EQUIVARIANT COHOMOLOGY IN ALGEBRAIC GEOMETRY
LECTURE SEVEN: EQUIVARIANT COHOMOLOGY OF GRASSMANNIANS

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1

In this lecture, we will see equivariant versions of the properties of Grassmannians discussed in the last lecture.

We will use the following notation (from K-theory): If A and B are vector bundles, set

$$\begin{aligned} c(B-A) = c(B)/c(A) &= \frac{1 + c_1(B) + c_2(B) + \cdots}{1 + c_1(A) + c_2(A) + \cdots} \\ &= 1 + (c_1(B) - c_1(A)) \\ &\quad + (c_2(B) - c_1(A)c_1(B) + c_1(A)^2 - c_2(A)) \\ &\quad + \cdots, \end{aligned}$$

and let $c_p(B-A)$ be the term of degree p.

1.1. **Presentation.** For E a vector bundle of rank n on a base B, let $X = \mathbf{Gr}(k, E) \to B$ be the Grassmann bundle. Let $\ell = n - k$. With $\Lambda = H^*B$ and tautological sequence $0 \to S \to E \to Q \to 0$, we have

$$H^*X = \Lambda[c_1(Q), \ldots, c_\ell(Q)]/(s_{(1^r)}(c(Q-E)),\ k < r \leq n).$$

Since $s_{(1^r)}(c(Q-E)) = (-1)^r c_r(E-Q)$, the relations are also generated by $c_r(E-Q)$ for $k < r \leq n$. (This says $c_r(S) = 0$ for $k < r \leq n$.)

For $B = BGL(V)$ (or approximations $B = B_m$), this gives $H^*_G Gr(k,V)$ for $G = GL(V)$. Note that S and Q come from the equivariant sub- and quotient bundles on $Gr(k,V)$ (so $c_i(Q) = c_i^G(Q)$).

1.2. **Schubert basis.** To get more information, we must restrict to a torus. Take $V = \mathbb{C}^n$, and let T be the subgroup of diagonal matrices in $GL_n\mathbb{C}$. We have the same description of $H^*_T X$, where $X = Gr(k,n)$, but now $\Lambda = \Lambda_T = \mathbb{Z}[t_1, \ldots, t_n]$ and $c(E) = \prod_{i=1}^n (1 + t_i)$. Taking a T-invariant flag $F_\bullet$, we have T-invariant Schubert varieties $\Omega_\lambda(F_\bullet)$. (In this section, we always assume a partition λ is contained in the k by ℓ rectangle.) In fact, the T-invariant

Date: March 18, 2007.

flags are exactly $F_\bullet(w)$, for $w \in \Sigma_n$, where $F_i(w) = \text{Span}\{e_{w(1)}, \dots, e_{w(i)}\}$. Thus we have classes

$$\sigma_\lambda(w) = [\Omega_\lambda(F_\bullet(w))]^T \in H_T^* X.$$

For any fixed w, the $\sigma_\lambda(w)$ form a basis for H_T^*X over Λ. The main cases will be $w = id$ and $w = w_0$; write $F_\bullet = F_\bullet(id)$, $\widetilde{F}_\bullet = F_\bullet(w_0)$ (so $\widetilde{F}_i = \text{Span}\{e_n, e_{n-1}, \dots, e_{n+1-i}\}$), $\sigma_\lambda = \sigma_\lambda(id)$, and $\widetilde{\sigma}_\lambda = \sigma_\lambda(w_0)$.

1.3. **Kempf-Laksov formula.** Generally, if there is a filtration of vector bundles $F_1 \subset \cdots \subset F_n = E$ on a base B, then in $\mathbf{Gr}(k, E) \to B$ there are loci $\boldsymbol{\Omega}_\lambda(F_\bullet)$ of codimension $|\lambda|$, which restrict to the usual Schubert varieties in each fiber. Equivalently, $\boldsymbol{\Omega}_\lambda(F_\bullet)$ is the locus where

$$\text{rk}(F_{\ell+i-\lambda_i} \to Q) \le \ell - \lambda_i \quad \text{for } 1 \le i \le k.$$

(The kernel of the map is $F_{\ell+i-\lambda_i} \cap S$, and this says it has dimension at least i.) There is a general degeneracy locus formula for such loci, given by Kempf and Laksov (generalizing the Giambelli-Thom-Porteous formula) [Kem-Lak74]:

$$[\boldsymbol{\Omega}_\lambda(F_\bullet)] = \begin{vmatrix} c_{\lambda_1}(1) & c_{\lambda_1+1}(1) & \cdots & \\ c_{\lambda_2-1}(2) & c_{\lambda_2}(2) & \ddots & \\ \vdots & \ddots & \ddots & \\ & & & c_{\lambda_k}(k) \end{vmatrix},$$

where $c_p(i) = c_p(Q - F_{\ell+i-\lambda_i})$. This is similar to a Schur polynomial – and equal to one if the F_j's have trivial Chern classes – but the rows come from different bundles. These polynomials are often called **factorial Schur polynomials**.

