In this lecture, we will see equivariant versions of the properties of Grassmannians discussed in the last lecture.

We will use the following notation (from $K$-theory): If $A$ and $B$ are vector bundles, set

$$c(B - A) = c(B)/c(A) = \frac{1 + c_1(B) + c_2(B) + \cdots}{1 + c_1(A) + c_2(A) + \cdots} = 1 + (c_1(B) - c_1(A))$$

$$+ (c_2(B) - c_1(A)c_1(B) + c_1(A)^2 - c_2(A))$$

and let $c_p(B - A)$ be the term of degree $p$.

1.1. **Presentation.** For $E$ a vector bundle of rank $n$ on a base $B$, let $X = \text{Gr}(k, E) \to B$ be the Grassmann bundle. Let $\ell = n - k$. With $\Lambda = H^*B$ and tautological sequence $0 \to S \to E \to Q \to 0$, we have

$$H^*X = \Lambda[c_1(Q), \ldots, c_\ell(Q)]/(s_{(1^\ell)}(c(Q-E))), \ k < \ell \leq n).$$

Since $s_{(1^\ell)}(c(Q-E)) = (-1)^rc_r(E-Q)$, the relations are also generated by $c_r(E-Q)$ for $k < r \leq n$. (This says $c_r(S) = 0$ for $k < r \leq n$.)

For $B = BGL(V)$ (or approximations $B = B_m$), this gives $H^*_G\text{Gr}(k, V)$ for $G = GL(V)$. Note that $S$ and $Q$ come from the equivariant sub- and quotient bundles on $Gr(k, V)$ (so $c_i(Q) = e_i^G(Q)$).

1.2. **Schubert basis.** To get more information, we must restrict to a torus. Take $V = \mathbb{C}^n$, and let $T$ be the subgroup of diagonal matrices in $GL_n\mathbb{C}$. We have the same description of $H^*_T X$, where $X = Gr(k, n)$, but now $\Lambda = \Lambda_T = \mathbb{Z}[t_1, \ldots, t_n]$ and $c(E) = \prod_{i=1}^n(1 + t_i)$. Taking a $T$-invariant flag $F_\bullet$, we have $T$-invariant Schubert varieties $\Omega_\lambda(F_\bullet)$. (In this section, we always assume a partition $\lambda$ is contained in the $k$ by $\ell$ rectangle.) In fact, the $T$-invariant
flags are exactly \( F_\bullet(w) \), for \( w \in \Sigma_n \), where \( F_\bullet(w) = \text{Span}\{e_{w(1)}, \ldots, e_{w(i)}\} \).

Thus we have classes

\[
\sigma_\lambda(w) = [\Omega_\lambda(F_\bullet(w))]^* \in H_\ell^* T X.
\]

For any fixed \( w \), the \( \sigma_\lambda(w) \) form a basis for \( H_\ell^* T X \) over \( \Lambda \). The main cases will be \( w = id \) and \( w = w_0 \); write \( F_\bullet = F_\bullet(id) \), \( \widetilde{F}_\bullet = F_\bullet(w_0) \) (so \( \widetilde{F}_\bullet = \text{Span}\{e_n, e_{n-1}, \ldots, e_{n+1-i}\} \)), \( \sigma_\lambda = \sigma_\lambda(id) \), and \( \sigma_\lambda = \sigma_\lambda(w_0) \).

### 1.3. Kempf-Laksov formula.

Generally, if there is a filtration of vector bundles \( F_1 \subset \cdots \subset F_n = E \) on a base \( B \), then in \( \text{Gr}(k, E) \to B \) there are loci \( \Omega_\lambda(F_\bullet) \) of codimension \( |\lambda| \), which restrict to the usual Schubert varieties in each fiber. Equivalently, \( \Omega_\lambda(F_\bullet) \) is the locus where

\[
\text{rk}(F_{\ell+i-\lambda_i} \to Q) \leq \ell - \lambda_i \quad \text{for } 1 \leq i \leq k.
\]

(The kernel of the map is \( F_{\ell+i-\lambda_i} \cap S \), and this says it has dimension at least \( i \).) There is a general degeneracy locus formula for such loci, given by Kempf and Laksov (generalizing the Giambelli-Thom-Porteous formula) [Kem-Lak74]:

\[
[\Omega_\lambda(F_\bullet)] = \begin{vmatrix}
    c_{\lambda_1}(1) & c_{\lambda_{1}+1}(1) & \cdots \\
    c_{\lambda_2-1}(2) & c_{\lambda_{2}}(2) & \cdots \\
    \vdots & \ddots & \ddots \\
    c_{\lambda_k}(k)
\end{vmatrix},
\]

where \( c_p(i) = c_p(Q - F_{\ell+i-\lambda_i}) \). This is similar to a Schur polynomial – and equal to one if the \( F_j \)'s have trivial Chern classes – but the rows come from different bundles. These polynomials are often called **factorial Schur polynomials**.

