EQUIVARIANT COHOMOLOGY IN ALGEBRAIC GEOMETRY
LECTURE FIVE: PROJECTIVE SPACE; LOCALIZATION II

WILLIAM FULTON
NOTES BY DAVE ANDERSON

We begin with an answer to the challenge posed in the last lecture. For a torus T acting on $\mathbb{P}^{n-1}$ via characters $\chi_1, \ldots, \chi_n$, we have seen that $H_T^*\mathbb{P}^{n-1} = \Lambda[\zeta]/(\prod(\zeta + \chi_i))$ has basis $1, x_1, \ldots, x_{n-1}$, where

$$\begin{aligned} x_k &= [\{[0 : \cdots : 0 : * \cdots : *]\}]^T \text{ (first } k \text{ coordinates are 0)} \\ &= \prod_{i=1}^{k}(\zeta + \chi_i). \end{aligned}$$

Claim (D. Anderson)**.** In this basis, multiplication is given by

$$x_i \cdot x_j = x_{i+j} + \sum_{j \le k \le i+j} c_{ij}^k x_k,$$

where, setting $r = i + j - k$,

$$c_{ij}^k = \sum_{1 \le p_1 < \cdots < p_r \le i} \prod_{s=1}^{r} (\chi_{p_s} - \chi_{p_s + j + 1 - s}).$$

This can be proved by induction on i, using

$$x_i = x_1 \cdot x_{i-1} - (\chi_1 - \chi_i) x_{i-1}$$

and the identity

$$c_{ij}^k = c_{i-1,j}^{k-1} + (\chi_i - \chi_{k+1}) c_{i-1,j}^k,$$

for $i \le j \le k \le i + j - 2$.

It is also a special case of a theorem for Grassmann varieties [Knu-Tao03], which we will see later. Even in the case of projective space, though, we do not know a formula for c_{ij}^k which makes the symmetry $c_{ij}^k = c_{ji}^k$ evident.

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Example 1.1. Let $T = \mathbb{C}^*$ act on $\mathbb{P}^2$ with characters $0, t, 2t$; that is, $g \in T$ acts by the matrix $\mathrm{diag}(1, g, g^2)$. The fixed points are $p_1 = [1 : 0 : 0]$, $p_2 =$

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$[0:1:0]$, and $p_3 = [0:0:1]$. Every closed T-invariant curve is isomorphic to $\mathbb{P}^1$, containing two of these fixed points. Specifically, they are the lines

$$\begin{aligned} X_1 &= 0, \text{ with character } \pm t; \\ X_2 &= 0, \text{ with character } \pm 2t; \\ X_3 &= 0, \text{ with character } \pm t; \end{aligned}$$

and, for all $\lambda \neq 0$, the conics

$$X_2^2 - \lambda X_1 X_3 = 0, \text{ with character } \pm t,$$

passing through p_1 and p_3.

We have $H_T^*\mathbb{P}^2 = \Lambda[\zeta]/\zeta(\zeta+t)(\zeta+2t) \hookrightarrow \Lambda \oplus \Lambda \oplus \Lambda$, by the map $\zeta \mapsto (0, -t, -2t)$. The image consists of triples (u_1, u_2, u_3) such that

(i) $u_2 - u_1$ is divisible by t and $u_3 - u_1$ is divisible by $2t$, so $u_3 - u_2$ is divisible by t, and
(ii) $u_1 - 2u_2 + u_3$ is divisible by $2t^2$.

This is true for any coefficient ring R in which 2 is not a zerodivisor. (When 2 is a zerodivisor, the restriction map is not injective.) To see (i) must hold, consider the compositions $H_T^*\mathbb{P}^2 \to H_T^*(\{X_i = 0\}) \to \Lambda^{\oplus 3}$. Condition (ii) is also part of a general story, as we will see below.

Remark 1.2. Singular curves can occur as T-invariant curves. Indeed, if $\mathbb{C}^*$ acts by characters $0, t, 3t$, one has orbit closures given by the cuspidal curves $X_2^3 - \lambda X_1^2 X_3$.

If χ is a non-trivial character of T, then the action of T on $\mathbb{P}^1$ via $\chi, -\chi$ induces an action on the nodal curve $\mathbb{P}^1/(0 \sim \infty)$. Can this occur as an invariant curve in a smooth space X? This would be useful to know: for example, if the characters $\chi, -\chi, \chi_3, \ldots, \chi_n$ occur at some fixed point, could the first two characters come from a nodal T-invariant curve?

