 4.1.** Assume $H^i X$ is finitely generated and free over $R$ for $0 \leq i \leq m$, and suppose there are elements $x_{ij} \in H^i_GX$ that restrict to a basis for $H^i X$. Then every element of $H^m_G X$ has a unique expression $\sum x_{ij} c_{ij}$, for $c_{ij} \in H^{m-i}BG$. (There is also a unique expression $\sum c'_{ij} x_{ij}$, for possibly different elements $c'_{ij} \in H^{m-i}BG$.)

If in addition $H^i X = 0$ for $i > m$, and $H^k BG = 0$ for $k$ odd, then $H^*_G X$ is finitely generated and free over $\Lambda_G$, with basis $\{x_{ij}\}$. Moreover, the map $H^*_G X \otimes_{\Lambda_G} R \rightarrow H^*X$ is an isomorphism. In fact, for any homomorphism $G' \rightarrow G$, the map $H^*_G X \otimes_{\Lambda_G} \Lambda_{G'} \rightarrow H^*_G X$ is an isomorphism.

**Proof.** The first statement is simply an application of the Leray-Hirsch lemma to the fiber bundle $EG \times^G X \rightarrow BG$ (or $EG_m \times^G X \rightarrow BG_m$), which has fiber $X$.

For the second part, note that $H^k BG = 0$ for $k$ odd implies $\Lambda_G$ is a commutative ring, so $H^*_G X$ is a $\Lambda_G$-algebra, with basis $\{x_{ij}\}$ as above. In the fiber square

\[ \begin{array}{ccc}
EG' \times^{G'} X & \rightarrow & EG \times^G X \\
\downarrow & & \downarrow \\
BG' & \rightarrow & BG,
\end{array} \]

...
the vertical maps are fiber bundles with fiber \( X \), so the basis \( \{ x_{ij} \} \) for \( H^*_G X \) restricts to a basis \( \{ x'_{ij} \} \) for \( H^*_G X' \).

\[\square\]

5

Let \( G \) be a linear algebraic group acting on nonsingular algebraic varieties \( X \) and \( Y \), and let \( f : X \to Y \) be a proper, \( G \)-equivariant morphism. Then there are equivariant Gysin maps

\[ f^*_G : H^i_G X \to H^{i+2d}_G Y, \]

where \( d = \dim Y - \dim X \). (We will sometimes write \( f_* \) for \( f^*_G \).

These are constructed from the ordinary Gysin maps associated to the proper morphism

\[ EG_m \times^G X \to EG_m \times^G Y \]

of nonsingular algebraic varieties, with the difference in dimensions being \( d \). Identifying \( H^*_G X \) with \( H^*(EG_m \times^G X) \) for sufficiently large \( m \), and similarly for \( Y \), the equivariant Gysin map is defined to be the ordinary Gysin map \( H^*(EG_m \times^G X) \to H^{i+2d}(EG_m \times^G Y) \).

These Gysin maps have the following properties:

(i) (Functoriality) If \( g : Y \to Z \) is another proper \( G \)-equivariant morphism, then so is \( g \circ f \), and \((g \circ f)_* = g_* \circ f_*\).

(ii) (Projection formula) For \( b \in H^*_G Y \) and \( a \in H^*_G X \),

\[ f^*_G (f^* b \cdot a) = b \cdot f^*_G a. \]

(iii) (Naturality) Given a fiber square of nonsingular varieties and \( G \)-equivariant maps

\[
\begin{array}{ccc}
X' & \xrightarrow{g'} & X \\
\downarrow{f'} & & \downarrow{f} \\
Y' & \xrightarrow{g} & Y,
\end{array}
\]

with \( f \) and \( f' \) proper, and \( \dim Y - \dim X = \dim Y' - \dim X' \), then

\[ g^* \circ f^*_G = (f')^*_G \circ (g')^*. \]

(iv) (Embedding) If \( f : X \to Y \) is a closed \( G \)-equivariant embedding, its normal bundle \( N \) becomes a \( G \)-equivariant vector bundle on \( X \), and the composition \( f^* \circ f^*_G : H^*_G X \to H^{i+2d}_G X \) is multiplication by \( c^*_G(N) \).

(v) If \( g : X' \to Y \) is also \( G \)-equivariant, and \( g(X') \cap f(X) = \emptyset \), then \( g^* \circ f^*_G = 0 \). (This follows from property (iii).)
(vi) If $V \subset X$ is a $G$-invariant subvariety, and $W = f(V) \subset Y$, then $W$ is a $G$-invariant subvariety of $Y$, and

$$f^*_G[V]^G = \begin{cases} \deg(V/W)[W]^G & \text{if } \dim W = \dim V; \\ 0 & \text{if } \dim W < \dim V. \end{cases}$$

These properties follow directly from the corresponding properties for ordinary Gysin maps (see Appendix A), applied to approximation spaces. For example, to prove (vi), apply the non-equivariant Gysin map to $EG_m \times^G V \rightarrow EG_m \times^G W$, which has the same degree as $V \rightarrow W$.

