EQUIVARIANT COHOMOLOGY IN ALGEBRAIC GEOMETRY
LECTURE SIX: GRASSMANNIANS

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1

As an example of the localization theorem of last lecture, consider $X = Gr(k, \mathbb{C}^n)$, with the standard action of $T = (\mathbb{C}^*)^n$. There is a fixed point $p_A = \mathrm{Span}\{e_{a_1}, \ldots, e_{a_k}\}$ for each subset $A = \{a_1 < \cdots < a_k\} \subset \{1, \ldots, n\}$, and so

$$\# X^T = \mathrm{rk}\, H^* X = \binom{n}{k},$$

which can also be thought of as the number of paths from the lower-left corner to the upper-right corner of a k by $n-k$ box, using unit steps up or right. `insert Young diagram.`

The tangent space at p_A is $\mathrm{Hom}(S, Q)$, where S is the vector space corresponding to p_A, and $Q = \mathbb{C}^n/S$. This has basis $\{e_a^* \otimes e_b \,|\, a \in A, b \notin A\}$ so the characters of the torus action are $t_a - t_b$, for $a \in A$ and $b \notin A$. Given $a \in A$ and $b \notin A$, set $\chi = t_a - t_b$, and let $A' = (A \smallsetminus \{a\}) \cup \{b\}$, so $-\chi = t_b - t_a$ is a character at $p_{A'}$. The corresponding curve E_{χ, p_A} consists of points of the form

$$\mathrm{Span}(\{e_q \,|\, q \in A \smallsetminus \{a\}\} \cup \{s e_a + t e_b \,|\, [s:t] \in \mathbb{P}^1\}).$$

Write $(u_A) = (u_{p_A}) \in H_T^* X^T$. Thus the condition for (u_A) to be in $H_T^* X$ is that $u_A - u_{A'}$ be divisible by $t_a - t_b$, for all A, A' related by an exchange as above.

Example 1.1. For $X = Gr(2,4)$, this can be encoded in the following graph: `insert moment graph`.

2

The equivariant cohomology of the Grassmannian generalizes the classical story, so we review the key facts about its ordinary cohomology first, referring to [Ful97] for proofs and details. Let V be an n-dimensional vector space, set $\ell = n - k$, and let $X = Gr(k, V)$. We also write $X =$

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$\mathbb{G}r(k-1, \mathbb{P}(V)) \cong \mathbb{G}r(k-1, \mathbb{P}^{n-1})$. On X, there is the tautological sequence

$$0 \to S \to V_X \to Q \to 0,$$

where S has rank k and Q has rank ℓ.

2.1. **Presentation.** The cohomology ring is

$$H^*X = \mathbb{Z}[c_1(Q), \ldots, c_\ell(Q)]/(\ell \text{ relations}),$$

where the relations are given by the vanishing of the p by p determinants

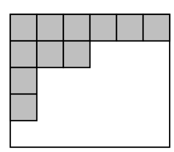
$$\begin{vmatrix} c_1 & c_2 & \cdots & \\ 1 & c_1 & \ddots & \\ 0 & \ddots & \ddots & c_2 \\ & 0 & 1 & c_1 \end{vmatrix},$$

for $k < p \le n$. (This says $c_p(S) = 0$ for $k < p \le n$.) Equivalently, we can write

$$H^*X = \mathbb{Z}[c_1(Q), \ldots, c_\ell(Q), c_1(S), \ldots, c_k(S)],$$

modulo the relations coming from $c(Q) \cdot c(S) = c(V) = 1$.

2.2. **Schubert basis.** There is a basis of **Schubert classes** σ_λ. These are indexed by **partitions** $\lambda = (\lambda_1 \ge \cdots \ge \lambda_k \ge 0)$, with $\lambda_1 \le \ell$; such a partition can be identified with its **Young diagram**, which is the collection of boxes inside a k by ℓ rectangle, with λ_i boxes in the ith row. For example, if $\lambda = (6, 3, 1, 1, 0)$, the corresponding diagram is shown below.



The **size** of a partition is $|\lambda| = \sum \lambda_i$, the number of boxes in its Young diagram. The **conjugate** partition is the partition λ' with λ'_j equal to the number of boxes in the jth column of λ. Final zeroes in a partition are often omitted.