Proposition 1.3. *Let X be a compact (proper) nonsingular algebraic variety which is equivariantly formal with respect to a T-action, and assume H_T^*X injects into $H_T^*X^T$. If $E \subset X$ is a T-invariant curve, not contained in X^T, then E cannot have a node.*

Proof. A note $p \in E$ would be a fixed point, and would be the only fixed point on E: resolving the singularity, we get $\tilde{E} \cong \mathbb{P}^1$, where we know the action by $\pm\chi$ has the two preimages of p for fixed points. Then $[E]^T \mapsto \chi - \chi = 0 \in H_T^*(p)$, and also $[E]^T \mapsto 0$ at the other components of X^T. It follows that $[E]^T = 0$ in H_T^*X, so $[E] = 0$ in H^*X.

For X proper and algebraic, this is impossible. In fact, there is always a hypersurface H meeting E properly, so the $H \cdot E = [H] \cdot [E]$ is nonzero, a contradiction. □

Does the proposition remain true if X is a compact complex manifold?

2. Localization (integration) formula

Proposition 2.1. *Let X be a nonsingular variety, equivariantly formal with respect to an action of a torus T, with finitely many fixed points. Let $\rho : X \to pt$ be the projection. Then for $u \in H_T^*X$, with images $u_p = i_p^*(u)$ in $H_T^*(p) = \Lambda$,*

$$\rho_*(u) = \sum_{p \in X^T} \frac{u_p}{c^T_{top}(T_pX)}.$$

Proof. Consider the diagram

$$\begin{tikzcd} H_T^*X^T \arrow[r, "i_*"] \arrow[d, equal] & H_T^*X \arrow[r, "i^*"] \arrow[d, "\rho_*"'] & H_T^*X^T = \bigoplus H_T^*(p) \\ \bigoplus H_T^*(p) \arrow[r] & H_T^*(pt), & \end{tikzcd}$$

where the bottom map is given by addition. The square commutes by functoriality of the Gysin map, for the composition $p \to X \to pt$.

It suffices to prove the proposition after inverting $\prod c^T_{top}(T_pX)$, when the top maps become isomorphisms. Thus we reduce to the case $u = (i_p)_*(v)$. In this case, $u_p = i_p^*(i_p)_*(v) = c^T_{top}(T_pX) \cdot v$, and $u_q = 0$ for $q \neq p$. Since $\rho_*(u) = \rho_*(i_p)_*(v) = v$, the proposition follows. □

Example 2.2. Let $X = \mathbb{P}^{n-1}$, with T acting by distinct characters $\chi_1, \dots, \chi_n$, so the fixed points are $p_1, \dots, p_n$, and $H_T^*X = \Lambda[\zeta]/(\prod(\zeta + \chi_i))$. Since $\rho_*(\zeta^k)$ is 0 if $k < n-1$ and 1 if $k = n-1$, we the following (nonobvious) algebraic identity:

$$\sum_{i=1}^{n} \frac{(-\chi_i)^k}{\prod_{j \neq i}(\chi_j - \chi_i)} = \begin{cases} 0 & \text{if } k < n-1; \\ 1 & \text{if } k = n-1. \end{cases}$$

For general $u \in H_T^*X$, this says

$$\rho_* u = \sum_{i=1}^{n} \frac{u_i}{\prod_{j \neq i}(\chi_j - \chi_i)}.$$

In the situation of Example 1.1, where $X = \mathbb{P}^2$ and the characters are 0, t, and $2t$, this is

$$\rho_*(u) = \frac{u_1}{2t^2} + \frac{u_2}{-t^2} + \frac{u_3}{2t^2} = \frac{u_1 - 2u_2 + u_3}{2t^2},$$

which gives another reason for the condition that $u_1 - 2u_2 + u_3$ be divisible by $2t^2$ (since $\rho_*(u) \in \Lambda$).

Say a nontrivial character χ of T has **coefficient** $c \in \mathbb{Z}_{>0}$ if, choosing an isomorphism $T \cong (\mathbb{C}^*)^m$ (so $M \cong \mathbb{Z}^m$), we have $\chi = \sum a_i t_i$, and $c = \gcd(a_1, \dots, a_m)$. Thus $\chi = c \cdot \eta$, where η is a character with $\ker \eta$ a subtorus, and $T \cong \ker \eta \times \mathbb{C}^*$.