In the equivariant case, for $F_\bullet(w)$, we have $c(F_r(w)) = \prod_{i=1}^r (1 + t_{w(i)})$. Similarly, we have formulas for $\sigma_\lambda(w)$, for any w. In particular, $\widetilde{\sigma}_\lambda$ is obtained from σ_λ by interchanging t_i and t_{n+1-i}.

1.4. **Poincaré duality.** The Poincaré dual basis to $\{\sigma_\lambda\}$ is $\{\widetilde{\sigma}_{\lambda^\vee}\}$. That is, for $\rho : X \to pt$,

$$\langle \sigma_\lambda, \widetilde{\sigma}_\mu \rangle = \rho_*(\sigma_\lambda \cdot \widetilde{\sigma}_\mu) = \begin{cases} 1 & \text{if } \mu = \lambda^\vee; \\ 0 & \text{otherwise.} \end{cases}$$

Proof. First note that if $|\lambda| + |\mu| < k\ell$, then $\langle \sigma_\lambda, \widetilde{\sigma}_\mu \rangle = 0$ by degree.

On the other hand, if $\mu \ne \lambda^\vee$ and $|\mu| + |\lambda| \ge k\ell$, then $\Omega_\lambda(F_\bullet) \cap \Omega_\mu(\widetilde{F}_\bullet) = \emptyset$. Indeed, if L is in both, then

$$\dim(L \cap F_{\ell+i-\lambda_i} \ge i \text{ and } \dim(L \cap \widetilde{F}_{\ell+(k+1-i)-\mu_{k+1-i}}) \ge k+1-i,$$

for $1 \le i \le k$. So the intersections $L \cap F_{\ell+i-\lambda_i} \cap F_{\ell+(k+1-i)-\mu_{k+1-i}}$ are nonempty; in particular, $F_{\ell+i-\lambda_i} \cap F_{\ell+(k+1-i)-\mu_{k+1-i}}$ is nonempty, so we must have

$$(\ell + i - \lambda_i) + (\ell + k + 1 - i - \mu_{k+1-i}) \ge n + 1,$$

i.e., $\lambda_i + \mu_{k+1-i} \leq \ell$ for $1 \leq i \leq k$. This says $\mu \subset \lambda^\vee$, and since $|\lambda| + |\mu| \geq k\ell$, it implies $\mu = \lambda^\vee$.

When $\mu = \lambda^\vee$, the intersection $\Omega_\lambda(F_\bullet) \cap \Omega_{\lambda^\vee}(\widetilde{F}_\bullet)$ consists of the single point $L = \mathrm{Span}\{e_{i_1}, \ldots, e_{i_k}\}$, where $i_a = \ell + a - \lambda_a$. We will see below that this is transverse. □

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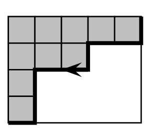
Our next goal is to describe multiplication in $H_T^* X$. Since the classes σ_λ form a basis, we have

$$\sigma_\lambda \cdot \sigma_\mu = \sum c_{\lambda\mu}^\nu \sigma_\nu.$$

Here the coefficients $c_{\lambda\mu}^\nu$ are homogeneous polynomials in t, of degree $|\lambda| + |\mu| - |\nu|$. In particular, many more of these are nonzero than in the ordinary (non-equivariant) case.

We will see a special case of an "equivariant Pieri rule" below, as one of several key properties of the coefficients $c_{\lambda\mu}^\nu$. General equivariant Littlewood-Richarson rules (due to Molev-Sagan and Knutson-Tao) will be discussed in the next lecture; here we will describe a characterization of the $c_{\lambda\mu}^\nu$ given in [Knu-Tao03].

