EQUIVARIANT COHOMOLOGY IN ALGEBRAIC GEOMETRY
LECTURE FOUR: LOCALIZATION 1

WILLIAM FULTON
NOTES BY DAVE ANDERSON

1

Recall that for $G = GL_n(\mathbb{C})$,

$$H_G^*\mathbb{P}^{n-1} = \Lambda_G[\zeta]/(\zeta^n + c_1\zeta^{n-1} + \cdots + c_n),$$

where $\zeta = c_1^G(\mathcal{O}(1))$. The following exercise gives an example of equivariant Poincaré duality.

Exercise 1.1. The Poincaré dual basis for $\{1, \zeta, \ldots, \zeta^{n-1}\}$ in $H_G^*\mathbb{P}^{n-1}$ is

$$\{\zeta^{n-1} + c_1\zeta^{n-2} + \cdots + c_{n-1},\ \zeta^{n-2} + c_1\zeta^{n-3} + \cdots + c_{n-2},\ \ldots,\\ \zeta^2 + c_1\zeta + c_2,\ \zeta + c_1,\ 1\}.$$

(Note that $p_*(\zeta^{n-1}) = 1$, and $p_*(\zeta^i) = 0$ for $i < n-1$.)

We also saw that if a torus T acts on $\mathbb{C}^n$ by characters $\chi_1, \ldots, \chi_n$, then $H_T^*\mathbb{P}^{n-1} = \Lambda_T[\zeta]/(\prod_{i=1}^n(\zeta + \chi_i))$.

Example 1.2. The torus $T = (\mathbb{C}^*)^n/\mathbb{C}^*$ acts on $\mathbb{P}^{n-1}$, but in this case $\mathcal{O}(-1)$, $\mathcal{O}(1)$, and $\mathbb{C}^n_{\mathbb{P}^{n-1}}$ are not equivariant with respect to the natural action. For $\chi \in \mathbb{Z}^n$, the line bundle L_χ is equivariant for T only if $\chi_i \in M = \{(a_1, \ldots, a_n) \mid \sum a_i = 0\}$. But if $\chi = (a_1, \ldots, a_n) = \sum a_i t_i$ with $\sum a_i = 1$, then

$$\mathcal{O}(1) \otimes L_\chi$$

has a trivial $\mathbb{C}^*$ action, so it is T-equivariant. If ζ' is its first Chern class, then

$$H_T^*\mathbb{P}^{n-1} = \Lambda[\zeta']/\textstyle\prod_{i=1}^n(\zeta' + t_i - \chi).$$

Equivalently, choose a splitting of $1 \to \mathbb{C}^* \to (\mathbb{C}^*)^n \to T \to 1$ given by such a χ; the corresponding map $\mathbb{Z}^n \to M$ takes t_i to $t_i - \chi$. Note that

$$\mathbb{Z}[t_1, \ldots, t_n][\zeta]/\textstyle\prod(\zeta + t_i) \to \Lambda[\zeta']/\prod(\zeta' + t_i - \chi)$$

is given by $t_i \mapsto t_i - \chi$ and $\zeta \mapsto \zeta'$.

Date: February 12, 2007.

Consider again the standard action of $T = (\mathbb{C}^*)^n$ on $\mathbb{P}^{n-1}$. There are invariant subvarieties $\mathbb{P}_I \subset \mathbb{P}^{n-1}$ for each subset $I \subset \{1, \ldots, n\}$, given by $\mathbb{P}_I = \{[X_1 : \cdots : X_n] \mid X_i = 0 \text{ for } i \in I\}$. We claim that

$$[\mathbb{P}_I] = \prod_{i \in I} (\zeta + t_i).$$

In fact, X_i is an equivariant section of $\mathcal{O}(1) \otimes L_{t_i}$, so $[X_i = 0]^T = c_1^T(\mathcal{O}(1) \otimes L_{t_i}) = \zeta + t_i$.

Setting $I_k = \{1, 2, \ldots, k\}$, the classes $x_k = [\mathbb{P}_{I_k}]^T$ form a basis for $H_T^* \mathbb{P}^{n-1}$, for $0 \leq k \leq n-1$ (with $x_0 = 1$). This is the simplest example of a *Schubert basis* in equivariant cohomology. Note that the $\mathbb{P}_{I_k}$'s are in fact invariant for the group B^- of lower-triangular matrices. We will see later that this corresponds to a certain kind of positivity in the multiplication of their classes.