In the equivariant case, for \( F_\bullet(w) \), we have \( c(F_r(w)) = \prod_{i=1}^r (1 + t_{w(i)}) \). Similarly, we have formulas for \( \sigma_\lambda(w) \), for any \( w \). In particular, \( \sigma_\lambda \) is obtained from \( \sigma_\lambda \) by interchanging \( t_i \) and \( t_{n+1-i} \).

### 1.4. Poincaré duality.

The Poincaré dual basis to \( \{\sigma_\lambda\} \) is \( \{\sigma_\lambda^\vee\} \). That is, for \( \rho : X \to pt \),

\[
(\sigma_\lambda, \sigma_\mu) = \rho_*(\sigma_\lambda \cdot \sigma_\mu) = \begin{cases} 
1 & \text{if } \mu = \lambda^\vee; \\
0 & \text{otherwise}.
\end{cases}
\]

**Proof.** First note that if \( |\lambda| + |\mu| < k \ell \), then \( (\sigma_\lambda, \sigma_\mu) = 0 \) by degree.

On the other hand, if \( \mu \neq \lambda^\vee \) and \( |\mu| + |\lambda| \geq k \ell \), then \( \Omega_\lambda(F_\bullet) \cap \Omega_\mu(\widetilde{F}_\bullet) = \emptyset \).

Indeed, if \( L \) is in both, then

\[
\text{dim}(L \cap F_{\ell+i-\lambda_i} \geq i \text{ and dim}(L \cap \widetilde{F}_{\ell+(k+1-i)-\mu_{k+1-i}}) \geq k+1-i),
\]

for \( 1 \leq i \leq k \). So the intersections \( L \cap F_{\ell+i-\lambda_i} \cap F_{\ell+(k+1-i)-\mu_{k+1-i}} \) are nonempty; in particular, \( F_{\ell+i-\lambda_i} \cap F_{\ell+(k+1-i)-\mu_{k+1-i}} \) is nonempty, so we must have

\[
(\ell + i - \lambda_i) + (\ell + k + 1 - i - \mu_{k+1-i}) \geq n + 1,
\]
i.e., $\lambda_i + \mu_{k+1-i} \leq \ell$ for $1 \leq i \leq k$. This says $\mu \subset \lambda^\vee$, and since $|\lambda| + |\mu| \geq k\ell$, it implies $\mu = \lambda^\vee$.

When $\mu = \lambda^\vee$, the intersection $\Omega_\lambda(F_{\bullet}) \cap \Omega_{\lambda^\vee}(\tilde F_{\bullet})$ consists of the single point $L = \text{Span}\{e_{i_1}, \ldots, e_{i_k}\}$, where $i_a = \ell + a - \lambda_a$. We will see below that this is transverse. □

Our next goal is to describe multiplication in $H^*_T X$. Since the classes $\sigma_\lambda$ form a basis, we have

$$\sigma_\lambda \cdot \sigma_\mu = \sum c^\nu_{\lambda\mu} \sigma_\nu.$$  

Here the coefficients $c^\nu_{\lambda\mu}$ are homogeneous polynomials in $t$, of degree $|\lambda| + |\mu| - |\nu|$. In particular, many more of these are nonzero than in the ordinary (non-equivariant) case.

We will see a special case of an “equivariant Pieri rule” below, as one of several key properties of the coefficients $c^\nu_{\lambda\mu}$. General equivariant Littlewood-Richardson rules (due to Molev-Sagan and Knutson-Tao) will be discussed in the next lecture; here we will describe a characterization of the $c^\nu_{\lambda\mu}$ given in [Knu-Tao03].

First we fix notation. Write $\Omega_\lambda = \Omega_\lambda(F_{\bullet})$. For a partition $\lambda$, let $I(\lambda) = \{\ell + 1 - \lambda_1, \ell + 2 - \lambda_2, \ldots, \ell + k - \lambda_k\}$. (This is the sequence of “jumping numbers” for $L \in \Omega_\lambda^0$; for $i \in I$, $\dim(L \cap F_i) = \dim(L \cap F_{i-1}) + 1$.) Let $J(\lambda)$ be the complement of $I(\lambda)$ in $\{1, \ldots, n\}$. One way to represent these sets is to consider identify $\lambda$ with a path from the NE corner to the SW corner of the $k$ by $\ell$ box; then $I(\lambda)$ (respectively, $J(\lambda)$) labels the vertical (resp., horizontal) steps in this path. An example is given below.

$$\lambda = (5, 3, 1, 1), \ k = 4, \ \ell = 5$$

$$I(\lambda) = \{1, 4, 7, 8\}$$

$$J(\lambda) = \{2, 3, 5, 6, 9\}$$

Set $\square = (1, 0, \ldots, 0)$ (so $\sigma_\square$ is the class of a divisor in $X$).