First we fix notation. Write $\Omega_\lambda = \Omega_\lambda(F_\bullet)$. For a partition λ, let $I(\lambda) = \{\ell + 1 - \lambda_1, \ell + 2 - \lambda_2, \ldots, \ell + k - \lambda_k\}$. (This is the sequence of "jumping numbers" for $L \in \Omega_\lambda^o$: For $i \in I$, $\dim(L \cap F_i) = \dim(L \cap F_{i-1}) + 1$.) Let $J(\lambda)$ be the complement of $I(\lambda)$ in $\{1, \ldots, n\}$. One way to represent these sets is to consider identify λ with a path from the NE corner to the SW corner of the k by ℓ box; then $I(\lambda)$ (respectively, $J(\lambda)$) labels the vertical (resp., horizontal) steps in this path. An example is given below.



$\lambda = (5, 3, 1, 1)$, $k = 4$, $\ell = 5$
$I(\lambda) = \{1, 4, 7, 8\}$
$J(\lambda) = \{2, 3, 5, 6, 9\}$

Set $\square = (1, 0, \ldots, 0)$ (so $\sigma_\square$ is the class of a divisor in X).

Let $p_\mu = p_{I(\mu)} = \mathrm{Span}\{e_{\ell+1-\mu_1}, \ldots, e_{\ell+k-\mu_k}\}$. Observe that

$$p_\mu \in \Omega_\lambda \Leftrightarrow \Omega_\mu \subset \Omega_\lambda \Leftrightarrow \mu \supset \lambda,$$

i.e., $\mu_i \geq \lambda_i$ for all i. Let us see what can be proved from the basic facts.

Let $\sigma_\lambda|_\mu$ be the image of σ_λ in $H_T^*(p_\mu) = \Lambda$. From the observation above, we have

$$\sigma_\lambda|_\mu = 0 \text{ unless } \lambda \subset \mu. \tag{1}$$

From the Giambelli formula, we have

$$\sigma_\square|_\mu = \sum_{j \in J(\mu)} t_j - \sum_{i=1}^{\ell} t_i. \tag{2}$$

We will use this frequently. Note that the RHS is nonzero if $\mu \neq \emptyset$.

In general, $\sigma_\lambda|_\mu = \det(c_{\lambda_i+j-i}(i))$, where

$$c(i) = \left(\prod_{j \in J(\mu)} (1+t_j)\right) / \left(\prod_{a=1}^{\ell+a-\lambda_a} (1+t_a)\right).$$

In principle, then, we know all of these.

$$\sigma_\lambda|_\lambda = \prod_{\substack{i \in I(\lambda) \\ j \in J(\lambda) \\ i<j}} (t_j - t_i). \tag{3}$$

Proof. Let $X^o = X \smallsetminus \bigcup \Omega_\mu$, where the union is over all μ properly containing λ. The Schubert cell is $\Omega^o_\lambda = \Omega_\lambda \cap X^o$, with inclusion $\iota : \Omega^o_\lambda \to X^o$. Consider the diagram

$$\begin{array}{ccccc} H^*_T(\Omega^o_\lambda) & \xrightarrow{\iota_*} & H^*_T X^o & \xrightarrow{\iota^*} & H^*_T(\Omega^o_\lambda) \\ & & \uparrow & & \\ & & H^*_T X, & & \end{array}$$

where the first horizontal map is the Gysin pushforward, and the others are restrictions. The Gysin map takes 1 to the class $[\Omega^o_\lambda]^T$, which is the restriction of $[\Omega_\lambda]^T$. The composition $\iota^*\iota_*$ is multiplication by the top equivariant Chern class of the normal bundle $N = N_{\Omega^o/X^o}$, so the restriction of $[\Omega_\lambda]^T$ to $H^*_T(\Omega^o_\lambda)$ is $c^T_{|\lambda|}(N)$.

In $H^*_T(p_\lambda)$, this restricts to the product of the weights of T on the normal space to Ω^o_λ in X^o at p_λ. To see what this is, note that the tangent space to X^o at p_λ has weights $t_j - t_i$ for $i \in I(\lambda)$ and $j \in J(\lambda)$; the tangent space to Ω^o_λ at p_λ has weights $t_j - t_i$ for those $i \in I(\lambda)$, $j \in J(\lambda)$ such that $i > j$. The normal space therefore has the remaining weights, as claimed. □

The claims about which weights appear are evident from an example.