Challenge 1.3. What is the multiplication table in this basis? That is, writing

$$x_i \cdot x_k = \sum c_{ij}^k x_k,$$

find a formula for the polynomials $c_{ij}^k \in \Lambda$.

Since the $\mathbb{P}_{I_k}$ are also $(\mathbb{C}^*)^n / \mathbb{C}^*$-invariant, the coefficients must be in the corresponding $\operatorname{Sym}^* M \subset \mathbb{Z}[t_1, \ldots, t_n]$.

Example 1.4. For $n = 2$, $H_T^* \mathbb{P}^1 = \Lambda[\zeta]/(\zeta + t_1)(\zeta + t_2)$ has basis $\{1, x_1 = \zeta + t_1\}$. We see

$$\begin{aligned} x_1^2 &= (\zeta + t_1)(\zeta + t_1) \\ &= (\zeta + t_1)((\zeta + t_2) + (t_1 - t_2)) \\ &= (t_1 - t_2) x_1. \end{aligned}$$

More generally,

$$x_1 \cdot x_p = x_{p+1} + (t_1 - t_{p+1}) x_p$$

for $1 \leq p \leq n-2$,

$$x_2 \cdot x_p = x_{p+2} + (t_1 - t_{p+1} + t_2 - t_{p+2}) x_{p+1} + (t_1 - t_{p+1})(t_2 - t_{p+1}) x_p$$

for $2 \leq p \leq n-3$, and so on.

Exercise 1.5. Let $T_\infty = \bigcup_n (\mathbb{C}^*)^n$ act on $\mathbb{P}^\infty = \bigcup_n \mathbb{P}^{n-1}$. Show that $\Lambda_{T_\infty} = \mathbb{Z}[t_1, t_2, \ldots]$, and $H_{T_\infty}^* \mathbb{P}^\infty = \Lambda[\zeta]$ has Λ-bases $\{1, \zeta, \zeta^2, \ldots\}$ and $\{1, x_1, x_2, \ldots\}$.

In the setup of the previous exercise, the challenge is to find the coefficients in the expansion $x_i \cdot x_j = \sum c_{ij}^k x_k$; this determines the coefficients in any $H_T^* \mathbb{P}^{n-1}$.

Exercise 1.6. Let T act on $\mathbb{C}^n$ by distinct characters $\chi_1, \ldots, \chi_n$, inducing an action on $\mathbb{P}^{n-1}$, so $H_T^*\mathbb{P}^{n-1} = \Lambda[\zeta]/(\prod(\zeta + \chi_i))$. The fixed points are $p_i = [0 : \cdots : 0 : 1 : 0 : \cdots : 0]$ (1 in the ith position). The restriction map $H_T^*\mathbb{P}^{n-1} \to H_T^*(p_i)$ takes ζ to $-\chi_i$, and the Gysin map $H_T^*(p_i) \to H_T^*\mathbb{P}^{n-1}$ takes 1 to $\prod_{j\neq i}(\zeta + t_j)$. Note that the composition

$$\Lambda^{\oplus n} = H_T^*((\mathbb{P}^{n-1})^T) \to H_T^*\mathbb{P}^{n-1} \to H_T^*((\mathbb{P}^{n-1})^T) = \Lambda^{\oplus n}$$

is diagonal.

Exercise 1.7. In the setup of the previous exercise, compute the matrix of the restriction map $H_T^*\mathbb{P}^{n-1} \to H_T^*((\mathbb{P}^{n-1})^T)$, using the basis $1, \zeta, \ldots, \zeta^{n-1}$ for $H_T^*\mathbb{P}^{n-1}$. What is its determinant?

Suppose a torus T acts on $\mathbb{P}^1$ with fixed points 0 and ∞.

Exercise 1.8. The action on the open set $\mathbb{C}$ containing $0 = [0 : 1]$ is by a character χ, so $g \cdot z = \chi(g)z$ for $g \in T$, $z \in \mathbb{C}$. If $\chi = 0$, then T acts trivially; otherwise $c_1^T(T_0\mathbb{P}^1) = \chi$ and $c_1^T(T_\infty\mathbb{P}^1) = -\chi$. The composite map

$$\Lambda_0 \oplus \Lambda_\infty \to H_T^*\mathbb{P}^1 \to \Lambda_0 \oplus \Lambda_\infty$$

has matrix $\begin{pmatrix} \chi & 0 \\ 0 & -\chi \end{pmatrix}$, and $H_T^*\mathbb{P}^1 = \Lambda[\zeta]/(\zeta + \chi)\zeta$. (Let T act on $\mathbb{C}^2$ by $g \cdot (z_1, z_2) = (\chi(g)z_1, z_2)$.)

