EQUIVARIANT COHOMOLOGY IN ALGEBRAIC GEOMETRY
LECTURE EIGHT: EQUIVARIANT COHOMOLOGY OF GRASSMANNIANS II

WILLIAM FULTON
NOTES BY DAVE ANDERSON

1

As before, let $X = Gr(k,n)$, let $\ell = n-k$, and let $0 \to S \to \mathbb{C}^n_X \to Q \to 0$ be the tautological sequence on X. We saw that $H^*_T X$ has a basis of Schubert classes

$$\sigma_\lambda = [\Omega_\lambda(F_\bullet)]^T,$$

where $F_i = \mathrm{Span}\{e_1, \ldots, e_i\}$ and

$$\Omega_\lambda(F_\bullet) = \{L \mid \dim(L \cap F_{\ell+i-\lambda_i}) \geq i \text{ for } 1 \leq i \leq k\}.$$

We also saw how to express σ_λ in terms of (equivariant) Chern classes, using the Kempf-Laksov formula:

$$\sigma_\lambda = |c^T_{\lambda_i+j-i}(Q - F_{\ell+i-\lambda_i})|_{1\leq i,j\leq k}.$$

These determinants are variations of Schur polynomials, which we will call **double Schur polynomials**[1] and denote $s_\lambda(x|y)$, where the two sets of variables are $x = (x_1, \ldots, x_k)$ and $y = (y_1, \ldots, y_n)$. (Here $k \leq n$, and the length of λ is at most k.) Setting the y variables to 0, one recovers the ordinary Schur polynomials: $s_\lambda(x|0) = s_\lambda(x)$. In fact, $s_\lambda(x|y)$ is symmetric in the x variables.

Here we give three descriptions of these double Schur polynomials, generalizing those for ordinary Schur polynomials. Set $(x_i|y)^p = (x_i - y_1)(x_i - y_2)\cdots(x_i - y_p)$.

(i) Generalizing the "bialternant" definition of Schur polynomials, we have

$$s_\lambda(x|y) = \frac{\left|(x_i|y)^{\lambda_j+k-j}\right|}{\left|(x_i|y)^{k-j}\right|}.$$

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[1]The terms "multi-Schur" and "factorial Schur" are also found in the literature, as well as several other variations.

(ii) Generalizing the tableaux definition, we have

$$s_\lambda(x|y) = \sum_{T\in SSYT(\lambda)} \prod_{(i,j)\in\lambda} (x_{T(i,j)} - y_{T(i,j)+j-i}),$$

where the sum is over semistandard (column-strict) tableaux with entries in $\{1,\ldots,k\}$.

(iii) Generalizing the Jacobi-Trudi formula,

$$s_\lambda(x|y) = \left|h_{\lambda_i+j-i}(x|\tau^{1-j}y)\right|_{1\le i,j\le k},$$

where

$$h_p(x|y) = s_{(p)}(x|y) = \sum_{1\le i_1\le\cdots\le i_p\le k} (x_{i_1} - y_{i_1})\cdots(x_{i_p} - y_{i_p+p-1})$$

and $(\tau^p y)_i = y_{i+p}$. We also have

$$s_\lambda(x|y) = \left|e_{\lambda'_i+j-i}(x|\tau^{j-1}y)\right|_{1\le i,j\le \ell},$$

where λ' is the conjugate partition to λ, and

$$e_p(x|y) = s_{(1^p)}(x|y) = \sum_{1\le i_1<\cdots<i_p\le k} (x_{i_1} - y_{i_1})\cdots(x_{i_p} - y_{i_p+1-p}).$$

See [Mac92] for more. (The term "generalized factorial Schur function" comes from the fact that the specialization $y = (k-1, k-2, \ldots, 1, 0)$ was studied first.)

The following has been well-known for some time, but it is hard to cite an original and complete source. (Thanks to L. Mihalcea and A. Molev for providing simple proofs, which we sketch below.)