Proposition 2.3. *Let T be a torus acting on $X = \mathbb{P}^{n-1}$ by characters $\chi_1, \dots, \chi_n$.*

(i) *X^T is finite if and only if $\chi_1, \dots, \chi_n$ are distinct, and in this case $X^T = \{p_1, \dots, p_n\}$.*

(ii) *Assuming X^T is finite, $H_T^* X \to H_T^* X^T$ is injective if and only if the coefficients of the characters $\chi_i - \chi_j$ are non-zerodivisors in R.*

(iii) *Assuming X^T is finite and $H_T^* X \to H_T^* X^T = \Lambda^{\oplus n}$ is injective, a tuple $(u_1, \dots, u_n)$ is in the image if and only if*

$$\sum_{i \notin \{p_1, \dots, p_r\}} \frac{u_i}{\prod_{j \notin \{i, p_1, \dots, p_r\}} (\chi_j - \chi_i)} \in \Lambda,$$

for all $0 \le r \le n-1$ and $1 \le p_1 < \cdots < p_r \le n$. In fact, it suffices that one such sum be in Λ for each $0 \le r \le n-1$.

Note that for $r = n-2$, with $\{p_1, \dots, p_r\} = \{1, \dots, n\} \smallsetminus \{k, \ell\}$, this says

$$\frac{u_k}{\chi_\ell - \chi_k} + \frac{u_\ell}{\chi_k - \chi_\ell} \in \Lambda;$$

that is, $u_k - u_\ell$ is divisible by $\chi_k - \chi_\ell$.

Proof. We have seen (i) (in ?), and the "if" direction of (ii) (in ?). For the converse in (ii), note that if $a(\chi_k - \chi_\ell) = 0$ for some nonzero $a \in R$, then for $u = a \cdot \prod_{i \neq \ell} (\zeta + \chi_i)$, $u_i = 0$ for $i \neq \ell$, and $u_\ell = a \cdot \prod_{i \neq \ell} (\chi_i - \chi_\ell) = 0$, but $u \neq 0$.

The "only if" part of (iii) simply says $\rho_*(u \cdot \prod_{i=1}^r (\zeta + \chi_{p_i}))$ is in Λ. (This is also the pushforward of the restriction of u to $Y = \{X_{p_1} = X_{p_2} = \cdots = X_{p_r} = 0\}$.)

For the converse, suppose $u \in S^{-1}\Lambda[\zeta]/\prod(\zeta + \chi_i)$, where S is the multiplicative set generated by $\chi_i - \chi_j$, and assume there exist monic polynomials $P_0, \dots, P_{n-1} \in \Lambda[\zeta]$ such that $\deg P_r = r$ and $\rho_*(u \cdot P_r) \in \Lambda$ for $0 \le r \le n-1$. Then $u \in \Lambda[\zeta]/\prod(\zeta + \chi_i)$. This is simple algebra: write $u = \sum_{k=0}^{n-1} a_k \zeta^k$ with $a_k \in S^{-1}\Lambda$, and use

$$\rho_*(\zeta^k) = \begin{cases} 1 & \text{for } k = n-1; \\ 0 & \text{for } k < n-1 \end{cases}$$

to see $a_{n-1-r} = \rho_*(u \cdot P_r) \in \Lambda$ for $0 \le r \le n-1$. (In fact, the case $r = n-1$ is redundant, since we assume all u_i are in Λ.) $\square$

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One can find many theorems in the literature describing the image of the inclusion $H_T^* X \hookrightarrow H_T^* X^T$ when X is formal and X^T is finite of cardinality at least (and therefore equal to) the rank of $H^* X$. Usually X is also assumed to be projective. For example, if $S \subset T$ is a subtorus (isomorphic to $(\mathbb{C}^*)^k$), so $Y = X^S$ is nonsingular (see [Ive72]) and T invariant; since

$$\begin{tikzcd} H_T^*X \arrow[rr] \arrow[dr] & & H_T^*X^T \arrow[dl] \\ & H_T^*Y & \end{tikzcd}$$

commutes, a class in $H_T^*X^T$ which comes from H_T^*X must lift to H_T^*Y for each such Y (see [Cha-Skj74]).

If Y is a disjoint union of $\mathbb{P}^1$'s and isolated fixed points, then $(u_p) \in H_T^*X^T$ lifts to H_T^*Y if and only if $u_p - u_q$ is divisible by χ whenever p and q are on the same $\mathbb{P}^1$, where the action is by χ. According to [Gor-Kot-Mac98], this condition is also sufficient for (u_p) be in the image of H_T^*X, if there are only a finite number of T-invariant curves. There are some recent generalizations of this result, allowing infinitely many invariant curves, and using $\mathbb{Z}$ coefficients [Bra-Chen-Sot07, Eva05].