Example 2.1. Let $k = 4$, $\ell = 5$, $\lambda = (5,3,1,1)$, so $I(\lambda) = \{1,4,7,8\}$ and $J(\lambda) = \{2,3,5,6,9\}$. The Schubert cell Ω^o_λ is identified with affine space as follows:

$$\Omega^o_\lambda = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & * & * & 0 & * & * & 1 & 0 & 0 \\ 0 & * & * & 0 & * & * & 0 & 1 & 0 \end{pmatrix}.$$

An element $g = (g_1, \dots, g_n) \in T$ acts on the entry in row a and column b via multiplication by g_b/g_a, so the corresponding weight is $t_b - t_a$.

$$c_{\lambda\mu}^{\nu} = 0 \text{ unless } \lambda \subset \nu \text{ and } \mu \subset \nu. \tag{4}$$

Proof. The classes σ_α, for $\alpha \not\supset \lambda$ (so $\Omega_\alpha \not\subset \Omega_\lambda$), give a basis for $H_T^*(X \smallsetminus \Omega_\lambda)$. Since σ_λ restricts to 0 in $H_T^*(X \smallsetminus \Omega_\lambda)$, $\sigma_\lambda \cdot \sigma_\mu \mapsto 0$ also. So only those σ_ν with $\nu \supset \lambda$ can appear. □

$$c_{\lambda\mu}^{\mu} = \sigma_\lambda|_\mu. \tag{5}$$

Proof. Restrict the equation $\sigma_\lambda \cdot \sigma_\mu = \sum c_{\lambda\mu}^{\nu}\sigma_\nu$ to p_μ. By (1), $\sigma_\nu \mapsto 0$ unless $\nu \subset \mu$, and by (4), $c_{\lambda\mu}^{\nu} = 0$ unless $\lambda, \mu \subset \nu$. Thus the only term that appears is $\nu = \mu$, and

$$\sigma_\lambda|_\mu \cdot \sigma_\mu|_\mu = c_{\lambda\mu}^{\mu}\sigma_\mu|_\mu.$$

Since $\sigma_\mu|_\mu \neq 0$ by (3), these factors cancel, and the claim follows. □

$$c_{\lambda\lambda}^{\lambda} = \sigma_\lambda|_\lambda = \prod_{\substack{i\in I(\lambda)\\ j\in J(\lambda)\\ i<j}} (t_j - t_i). \tag{6}$$

This is immediate from (5) and (3).

The next property is a “Pieri-Monk” rule for multiplication by a divisor class:

$$\sigma_{\square} \cdot \sigma_\lambda = \sum_{\lambda^+} \sigma_{\lambda^+} + (\sigma_{\square}|_\lambda)\sigma_\lambda, \tag{7}$$

the sum over partitions λ^+ obtained by adding one box to λ.

Proof. We know that the only classes σ_ν which can occur on the RHS are those with $\nu \supset \lambda$ and $|\nu| \leq |\lambda| + 1$ thus $\nu = \lambda^+$ or $\nu = \lambda$. For $\nu = \lambda^+$, the classical formula applies. For $\nu = \lambda$, we know $c_{\square\lambda}^{\lambda} = \sigma_{\square}|_\lambda$ by (5). □

$$(\sigma_{\square}|_\lambda - \sigma_{\square}|_\mu)c_{\lambda\mu}^{\lambda} = \sum c_{\lambda\mu^+}^{\lambda}, \tag{8}$$

the sum over μ^+ obtained from μ by adding one box.

Proof. Using (5) and commutativity ($c_{\lambda\mu}^{\lambda} = c_{\mu\lambda}^{\lambda}$), the LHS is $(\sigma_{\square}|_\lambda - \sigma_{\square}|_\mu)\sigma_\mu|_\lambda$, and the RHS is $\sum \sigma_{\mu^+}|_\lambda$. The equality follows from the restriction of Pieri-Monk (7) to p_λ. □

Finally, for any λ, μ, ν, we have

$$(\sigma_{\square}|_\nu - \sigma_{\square}|_\lambda)c_{\lambda\mu}^{\nu} = \sum_{\lambda^+} c_{\lambda^+\mu}^{\nu} - \sum_{\nu^-} c_{\lambda\mu}^{\nu^-}, \tag{9}$$

the sums over λ^+ obtained by adding one box to λ, and ν^- obtained by removing one box from ν.