Fix a complete flag $F_\bullet$ in V. The classes σ_λ are

$$\sigma_\lambda = [\Omega_\lambda(F_\bullet)] \in H^{2|\lambda|}X,$$

where $\Omega_\lambda(F_\bullet)$ is the **Schubert variety**, defined by

$$\Omega_\lambda(F_\bullet) = \{L \subset V \mid \dim(L \cap F_{\ell+i-\lambda_i}) \ge i \text{ for } 1 \le i \le k\}.$$

(One way to remember this definition is to note that $|\lambda|$ is the codimension of Ω_λ, so $\lambda = (0, 0, \ldots, 0)$ should correspond to trivial conditions on L.) In fact, only the "corners" of λ are needed, i.e., the conditions coming from those i such that $\lambda_i > \lambda_{i+1}$ are sufficient.

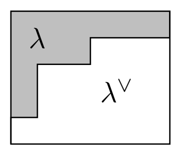
The Schubert variety $\Omega_\lambda(F_\bullet)$ is the closure of the set $\Omega^\circ_\lambda(F_\bullet)$ where the dimension of each intersection $L \cap F_j$ is as large as possible — so jumps in dimensions occur exactly at $F_{\ell+1-\lambda_1}$, $F_{\ell+2-\lambda_2}$, etc. By elementary linear algebra ("echelon form"), Ω_λ^o is isomorphic to affine space, so this is a cell.

2.3. **"Giambelli" formula.** A Schubert class can be expressed in terms of Chern classes as follows:

$$\sigma_\lambda = s_\lambda(c(Q)) := \det(c_{\lambda_i+j-i})_{1\le i,j,\le k}.$$

The determinant $s_\lambda(c(Q))$ occuring here is a *Schur determinant*. (This is also equal to $s_{\lambda'}(c(S^\vee))$, where $S^\vee$ is the dual of S.) Note that this determinant is unchanged by appending zeroes to the end of λ.

2.4. **Poincaré duality.** For a partition λ let $\lambda^\vee$ be its complement inside the k by ℓ rectangle, so $\lambda^\vee = (\ell-\lambda_k, \ell-\lambda_{k-1}, \dots, \ell-\lambda_1)$. Poincaré duality



on X then has the following form:

$$\langle \sigma_\lambda, \sigma_\mu \rangle = \int_X \sigma_\lambda \cdot \sigma_\mu = \begin{cases} 1 & \text{if } \mu = \lambda^\vee; \\ 0 & \text{otherwise.} \end{cases}$$

Thus the Schubert basis is self-dual.

Remark 2.1. Schubert was one of the first to emphasize the utility of self-dual bases in enumerative geometry. In particular, for $Y \subset X$ the expansion of the class $[Y] = \sum a_\lambda \sigma_\lambda$ is determined by

$$a_\lambda = [Y] \cdot \sigma_{\lambda^\vee} = \#(Y \cap \Omega_{\lambda^\vee}(F_\bullet)),$$

for a general flag $F_\bullet$. Schubert used some version of this (and even of the Künneth decomposition of the diagonal) for all the spaces he studied.

2.5. **Pieri formula.** For $\sigma_i = \sigma_{(i,0,\dots,0)} = c_i(Q)$,

$$\sigma_i \cdot \sigma_\lambda = \sum \sigma_\mu,$$

where the sum is over all μ obtained by adding i boxes to λ, with no two in the same column.

(Similarly, for $\sigma_{(1^i)} = \sigma_{(1,\dots,1,0,\dots,0)} = c_i(S^\vee)$,

$$\sigma_{(1^i)} \cdot \sigma_\lambda = \sum \sigma_\mu,$$

the sum over μ obtained by adding i boxes to λ with no two in a row.)

2.6. **Littlewood-Richardson rule.** The Pieri formula is a special case of a general rule for multiplying Schubert classes:

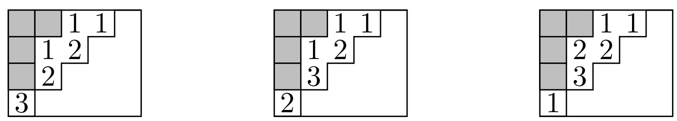
$$\sigma_\lambda \cdot \sigma_\mu = \sum c^\nu_{\lambda\mu} \sigma_\nu,$$

where $c^\nu_{\lambda\mu}$ is the number of ways to fill the boxes of $\nu \smallsetminus \lambda$ with μ_1 1's, μ_2 2's, etc., such that