At 0, the restriction map takes ζ to $-\chi$; at ∞, ζ maps to 0. By the above exercise, the Gysin inclusion takes $(1, 0)$ to ζ and $(0, 1)$ to $\zeta + \chi$.

Exercise 1.9. The image of $H_T^*\mathbb{P}^1$ in $\Lambda_0 \oplus \Lambda_\infty$ consists of pairs (u_1, u_2) such that $u_2 - u_1$ is divisible by χ.

Remark 1.10. If T acts on a nonsingular curve C with exactly two fixed points, then C is isomorphic to $\mathbb{P}^1$ with the action described above, by an isomorphism unique up to interchanging 0 and ∞. Thus the character $\pm\chi$ is intrinsic up to sign.

Now assume R is a UFD, and let T act on $\mathbb{P}^{n-1}$ with weights $\chi_1, \ldots, \chi_n$. Assume these are distinct, and for each i, assume the $n - 1$ weights $\chi_j - \chi_i$ are relatively prime in Λ.

Claim . The image of $H_T^*\mathbb{P}^{n-1}$ in $H_T^*((\mathbb{P}^{n-1})^T) = \bigoplus \Lambda_{p_i}$ consists of all n-tuples $(u_1, \ldots, u_n) \in \Lambda^{\oplus n}$ such that for all $i \neq j$, $f_i - f_j$ is divisible by $\chi_i - \chi_j$.

To see the sufficiency of this divisibility condition, suppose $(u_1, \ldots, u_n)$ satisfies it. Certainly $(u_1, \ldots, u_1) = u_1 \cdots (1, \ldots, 1)$ is in the image, so we can assume $u_1 = 0$. Now $(u_1, \ldots, u_n) = (0, (\chi_1 - \chi_2)v_2, \ldots, v_n)$. Since $(\chi_1 + \zeta)v_2$ maps to $(0, (\chi_1 - \chi_2)v_2, \ldots)$, we can assume $v_2 = 0$, and we have $(0, 0, (\chi_1 - \chi_3)(\chi_2 - \chi_3)w_3, \ldots, w_n)$. Subtract the image of $(\chi_1 + \zeta)(\chi_2 + \zeta)w_3$ from this to get zeros in the first three coordinates; continuing in this way, we arrive at $(0, \ldots, 0, \prod_{i=1}^{n-1}(\chi_i - \chi_n)z_n)$, which is the image of $\prod_{i=1}^{n-1}(\chi_i + \zeta)z_n$.

Exercise 1.11. Show that the condition that the $\chi_j - \chi_i$ be relatively prime is necessary: If they are not, the image is strictly smaller than predicted by the claim.

2

We now discuss the role of fixed points in more general situations. Assume X^G is finite, and consider the composition

$$\bigoplus_{p \in X^G} \Lambda = H^*_G X^G \xrightarrow{\text{Gysin}} H^*_G X \xrightarrow{\text{restr.}} H^*_G X^G = \bigoplus_{p \in X^G} \Lambda.$$

The composite map is diagonal (by property (v) of Gysin maps), and it is given by multiplication by $c^G_{top}(T_p X)$ on the summand Λ corresponding to $p \in X^G$ (by property (iv)).

Proposition 2.1. *Let T be a torus acting on X. If S is a multiplicatively closed set in the center of Λ, containing $c^T_{top}(T_p X)$ for all $p \in X^G$, then*

$$S^{-1} H^*_T X \to S^{-1} H^*_T X^T$$

is surjective. Moreover, the cokernel of $H^*_T X \to H^*_T X^T$ is annihilated by $\prod_{p \in X^T} c^T_{top}(T_p X)$.

If X is equivariantly formal with respect to R, so $H^* X$ is free over R, and if $\operatorname{rk} H^* X \leq \# X^T$, then $\operatorname{rk} H^* X = \# X^T$, and

$$S^{-1} H^*_T X \to S^{-1} H^*_T X^T$$

is an isomorphism.