Here we will give a version that is easy to prove and implies the result of [Gor-Kot-Mac98] in the case of finitely many invariant curves. Let R be any UFD. For a character χ with coefficient c, write $\chi = c \cdot \eta$, and say the **direction** of χ is $\pm\eta$. Two characters are **parallel** if they have the same direction. Note that two characters are relatively prime in Λ exactly when they are not parallel and their coefficients are relatively prime in R.

Assume T acts equivariantly formally on a smooth compact algebraic variety X, with a finite fixed point set X^T of cardinality equal to the rank of H^*X. For any two characters χ and χ' that occur in the tangent spaces at fixed points, assume that

(∗) if χ and χ' occur at the same fixed point, they are relatively prime in Λ.

It follows that for each χ occurring at any fixed point p, there is a unique T-invariant curve $E = E_{\chi,p}$ passing through p with tangent χ (see Lemma 3.2 below); such an E passes through one other fixed point q with tangent $-\chi$ (by Exercise (4.1.8)), so $E_{\chi,p} = E_{-\chi,q}$.

Theorem 3.1. *With these assumptions, an element $(u_p) \in H_T^*(X^T)$ is in the image of H_T^*X if and only if, for each T-invariant curve $E = E_{\chi,p} = E_{-\chi,q}$, $u_p - u_q$ is divisible by χ.*

Proof. The "only if" direction is clear. To see the condition is sufficient, consider an irreducible factor $f \in \Lambda$ of a character that occurs at a fixed point, so f is either a prime in R or a character with coefficient 1. Let $Y = Y_f$ be the union of all curves $E_{\chi,p}$ for all χ divisible by f and all p, together with the isolated fixed points where f does not divide the characters. Note that by the assumption (∗), Y_f is smooth: two curves in Y cannot pass through the same fixed point. We have maps

$$H_T^*X^T \to H_T^*Y_f \to H_T^*X \to H_T^*Y_f \to H_T^*X^T. \tag{1}$$

For any f, the determinant of the map $H_T^*Y_f \to H_T^*X \to H_T^*Y_f$ is multiplication by a product ψ of characters, all of which are relatively prime to f. Indeed, the characters occurring in the normal bundle to Y_f are the other characters at fixed points contained in curves E in Y_f, and all the characters at isolated points, not divisible by f. The hypothesis that $u_p - u_q$ be divisible by χ implies that (u_p) comes by restriction from $H_T^*Y_f$, and it follows that

(a) $\psi \cdot (u_p)$ comes from H_T^*X (in fact, from the Gysin image of $H_T^*Y_f$).

We also know that

(b) $\varphi \cdot (u_p)$ comes from H_T^*X (in fact, as a Gysin push-forward from $H_T^*X^T$), where φ is the product of all the characters at all the fixed points.

Now simple algebra implies that φ is a denominator for (u_p), but that no character χ occurring in φ is actually needed. Indeed, take a basis $\{e_\alpha\}$ of H_T^*X over Λ, and write $u = \sum r_\alpha e_\alpha$, with r_α in the quotient field of Λ. By (b), the denominator of r_α is a product of characters χ, but by (a), no χ can occur. □

Lemma 3.2. *Let a torus T act on a nonsingular algebraic variety, with finitely many fixed points. If p is an isolated fixed point (so the weights $\chi_1, \dots, \chi_n$ are nonzero), then there are finitely many T-invariant curves through p if and only if no two of the weights at p are parallel. In this case, there are n such curves, each nonsingular at p, and tangent to a corresponding χ_i.*

Proof. There is a neighborhood of p in X equivariantly isomorphic to a neighborhood of 0 in T_pX. (One way to see this is to find an analytic neighborhood by using a maximal compact torus $K \cong (S^1)^r \subset T$, choose a K-invariant Hermitian metric on X, and use the exponential mapping. Another is to find an étale neighborhood using the "étale slice theorem": see [GIT, p. 198].) Thus it suffices to prove the lemma for $X = \mathbb{C}^n$, with T acting by characters $\chi_1, \dots, \chi_n$. If, say, χ_1 and χ_2 are parallel, one can find infinitely many (possibly singular) T-invariant curves in the (X_1, X_2)-plane, as in Example 1.1. Conversely, if no two are parallel, any point $(X_1, \dots, X_n)$ with two or more nonzero coordinates has a T-orbit of dimension at least 2, so the T-invariant curves are just the n axes. □