Proposition 1.1. (i) *Let $y_1,\ldots,y_k$ be Chern roots for $S^\vee$, i.e., $c^T(S) = \prod_{i=1}^k (1-y_i)$. Let $u_i = -t_{n+1-i}$. Then*

$$\sigma_\lambda = s_\lambda(y|u).$$

(ii) *Let $c^T(Q) = \prod_{i=1}^\ell (1+x_i)$. Then*

$$\sigma_\lambda = s_{\lambda'}(x|t).$$

To prove the first part, verify that

$$h_{r+j-i}(y|\tau^{1-j}u) = c^T_{r+j-i}(Q - F_{\ell+i-r}).$$

Indeed, we have

$$h_{r+j-i}(y|\tau^{1-j}u) = \sum_{a+b=r+j-i} h_a(y_1,\ldots,y_k)(-1)^a e_b(u_1,\ldots,u_{k+r-i})$$

and

$$c^T_{r+j-i}(Q - F_{\ell+i-r}) = c^T_{r+j-i}(\widetilde{F}_{k-i+r} - S).$$

(This comes from the exact sequences $0 \to F_p \to \mathbb{C}^n \to \widetilde{F}_{n-p} \to 0$ and $0 \to S \to \mathbb{C}^n_X \to Q \to 0$.) The RHS's are equal.

The second formula can be proved dually. It can also be deduced geometrically, using Grassmann duality. A point of $Gr(k,n)$ is $L = \mathrm{im}(A) = \ker(B)$ for an $n \times k$ matrix A and an $\ell \times n$ matrix B such that $B \cdot A = 0$. The isomorphism

$$\varphi : Gr(k,n) \to Gr(\ell,n)$$

takes L to $\varphi(L) = \ker({}^tA) = \mathrm{im}({}^tB)$. The group $G = GL_n(\mathbb{C})$ acts by $L \mapsto g \cdot L$, so $A \mapsto g \cdot A$ and $B \mapsto B \cdot g^{-1}$. Therefore φ is equivariant with respect to the homomorphism $G \to G$ given by $g \mapsto ({}^tg)^{-1}$. Restricting to the torus, the map $T \to T$ is $(g_1, \ldots, g_n) \mapsto (g_1^{-1}, \ldots, g_n^{-1})$, so the weights are mapped by $t_i \mapsto -t_i$.

Exercise 1.2. For $\lambda \subset (k^\ell)$, φ maps $\Omega_\lambda(F_\bullet)$ isomorphically onto $\Omega_{\lambda'}(\widetilde{F}_\bullet)$. (We will see a generalization of this later.) It follows that $\varphi^*\sigma_\lambda = \widetilde{\sigma}_{\lambda'}$.

Note that the duality map φ takes $0 \to S \to \mathbb{C}^n_{Gr(k,n)} \to Q \to 0$ to $0 \to Q^\vee \to \mathbb{C}^n_{Gr(\ell,n)} \to S^\vee \to 0$, so $\sigma_\lambda(x_1, \ldots, x_k, t_1, \ldots, t_n)$ maps to $\sigma_{\lambda'}(y_1, \ldots, y_\ell, -t_n, \ldots, -t_1)$. Also, passing from $F_\bullet$ to $\widetilde{F}_\bullet$ interchanges t_i and t_{n+1-i}. As a consequence, we have the following:

Corollary 1.3. *$c_{\lambda'\mu'}^{\nu'}$ is obtained from $c_{\lambda\mu}^{\nu}$ by interchanging t_i and $-t_{n+1-i}$ (as well as k and ℓ).*

Molev and Sagan [Mol-Sag99] give a nice combinatorial formula for multiplying double Schur polynomials: for

$$s_\lambda(y|u) \cdot s_\mu(y|u) = \sum c_{\lambda\mu}^{\nu} s_\nu(y|u),$$

they express the degree $(|\lambda|+|\mu|-|\nu|)$ polynomials $c_{\lambda\mu}^{\nu}$ as sums of products of factors $(t_i - t_j)$, indexed by tableau-like objects, in the spirit of the classical Littlewood-Richardson rule. Since these double Schur functions represent equivariant Schubert classes by Proposition 1.1, the Molev-Sagan rule gives a formula for multiplying Schubert classes in $H_T^*Gr(k,n)$.