Proof. By the Pieri-Monk rule, we have

$$\begin{aligned}\sigma_{\square}\cdot(\sigma_\lambda\cdot\sigma_\mu) &= \sum_\nu c_{\lambda\mu}^\nu\,\sigma_{\square}\cdot\sigma_\nu\\ &= \sum_{\nu^+} c_{\lambda\mu}^\nu\,\sigma_{\nu^+} + \sum_\nu c_{\lambda\mu}^\nu\,(\sigma_{bx}|_\nu)\,\sigma_\nu,\end{aligned}$$

and

$$\begin{aligned}(\sigma_{\square}\cdot\sigma_\lambda)\cdot\sigma_\mu &= \sum_{\lambda^+}\sigma_{\lambda^+}\cdot\sigma_\mu + (\sigma_{\square}|_\lambda)\,\sigma_\lambda\cdot\sigma_\mu\\ &= \sum_{\lambda^+} c_{\lambda^+\mu}^\nu\,\sigma_\nu + (\sigma_{\square}|_\lambda)\sum_\nu c_{\lambda\mu}^\nu\,\sigma_\nu.\end{aligned}$$

Using associativity, these are equal. The claim follows by equating coefficients of σ_ν. □

Proposition 2.2 ([Knu-Tao03]). *The polynomials $c_{\lambda\mu}^\nu$, homogeneous of degree $|\lambda|+|\mu|-|\nu|$ in Λ, satisfy and are uniquely determined by properties (6), (8), and (9); that is,*

(i) $c_{\lambda\lambda}^\lambda = \sigma_\lambda|_\lambda = \prod_{\substack{i\in I(\lambda)\\ j\in J(\lambda)\\ i<j}} (t_j - t_i)$;

(ii) $(\sigma_{\square}|_\lambda - \sigma_{\square}|_\mu)c_{\lambda\mu}^\lambda = \sum c_{\lambda\mu^+}^\lambda$; *and*

(iii) $(\sigma_{\square}|_\nu - \sigma_{\square}|_\lambda)c_{\lambda\mu}^\nu = \sum_{\lambda^+} c_{\lambda^+\mu}^\nu - \sum_{\nu^-} c_{\lambda\mu}^{\nu^-}$.

Note that each $\sigma_{\square}|_\lambda$ is a known linear polynomial, and

$$\sigma_{\square}|_\lambda - \sigma_{\square}|_\mu = \sum_{j\in J(\lambda)} t_j - \sum_{j\in J(\mu)} t_j$$

vanishes if and only if $\lambda=\mu$. Note also that this characterization of the coefficients includes the classical Littlewood-Richardson coefficients, but all the equations reduce to $0=0$ when the t_i's are set to 0!

Proof. We have seen above that (i), (ii), and (iii) are satisfied. For uniqueness, we assume the polynomials $c_{\lambda\mu}^\nu$ satisfy these properties, and proceed by induction.

Step 1: We claim $c_{\lambda\mu}^\lambda = \sigma_\mu|_\lambda$, which vanishes unless $\mu\subset\lambda$. (By the Pieri-Monk formula, the polynomials $\sigma_\mu|_\lambda$ satisfy (i) and (ii).) To see this, use induction on $|\lambda|-|\mu|$. The base case ($\lambda=\mu$) is true by property (i).

For $\lambda \neq \mu$, use (ii) and induction, noticing that all the terms on the RHS have $|\lambda| - |\mu^+| = |\lambda| - |\mu| - 1$.

Step 2: To determine $c_{\lambda\mu}^{\nu}$, use induction on $|\nu| - |\lambda|$. (We know $c_{\lambda\mu}^{\nu} = 0$ if $|\nu| - |\lambda| > |\mu|$.) The base case $\nu = \lambda$ is done by (ii) and Step 1. If $\nu \neq \lambda$, use (iii), noticing once again that the terms on the RHS have $|\nu| - |\lambda^+| = |\nu^-| - |\lambda| = |\nu| - |\lambda| - 1$. (We also see that $c_{\lambda\mu}^{\nu} = 0$ unless λ and μ are contained in ν.) □

Remark 2.3. All of the above will hold for an arbitrary Grassmann bundle $\mathbf{Gr}(k, E) \to B$, with a flag of bundles $F_1 \subset \cdots \subset F_n = E$ on B.

Remark 2.4. The fact that $\sigma_\lambda|_\mu = 0$ unless $\lambda \subset \mu$ and the expression for $\sigma_\lambda|_\lambda$ (properties (1) and (3)) can be found in [Mol-Sag99] and [Oko96]. A version of the Pieri-Monk formula (7) can be found in [Oko-Ols97]. The recursion in (9) (Property (iii) in Proposition 2.2) is due to Molev and Sagan [Mol-Sag99].

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