Let $p_\mu = p_{I(\mu)} = \text{Span}\{e_{\ell+1-\mu_1}, \ldots, e_{\ell+k-\mu_k}\}$. Observe that

$$p_\mu \in \Omega_\lambda \iff \Omega_\mu \subset \Omega_\lambda \iff \mu \supset \lambda,$$

i.e., $\mu_i \geq \lambda_i$ for all $i$. Let us see what can be proved from the basic facts.

Let $\sigma_\lambda|_{p_\mu}$ be the image of $\sigma_\lambda$ in $H^*_T(p_\mu) = \Lambda$. From the observation above, we have

$$(1) \quad \sigma_\lambda|_{p_\mu} = 0 \text{ unless } \lambda \subset \mu.$$

From the Giambelli formula, we have

$$(2) \quad \sigma_\square|_{p_\mu} = \sum_{j \in J(\mu)} t_j - \sum_{i=1}^\ell t_i.$$
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We will use this frequently. Note that the RHS is nonzero if \( \mu \neq \emptyset \).

In general, \( \sigma|_\mu = \det(c_{\lambda + j - i}(i)) \), where

\[
c(i) = \left( \prod_{j \in J(\mu)} (1 + t_j) \right) / \left( \prod_{a=1}^{\ell + a - \lambda_a} (1 + t_a) \right).
\]

In principle, then, we know all of these.

\[
(3) \qquad \sigma|_\lambda = \prod_{i \in I(\lambda), j < j \in J(\lambda)} (t_j - t_i).
\]

**Proof.** Let \( X^o = X \setminus \bigcup \Omega_\mu \), where the union is over all \( \mu \) properly containing \( \lambda \). The Schubert cell is \( \Omega^o_\lambda = \Omega_\lambda \cap X^o \), with inclusion \( \iota : \Omega^o_\lambda \to X^o \). Consider the diagram

\[
H^*_T(\Omega^o_\lambda) \xrightarrow{t^*} H^*_T X^o \xrightarrow{\iota^*} H^*_T(\Omega^o_\lambda) \xrightarrow{\iota^*} H^*_T X,
\]

where the first horizontal map is the Gysin pushforward, and the others are restrictions. The Gysin map takes 1 to the class \([\Omega_\lambda]^T\), which is the restriction of \([\Omega_\lambda]^T\). The composition \( t^* \iota^* \) is multiplication by the top equivariant Chern class of the normal bundle \( N = N_{\Omega^o_\lambda/X^o} \), so the restriction of \([\Omega_\lambda]^T\) to \( H^*_T(\Omega^o_\lambda) \) is \( c^T_{[\lambda]}(N) \).

In \( H^*_T(p_\lambda) \), this restricts to the product of the weights of \( T \) on the normal space to \( \Omega^o_\lambda \) in \( X^o \) at \( p_\lambda \). To see what this is, note that the tangent space to \( X^o \) at \( p_\lambda \) has weights \( t_j - t_i \) for \( i \in I(\lambda) \) and \( j \in J(\lambda) \); the tangent space to \( \Omega^o_\lambda \) at \( p_\lambda \) has weights \( t_j - t_i \) for those \( i \in I(\lambda) \), \( j \in J(\lambda) \) such that \( i > j \). The normal space therefore has the remaining weights, as claimed. \( \square \)

The claims about which weights appear are evident from an example.

**Example 2.1.** Let \( k = 4 \), \( \ell = 5 \), \( \lambda = (5, 3, 1, 1) \), so \( I(\lambda) = \{1, 4, 7, 8\} \) and \( J(\lambda) = \{2, 3, 5, 6, 9\} \). The Schubert cell \( \Omega^o_\lambda \) is identified with affine space as follows:

\[
\Omega^o_\lambda = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & * & * & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & * & * & 0 & * & * & 1 & 0 & 0 \\
0 & * & * & 0 & * & * & 0 & 1 & 0
\end{pmatrix}.
\]

An element \( g = (g_1, \ldots, g_n) \in T \) acts on the entry in row \( a \) and column \( b \) via multiplication by \( g_b / g_a \), so the corresponding weight is \( t_b - t_a \).
The classes $\sigma_\alpha$, for $\alpha \nsubseteq \lambda$ (so $\Omega_\alpha \not\subset \Omega_\lambda$), give a basis for $H_\ast^T(X \setminus \Omega_\lambda)$. Since $\sigma_\lambda$ restricts to 0 in $H_\ast^T(X \setminus \Omega_\lambda)$, $\sigma_\lambda \cdot \sigma_\mu \mapsto 0$ also. So only those $\sigma_\nu$ with $\nu \supset \lambda$ can appear. 