The degree 0 terms in the Molev-Sagan rule are easily identified with Littlewood-Richardson coefficients. For higher-order terms, though, there is cancellation. We refer to [Mol-Sag99] for the formulation of the rule, but here is an example.

Example 1.4. For $k = 2$ and $\ell = 3$ (so $n = 5$), $\lambda = (2)$, and $\mu = \nu = (2,1)$, the Molev-Sagan formula gives

$$\begin{aligned} c_{\lambda\mu}^{\nu} &= (u_4 - u_1)(u_4 - u_2) + (u_4 - u_1)(u_2 - u_3) + (u_2 - u_2)(u_2 - u_3) \\ &= (u_4 - u_1)(u_4 - u_3) \\ &= (t_5 - t_2)(t_3 - t_2). \end{aligned}$$

The result is positive in $t_j - t_i$ (for $i < j$), but the original sum has terms which are positive, negative, and zero in these variables.

In this example, since $\mu = \nu$, we know $c^{\mu}_{\lambda\mu} = \sigma_\lambda|_\mu$, with $\sigma_\lambda = s_{(1,1)}(x|t)$, so we can check the above formula. The three tableaux on $(1,1)$ using entries in $\{1,2,3\}$ give

$$\sigma_\lambda = s_{(1,1)}(x|t) = (x_1 - t_1)(x_2 - t_1) + (x_1 - t_1)(x_3 - t_2) + (x_2 - t_2)(x_3 - t_2).$$

Since $J(\mu) = \{1,3,5\}$ the restriction to p_μ is given by $x_1 \mapsto t_1$, $x_2 \mapsto t_3$, $x_3 \mapsto t_5$. Thus

$$\begin{aligned} \sigma_\lambda|_\mu &= (t_1 - t_1)(t_3 - t_1) + (t_1 - t_1)(t_5 - t_2) + (t_3 - t_2)(t_5 - t_2) \\ &= (t_3 - t_2)(t_5 - t_2), \end{aligned}$$

as predicted.

2

We now describe the Knutson-Tao rule for equivariant Schubert calculus. First, recall the characterization from Lecture 7:

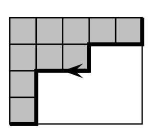
Proposition 2.1. *The polynomials $c^{\nu}_{\lambda\mu}$, homogeneous of degree $|\lambda|+|\mu|-|\nu|$ in Λ, satisfy and are uniquely determined by the following properties:*

(i) $c^{\lambda}_{\lambda\lambda} = \sigma_\lambda|_\lambda = \prod_{\substack{i\in I(\lambda)\\ j\in J(\lambda)\\ i<j}} (t_j - t_i)$;

(ii) $(\sigma_{\square}|_\lambda - \sigma_{\square}|_\mu) c^{\lambda}_{\lambda\mu} = \sum c^{\lambda}_{\lambda\mu^+}$; *and*

(iii) $(\sigma_{\square}|_\nu - \sigma_{\square}|_\lambda) c^{\nu}_{\lambda\mu} = \sum_{\lambda^+} c^{\nu}_{\lambda^+\mu} - \sum_{\nu^-} c^{\nu^-}_{\lambda\mu}$.

Our goal is to find a *positive* formula for $c^{\nu}_{\lambda\mu}$ in $\mathbb{Z}_{\geq 0}[t_2 - t_1, \ldots, t_n - t_{n-1}]$. First we introduce some notation. Partitions λ fitting inside the $r \times (n-r)$ box correspond bijectively to sequences of r 1's and $n-r$ 0's, as follows. Starting in the northeast corner of the box, trace the border of λ; record a 0 for each step left, and a 1 for each step down. For example, the partition $\lambda = (5,3,1,1)$ corresponds to the sequence 1 0 0 1 0 0 1 1 0:



Equivalently, the 1's in this sequence appear in positions $I(\lambda)$, and the 0's appear in positions $J(\lambda)$. (Note that this encoding depends not only on λ, but also on n and k.)