(i) the filling is weakly increasing across rows;
(ii) the filling is strictly increasing down columns; and
(iii) when the numbers are read from right to left in rows, starting with the first row, the numbers read up to any point satisfy

$$\#1's \geq \#2's \geq \#3's \geq \cdots .$$

Example 2.2. For $\lambda = (2, 1, 1)$, $\mu = (3, 2, 1)$, and $\nu = (4, 3, 2, 1)$, one can check that there are three fillings satisfying the above conditions:



Thus $c^\nu_{\lambda\mu} = 3$. (Anders Buch's "Littlewood-Richardson calculator", available at `http://www.math.rutgers.edu/~asbuch/lrcalc/`, is very useful for such computations.)

Remark 2.3. The Littlewood-Richardson rule was originally formulated in the context of the representation theory of GL_k (and of the symmetric group). Here one has

$$V_\lambda \otimes V_\mu = \bigoplus V_\nu^{\oplus c^\nu_{\lambda\mu}},$$

where V_λ is the irreducible representation of GL_k with highest weight λ. (The original proofs of this were seriously flawed, though.)

It is somewhat mysterious that the same numbers should show up in geometry. One reason is the role of *Schur polynomials* in both contexts: the symmetric polynomial $s_\lambda(x_1, \ldots, x_k)$ is the character of V_λ, and these polynomials satisfy $s_\lambda \cdot s_\mu = \sum c^\nu_{\lambda\mu} s_\nu$. By the Pieri and Giambelli formulas, there is a homomorphism from the ring of symmetric polynomials (which has a basis of Schur polynomials) to $H^*(Gr(k, n))$ taking s_λ to σ_λ.

3

Now consider equivariant cohomology $H^*_G X$, for $G = GL(V)$ and $X = Gr(k, V)$. Recall that we have approximation spaces $E_m = \mathrm{Hom}^o(V, \mathbb{C}^m)$ and $B_m = E_m/G = Gr(n, \mathbb{C}^m)$, with

$$\begin{array}{ccc} E_m \times^G X = & \mathbf{Gr}(k, E) \\ \downarrow & \downarrow \\ B_m & = Gr(n, \mathbb{C}^m), \end{array}$$

where $E \subset \mathbb{C}^m_{B_m}$ is the tautological subbundle of rank n, and the identification in the top row is given by $(\Phi, L) \mapsto \Phi(L)$. Thus we are reduced to studying a Grassmann bundle $\mathbf{Gr}(k, E)$ over a base variety B, instead of just a Grassmann variety. Recall also that $\Lambda = \Lambda_G = \mathbb{Z}[c_1, \ldots, c_n]$.

On $\mathbf{Gr}(k, E)$, there is the tautological sequence

$$0 \to Sub^k \to E_{\mathbf{Gr}} \to Quot^{n-k} \to 0.$$

3.1. **Presentation.** We have

$$H^*\mathbf{Gr}(k, E) = \Lambda[c_1(Quot), \ldots, c_\ell(Quot)]/(\ell \text{ relations}),$$

where $\Lambda = H^*B$, and the relations say $c_i(Sub) = 0$ for $k < i \le n$, as before.

On X, the bundles in the tautological sequence

$$0 \to S^k \to V_X \to Q^{n-k} \to 0$$

are equivariant, and V becomes the universal subbundle on B_m. In particular, we have

$$H^*_G X = \Lambda[c_1^G(Q), \ldots, c_\ell^G(Q)]/(\ell \text{ relations}),$$

where $\Lambda = \mathbb{Z}[c_1, \ldots, c_n]$ with $c_i = c_i^G(V)$, and the relations come from $c^G(S) \cdot c^G(Q) = c^G(V) = 1 + c_1 + \cdots + c_n$.

3.2. **Basis.** Once again, there is an algebraic basis of the form $s_\lambda(c^G(Q))$, for $\ell \ge \lambda_1 \ge \cdots \ge \lambda_k \ge 0$. However, since $G = GL(V)$ acts transitively on X, there are no invariant subvarieties, and we cannot do much more with this group. If instead we consider the maximal torus T, or a Borel subgroup B, there is more equivariant geometry available; we will see this in the next lecture.

References

[Ful97] W. Fulton, *Young Tableaux*, Cambridge Univ. Press, 1997.