Corollary 3.3 (GKM)**.** *Let T act equivariantly formally on a smooth compact algebraic variety X, with finite fixed point set of cardinality equal to the rank of H^*X, and a finite number of T-invariant curves. Let R be a field containing $\mathbb{Q}$. Then (u_p) is in the image of H_T^*X if and only if, for each T-invariant curve E passing through fixed points p and q with character $\pm\chi$, $u_p - u_q$ is divisible by χ.*

Proof. It suffices to verify condition $(*)$; this follows from Lemma 3.2. □

4. (Continuation, to be integrated into the above)

When a torus T acts on a nonsingular variety X, each connected component P of X^T is nonsingular. The composition

$$H_T^*P \to H_T^*X \to H_T^*P$$

is multiplication by the equivariant top Chern class $c_d^T(N_{P/X})$ of the normal bundle. This can be written as

$$c_d^T(N_{P/X}) = \chi_1 \cdot \dots \cdot \chi_d + \sum_{i=1}^{d} a_i c_i,$$

where $\chi_1, \dots, \chi_d$ are the (nonzero) characters of T on a fiber of $N_{P/X}$ at any point of P, $c_i \in H^{2i}(P)$, and $a_i \in \Lambda^i$. Since the c_i are nilpotent, it follows that $c_d^T(N_{P/X})$ is ivertible in $S^{-1}H_T^*P$, where S is the multiplicative set in Λ generated by $M \smallsetminus \{0\}$. Therefore

$$S^{-1}H_T^*X^T \to S^{-1}H_T^*X \to S^{-1}H_T^*X^T$$

is an isomorphism. In particular,

Lemma 4.1. *The following are equivalent:*

(i) $H_T^*X^T \to H_T^*X$ *becomes surjective after localizing at* S.
(ii) $H_T^*X^T \to H_T^*X$ *becomes an isomorphism after localizing at* S.
(iii) $H_T^*X \to H_T^*X^T$ *becomes injective after localizing at* S.
(iv) $H_T^*X \to H_T^*X^T$ *becomes an isomorphism after localizing at* S.

Let $f : X \to Y$ be a proper, T-equivariant morphism of nonsingular varieties. For each connected component P of X^T, $f(P)$ is contained in a unique component Q of Y^T; let $f_P : P \to Q$ be the induced map. For $x \in H_T^*X$, let $x|_P$ denote the restriction of x to H_T^*P, and similarly write $y|_Q$ for $y \in H_T^*Y$.

Proposition 4.2 (Localization formula). *Assume X satisfies the conditions of Lemma 4.1. Then for all x in H_T^*X, and all components Q of Y^T,*

$$f_*(x)|_Q = c_{top}^T(N_{Q/Y}) \sum_{f(P)\subset Q} (f_P)_* \left(\frac{x|_P}{c_{top}^T(N_{P/X})} \right)$$

in $S^{-1}H_T^*Q$.

Proof. It suffices to prove the formula for x of the form $(\iota_P)_*(z)$, for $z \in H_T^*P$, P a component of X^T. The LHS is $\iota_Q^*((\iota_Q)_*(f_P)_*(z))$, which is $c_{top}^T(N_{Q/Y}) \cdot (f_P)_*(z)$ when $f(P) \subset Q$, and is 0 otherwise. Since $x|_P = (\iota_P)^*(\iota_P)_*(z) = c_{top}^T(N_{P/X}) \cdot z$, and $x|_{P'} = 0$ for $P' \neq P$, the RHS is also equal to $c_{top}^T(N_{Q/Y}) \cdot (f_P)_*(z)$. $\square$

When H^*Q is torsion-free, $H_T^*Q \to S^{-1}H_T^*Q$ is an embedding, and the formula is an identity in $H_T^*Q = \Lambda \otimes_{\mathbb{Z}} H^*Q$.

When Y is a point, one obtains the "integration formula"

$$f_*(x) = \sum_P (f_P)_* \left(\frac{x|_P}{c^T_{top}(N_{P/X})} \right).$$

When P is a point, $c^T_{top}(N_{P/X})$ is the product of the (nonzero) characters of T on the tangent space to X at P.

Exercise 4.3. When f is a closed embedding, and $f(P) \subset Q$, $N_{P/X}$ is a subbundle of $N_{Q/Y}|_P$. If $E_{P/Q}$ denotes the quotient bundle, then

$$f_*(x)|_Q = \sum_{f(P)\subset Q} (f_P)_*(c^T_{top}(E_{P/Q}) \cdot x|_P).$$

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