(4) $c_{\lambda\mu}^\nu = 0$ unless $\lambda \subset \nu$ and $\mu \subset \nu$.

**Proof.**

The classes $\sigma_\alpha$, for $\alpha \nsubseteq \lambda$ (so $\Omega_\alpha \not\subset \Omega_\lambda$), give a basis for $H_\ast^T(X \setminus \Omega_\lambda)$. Since $\sigma_\lambda$ restricts to 0 in $H_\ast^T(X \setminus \Omega_\lambda)$, $\sigma_\lambda \cdot \sigma_\mu \mapsto 0$ also. So only those $\sigma_\nu$ with $\nu \supset \lambda$ can appear. 

(5) $c_{\lambda\mu}^\nu = \sigma_\lambda|_\mu$.

**Proof.**

Restrict the equation $\sigma_\lambda \cdot \sigma_\mu = \sum c_{\lambda\mu}^\nu \sigma_\nu$ to $p_\mu$. By (1), $\sigma_\nu \mapsto 0$ unless $\nu \subset \mu$, and by (4), $c_{\lambda\mu}^\nu = 0$ unless $\lambda, \mu \subset \nu$. Thus the only term that appears is $\nu = \mu$, and

$$\sigma_\lambda|_\mu \cdot \sigma_\mu|_\mu = c_{\lambda\mu}^\mu \sigma_\mu|_\lambda.$$ 

Since $\sigma_\mu|_\mu \neq 0$ by (3), these factors cancel, and the claim follows. 

(6) $c_{\lambda\lambda}^\lambda = \sigma_\lambda|_\lambda = \prod_{i \in J(\lambda)} (t_j - t_i)$. 

This is immediate from (5) and (3).

The next property is a “Pieri-Monk” rule for multiplication by a divisor class:

(7) $\sigma_\square \cdot \sigma_\lambda = \sum_{\lambda^+} \sigma_{\lambda^+} + (\sigma_\square|_\lambda)\sigma_\lambda,$

the sum over partitions $\lambda^+$ obtained by adding one box to $\lambda$.

**Proof.**

We know that the only classes $\sigma_\nu$ which can occur on the RHS are those with $\nu \supset \lambda$ and $|\nu| \leq |\lambda| + 1$ thus $\nu = \lambda^+$ or $\nu = \lambda$. For $\nu = \lambda^+$, the classical formula applies. For $\nu = \lambda$, we know $c_{\lambda\lambda}^\lambda = \sigma_\square|_\lambda$ by (5).

(8) $(\sigma_\square|_\lambda - \sigma_\square|_\mu) c_{\lambda\mu}^\nu = \sum_{\lambda^+} c_{\lambda^+\mu}^\nu,$

the sum over $\mu^+$ obtained from $\mu$ by adding one box.

**Proof.**

Using (5) and commutativity ($c_{\lambda\mu}^\nu = c_{\mu\lambda}^\nu$), the LHS is $(\sigma_\square|_\lambda - \sigma_\square|_\mu)\sigma_\mu|_\lambda$, and the RHS is $\sum \sigma_\mu^+|_\lambda$. The equality follows from the restriction of Pieri-Monk (7) to $p_\lambda$. 

Finally, for any $\lambda, \mu, \nu$, we have

(9) $(\sigma_\square|_\nu - \sigma_\square|_\lambda) c_{\lambda\mu}^\nu = \sum_{\lambda^+} c_{\lambda^+\mu}^\nu - \sum_{\nu^+} c_{\lambda\mu}^{\nu^+},$
the sums over \( \lambda^+ \) obtained by adding one box to \( \lambda \), and \( \nu^- \) obtained by removing one box from \( \nu \).