Note that the ordinary version of (iii) implies that the definition of $f^*_G$ is independent of the choice of $EG_m$ or $EG$. Indeed, if $E \rightarrow E'$ is an equivariant map of two such choices, then applying the non-equivariant version of (iii) to the fiber square

$$\begin{CD} E \times^G X @>>> E' \times^G X \\ @VVV @VVV \\ E \times^G Y @>>> E' \times^G Y \end{CD}$$

shows that the pushforwards $f^*_G$ for the two choices agree.

One consequence of (iv) which will be useful later is the following:

**Proposition 5.1.** Let $Y \subset X$ be a closed $G$-invariant subvariety of codimension $d$, and let $p \in Y$ be a nonsingular fixed point. Then the image of $[Y]^G$ under the restriction map $H^*_G X \rightarrow H^*_G(p)$ is $c_d^G(N(p))$, where $N(p)$ is the normal space to $Y$ in $X$ at $p$.

When $f$ is the projection of a fiber bundle, in the non-equivariant case and with coefficients in $\mathbb{R}$, the Gysin map $f^*$ can be interpreted as integration over the fiber. The equivariant Gysin map can be interpreted similarly as integration over the fibers of $EG_m \times^G X \rightarrow EG_m \times^G Y$.

There are generalizations of properties (iii), (iv), and (v) involving excess normal bundles. See Appendix A, or [Ful-Mac81, §4].

A special case of the Gysin construction comes from the map $X \overset{p}{\rightarrow} pt$, when $X$ is compact. The ordinary Gysin map is $p_* : H^{2N}X \rightarrow H^0(pt) = \mathbb{Z}$ (where $N = \dim X$), while $p_*(H^jX) = 0$ for $j \neq 2N$. In the equivariant situation, however, the map is

$$p^*_G : H^i_G X \rightarrow H^{i-2N}_G(pt),$$

which can be nonzero for $i \geq 2N$. These maps are sometimes written as $a \mapsto \int_X a$.

Using the equivariant pushforward to a point, we have an equivariant version of Poincaré duality:

**Proposition 5.2.** Let $\{x_a\}$ be a (homogeneous, right) basis for $H^*_G X$ over $\Lambda_G$, and assume $X$ is formal with respect to the coefficient ring $R$. Then
there is a unique (homogeneous, left) basis \{y_\alpha\} of \(H^*_G X\) over \(\Lambda_G\) such that
\[ p_*(y_\beta \cdot x_\alpha) = \delta_{\beta\alpha} \]
in \(\Lambda_G\).

This follows from a general fact about fiber bundles; see Appendix A. The bases \(\{x_\alpha\}\) and \(\{y_\alpha\}\) are sometimes called (equivariant) Poincaré dual bases.

For \(x, y \in H^*_G X\), set \(\langle y, x \rangle = p_*(y \cdot x) \in \Lambda_G\). Then with \(\{x_\alpha\}\) and \(\{y_\beta\}\) as in Proposition 5.2, for any \(y \in H^*_G X\), we have \(y = \sum c_\alpha y_\alpha\), where \(c_\alpha = \langle y, x_\alpha \rangle\). Indeed, if we write \(y = \sum c_\beta y_\beta\), then by the projection formula we have
\[
\langle y, x_\alpha \rangle = \sum c_\beta \langle y_\beta, x_\alpha \rangle = \sum c_\beta \delta_{\beta\alpha} = c_\alpha.
\]

Example 5.3. Let \(x_\alpha = [V_\alpha]^G\) for \(G\)-invariant subvarieties \(V_\alpha \subset X\), and suppose there are \(G\)-invariant subvarieties \(W_\alpha\) such that \(V_\alpha\) and \(W_\alpha\) intersect transversally in a point, and \(V_\alpha \cap W_\beta = \emptyset\) if \(\alpha \neq \beta\) and \(\text{codim}(V_\alpha) + \text{codim}(W_\beta) \leq \dim X\). Then we can take \(y_\alpha = [W_\alpha]^G\) to form a dual basis.

Example 5.4. If \(X = \mathbb{P}^{n-1}\), \(G = (\mathbb{C}^*)^n\), and
\[ V_\alpha = \{[* : \cdots : * : 0 : \cdots : 0]_a\} \subset \mathbb{P}^{n-1}, \]
then \(x_\alpha = [V_\alpha]^G\) is a basis for \(H^*_{\mathbb{P}^{n-1}}\), for \(0 \leq a \leq n - 1\). If we take
\[ W_\alpha = \{[0 : \cdots : 0 : * : \cdots : *]_{a+1}\}, \]
then \(y_\alpha = [W_\alpha]^G\) is the Poincaré dual basis.

References