Example 2.2. For $T = (\mathbb{C}^*)^n$ acting on $X = \mathbb{P}^{n-1}$, we have $X^T = \{p_1, \ldots, p_n\}$ (with p_i having all but the ith coordinate zero, as before), so there are $n = \operatorname{rk} H^* X$ fixed points. One sees an isomorphism as in the proposition after localizing at the multiplicative set S generated by the elements $t_i - t_j$, for $i \neq j$.

More generally, if G is a semisimple (reductive) group acting on a nonsingular variety X of dimension n, a fixed point x is isolated if and only if the corresponding representation of G on $T_x X$ does not contain the trivial representation. When $G = T$ is a torus acting by characters $\chi_1, \ldots, \chi_n$, this means $\chi_i \neq 0$ for all i; therefore $c^T_n(T_x X) = \chi_1 \cdots \chi_n \neq 0$ (at least if $R = \mathbb{Z}$). If G is not a torus, however, one can still have $c^G_n(T_x X) = 0$.

Example 2.3 (J. de Jong)**.** Let $G = SL_n\mathbb{C}$ act on $X = G$ by conjugation. The fixed point set is then the center of G, which corresponds to the nth roots of unity, so it is finite (and in particular, isolated). The corresponding representation is the adjoint action on $\mathfrak{g} = T_e G$, which is irreducible. The diagonal torus acts trivially on its Lie algebra $\mathfrak{h} \subset \mathfrak{g}$, though, so e is not an isolated fixed point for T. Since $c^G_{top}(T_e G)$ maps to $c^T_{top}(T_e G) = 0$ under the inclusion $\Lambda_G \hookrightarrow \Lambda_T$, we see $c^G_{top}(T_e G) = 0$.

Example 2.4. For $G = B^+$ acting on $X = \mathbb{P}^{n-1}$, $X^G = \{p_1\}$ consists of only one point, so in this case there is no isomorphism after localizing.

Remark 2.5. When X is formal, so H_T^*X is free over Λ, it follows that $H_T^*X \hookrightarrow H_T^*X^T = \bigoplus_{p \in X^T} \Lambda$ is injective when X^T is finite. We will describe the image later.

3

If X^G is not finite, and Y is a component, then the composition

$$H_G^*Y \to H_G^*X \to H_G^*Y$$

is given by multiplication by $c_d^G(N)$, where $d = \operatorname{codim}(Y, X)$ and $N = N_{Y/X}$ is the normal bundle. (This makes sense, since Y is always smooth; see [Ive72] for more general conditions.) When $H^kBG = 0$ for k odd, we have $H_G^*Y = \Lambda \otimes H^*Y$, and $c_d^G(N) = \sum_{i=0}^d c_i \otimes y_i$, with $c_i \in \Lambda^{2i}$ and $y_i \in H^{2d-2i}Y$; in fact, $y_d = 1$. Restricting to $p \in Y$, then, $c_d^G(N)$ restricts to $c_d = c_d^G(N_p)$. In fact, this is independent of the choice of p, since N is locally trivial as a G-bundle.

When $G = T$ is a torus, we claim that if c_d is contained in a multiplicative set S in Λ, then $c_d^T(N)$ is invertible in $S^{-1}H_T^*Y$. Indeed, the elements y_i are nilpotent for $i < d$ (under mild hypotheses on Y guaranteeing $H^jY = 0$ for $j \gg 0$, e.g., Y is an algebraic variety). As a consequence, we have the following:

Proposition 3.1. *If H^*X^T is free, with $\operatorname{rk} H^*X^T \geq \operatorname{rk} H^*X$, then equality holds, and the maps*

$$S^{-1}H_T^*X^T \to S^{-1}H_T^*X \to S^{-1}H_T^*X^T$$

are isomorphisms, for any S containing $c_{top}^T(N_p)$ for all $p \in X^T$.

References

[Gor-Kot-Mac98] M. Goresky, R. Kottwitz, and R. MacPherson, "Equivariant cohomology, Koszul duality, and the localization theorem," *Invent. Math.* **131** (1998), no. 1, 25–83.

[Ive72] B. Iversen, "A fixed point formula for action of tori on algebraic varieties," *Invent. Math.* **16** (1972) 229–236.