**Proof.** By the Pieri-Monk rule, we have

\[
\sigma_{\lambda} \cdot (\sigma_{\lambda} \cdot \sigma_{\mu}) = \sum_{\nu} c^{\nu}_{\lambda \mu} \sigma_{\nu} \cdot \sigma_{\nu}
\]

\[
= \sum_{\nu^+} c^{\nu^+}_{\lambda \mu} \sigma_{\nu^+} + \sum_{\nu^-} c^{\nu^-}_{\lambda \mu} (\sigma_{\nu^+} | \nu) \sigma_{\nu},
\]

and

\[
(\sigma_{\lambda} \cdot \sigma_{\lambda}) \cdot \sigma_{\mu} = \sum_{\lambda^+} \sigma_{\lambda^+} \cdot \sigma_{\mu} + (\sigma_{\lambda} | \lambda) \sigma_{\lambda} \cdot \sigma_{\mu}
\]

\[
= \sum_{\lambda^+} c^{\nu^+}_{\lambda^+ \mu} \sigma_{\nu^+} + (\sigma_{\lambda} | \lambda) \sum_{\nu} c^{\nu}_{\lambda \mu} \sigma_{\nu}.
\]

Using associativity, these are equal. The claim follows by equating coefficients of \( \sigma_{\nu} \).

\[\square\]

**Proposition 2.2** ([Knu-Tao03]). The polynomials \( c^{\nu}_{\lambda \mu} \), homogeneous of degree \( |\lambda| + |\mu| - |\nu| \) in \( \Lambda \), satisfy and are uniquely determined by properties (6), (8), and (9); that is,

(i) \( c^{\lambda}_{\lambda \lambda} = \sigma_{\lambda} | \lambda = \prod_{i \in I(\lambda)} \left( t_j - t_i \right) \);

(ii) \( (\sigma_{\lambda} | \lambda - \sigma_{\lambda} | \mu)c^{\lambda}_{\lambda \mu} = \sum_{\lambda^+} c^{\lambda}_{\lambda^+ \mu} \); and

(iii) \( (\sigma_{\lambda} | \nu - \sigma_{\lambda} | \lambda)c^{\mu}_{\lambda \mu} = \sum_{\lambda^+} c^{\nu^+}_{\lambda^+ \mu} - \sum_{\nu^-} c^{\nu^-}_{\lambda \mu} \).

Note that each \( \sigma_{\lambda} | \lambda \) is a known linear polynomial, and

\[
\sigma_{\lambda} | \lambda - \sigma_{\lambda} | \mu = \sum_{j \in J(\lambda)} t_j - \sum_{j \in J(\mu)} t_j
\]

vanishes if and only if \( \lambda = \mu \). Note also that this characterization of the coefficients includes the classical Littlewood-Richardson coefficients, but all the equations reduce to \( 0 = 0 \) when the \( t_i \)'s are set to 0!

**Proof.** We have seen above that (i), (ii), and (iii) are satisfied. For uniqueness, we assume the polynomials \( c^{\nu}_{\lambda \mu} \) satisfy these properties, and proceed by induction.

**Step 1:** We claim \( c^{\lambda}_{\lambda \mu} = \sigma_{\mu} | \lambda \), which vanishes unless \( \mu \subset \lambda \). (By the Pieri-Monk formula, the polynomials \( \sigma_{\mu} | \lambda \) satisfy (i) and (ii).) To see this, use induction on \( |\lambda| - |\mu| \). The base case (\( \lambda = \mu \)) is true by property (i).
For $\lambda \neq \mu$, use (ii) and induction, noticing that all the terms on the RHS have $|\lambda| - |\mu^+| = |\lambda| - |\mu| - 1$.

**Step 2:** To determine $c^\nu_{\lambda\mu}$, use induction on $|\nu| - |\lambda|$. (We know $c^\nu_{\lambda\mu} = 0$ if $|\nu| - |\lambda| > |\mu|$.) The base case $\nu = \lambda$ is done by (ii) and Step 1. If $\nu \neq \lambda$, use (iii), noticing once again that the terms on the RHS have $|\nu| - |\lambda^+| = |\nu^-| - |\lambda| = |\nu| - |\lambda| - 1$. (We also see that $c^\nu_{\lambda\mu} = 0$ unless $\lambda$ and $\mu$ are contained in $\nu$.)

**Remark 2.3.** All of the above will hold for an arbitrary Grassmann bundle $Gr(k, E) \to B$, with a flag of bundles $F_1 \subset \cdots \subset F_n = E$ on $B$.

**Remark 2.4.** The fact that $\sigma|_{\lambda\mu} = 0$ unless $\lambda \subset \mu$ and the expression for $\sigma|_{\lambda\lambda}$ (properties (1) and (3)) can be found in [Mol-Sag99] and [Oko96]. A version of the Pieri-Monk formula (7) can be found in [Oko-Ols97]. The recursion in (9) (Property (iii) in Proposition 2.2) is due to Molev and Sagan [Mol-Sag99].

**References**