A *puzzle of type* $\Delta^{\nu}_{\lambda\mu}$ is described as follows. Write the sequences corresponding to λ, μ, and ν around the border of an equilateral triangle of side length n as indicated in Figure 1: λ is written along the northwest edge from SW to NE, μ is written along the northeast edge from NW to SE, and ν is written along the bottom edge from left to right. To complete the puz-

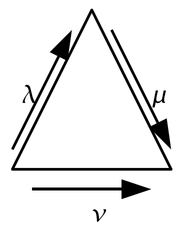


FIGURE 1.

zle, fill the triangle with the pieces shown in Figures 2 and 3, in a "jigsaw" fashion: shared edges must share the same label (0 or 1). The "classical pieces" of Figure 2 may be rotated; the "equivariant piece" of Figure 3 may only appear in the displayed orientation. An equivariant piece is said to

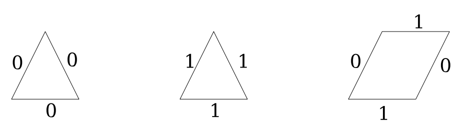


FIGURE 2. Classical puzzle pieces.

be in position (i, j) if a line drawn SW from the piece meets the bottom edge i units from the left, and a line drawn SE from the piece meets the bottom edge j units from the left, as in Figure 4. The *weight* of a puzzle is $\prod(t_j - t_i)$, the product being taken over all (i, j) with an equivariant piece in position (i, j).

The main theorem of [Knu-Tao03] is the following:

Theorem 2.2. *The polynomial $c_{\lambda\mu}^{\nu}$ is the sum of the weights of all puzzles of type $\Delta_{\lambda\mu}^{\nu}$:*

$$c_{\lambda\mu}^{\nu} = \sum_{\textit{puzzles}} \prod_{\substack{\textit{equivariant}\\ \textit{pieces}}} (t_j - t_i).$$

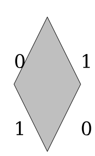


FIGURE 3. The equivariant puzzle piece.

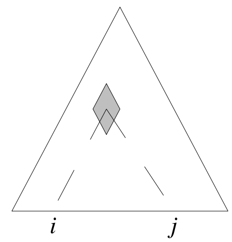


FIGURE 4. An equivariant piece in position (i, j).

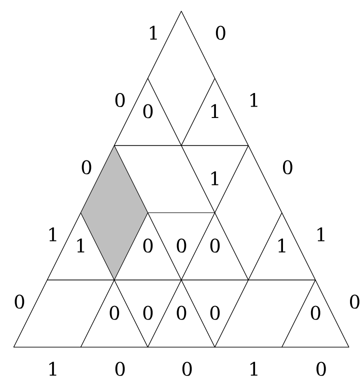


FIGURE 5. A puzzle of type $\Delta^{10010}_{01001,01010}$ and weight $t_3 - t_1$.

Example 2.3. The puzzle in Figure 5 contributes $t_3 - t_1$ to the coefficient of $\sigma_{(3,1)}$ in $\sigma_{(2)} \cdot \sigma_{(2,1)}$.

Exercise 2.4. Show that

$$\begin{aligned} \sigma_2 \cdot \sigma_{(2,1)} &= \sigma_{(3,2)} + (t_5 - t_4 + t_3 - t_2)\sigma_{(2,2)} \\ &\quad + (t_5 - t_4 + t_4 - t_3 + t_3 - t_1)\sigma_{(3,1)} \\ &\quad + (t_3 - t_2)(t_5 - t_4 + t_3 - t_2)\sigma_{(2,1)}. \end{aligned}$$

As indicated, there are eight puzzles computing this product, but there is cancellation in the result. (E.g., the coefficient of $\sigma_{(3,1)}$ reduces to $(t_5 - t_1)$.)

Exercise 2.5. Use the puzzle rule to deduce the formula given in Lecture 5 for the structure constants c_{ij}^k, in $H_T^*\mathbb{P}^{n-1}$.

To prove Theorem 2.2, one shows that the puzzle formula satisfies (i), (ii), and (iii) of Proposition 2.1. This involves some very pretty combinatorics. We refer to [Knu-Tao03] for the details, but we will sketch some of the ideas here.

The commutativity $c_{\lambda\mu}^\nu = c_{\mu\lambda}^\nu$ is far from obvious in the puzzle rule. On the other hand, another symmetry is clear: If one flips a puzzle left to right, and interchanges 0's and 1's, the result is again a puzzle. Note that swapping

0's and 1's corresponds to interchanging λ and λ', and flipping the puzzle left to right sends $t_j - t_i$ to $t_{n+i-i} - t_{n+1-j}$. This is just what we know about $c^{\nu}_{\lambda\mu}$ and $c^{\nu'}_{\mu'\lambda'}$ $(= c^{\nu'}_{\lambda'\mu'})$ from Grassmann duality. Knutson and Tao use this fact in lieu of commutativity.

To show (i) holds, use a "Green's theorem" argument. For a given puzzle, consider the (classical) rhombi which are oriented as . For each such rhombus, draw lines southeast from the south and southeast edges; if these lines meet the bottom of the puzzle in positions j and $j+1$, assign a "discrepancy" $t_{j+1} - t_j$ to this piece.

Claim . The sum of the discrepancies of all 's in a given puzzle is equal to $\sigma_{\square}|_{\nu} - \sigma_{\square}|_{\lambda}$.

Proof. To each edge labelled 1 of each piece in the puzzle, assign a "flux" as follows. Draw a line southeast from the edge, and suppose it meets the bottom of the puzzle in the jth position. If the edge is on the north or northwest side of the piece, its flux is t_j; if it is on the south or southeast side, its flux is $-t_j$; otherwise, the flux is 0. Let the total flux of a puzzle piece be the sum of the fluxes of its edges; one easily checks that the only piece with nonzero total flux is , which has flux equal to its discrepancy $t_{j+1} - t_j$. Note that the fluxes of internal edges in the puzzle cancel, so the summing the total flux of each piece gives the sum of the fluxes of the boundary edges, which one calculates to be $\sigma_{\square}|_{\nu} - \sigma_{\square}|_{\lambda}$, as desired. $\square$

The situation is similar for pieces oriented as . Several interesting facts follow immediately:

Corollary 2.6. *There must be the same number of* 1*'s on all three sides of the boundary.*

(Set $t_j = 1$.)

Corollary 2.7. *The number of pieces in a puzzle is* $|\nu| - |\lambda|$*, and the number of pieces is* $|\nu| - |\mu|$*.*

(Set $t_i = n + 1 - i$.)

Corollary 2.8. *There are no puzzles with boundary given by* λ*,* μ *and* ν *unless* $\lambda \subset \nu$ *and* $\mu \subset \nu$*.*

($\nu \supset \lambda$ if and only if $\sigma_{\square}|_{\nu} - \sigma_{\square}|_{\lambda}$ is positive.)

Corollary 2.9. *Property (i) holds for puzzles.*

Since $\lambda = \mu = \nu$, there are no or pieces. One then finds a unique puzzle with λ on each boundary side, and checks that its weight is as required.

The next step is to show that Property (ii) follows from Property (iii) and two facts which we have seen:

- $c^{\nu}_{\lambda\mu} = 0$ unless λ and μ are contained in ν;

- Grassmann duality, relating $c_{\lambda\mu}^{\nu}$ and $c_{\lambda'\mu'}^{\nu'}$.

It remains to prove Property (iii) for puzzles. Knutson and Tao do this by introducing puzzles with "gashes" – edges where the adjoining puzzle pieces have different labels – and seeing how to propagate and remove gashes.

Remark 2.10. In the nonequivariant case, Vakil constructed a sequence of degenerations, starting with "$\Omega_\lambda(F_\bullet) \cdot \Omega_\mu(\widetilde{F_\bullet})$" and ending with "$\Omega_\lambda(F_\bullet) \cdot \Omega_\mu(F_\bullet)$", from which one can see directly a contribution from each (nonequivariant) puzzle [Vak03]. Vakil and Coskun [Cos-Vak06] have a conjectured procedure to do the same in the equivariant setting. At their best, the combinatorics and geometry serve each other here.

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