Remark 0.1. We saw that the study of cohomology and equivariant cohomology of Grassmannians leads to interesting symmetric polynomials, namely, the Schur polynomials \( s_\lambda(x) \) and \( s_\lambda(x|t) \). These arise in contexts other than intersection theory and representation theory. For example, Griffiths asked which polynomials \( P \) in \( c_1(E), \ldots, c_n(E) \) are positive whenever \( E \) is an ample vector bundle on an \( n \)-dimensional variety. (That is, \( \int_X P(c_1(E), \ldots, c_n(E)) > 0; P \) should be homogeneous of degree \( n \).)

Bloch showed that the Chern classes \( c_1, \ldots, c_n \) are positive in this sense; Griffiths gave other examples, such as \( c_2^2 - c_2, \ldots \).

The complete answer was given by Fulton-Lazarsfeld [Ful-Laz83]. Write \( P = \sum a_\lambda s_\lambda \), where \( s_\lambda \) is the Schur polynomial \( \det(c_{\lambda_i+j-i}) \) and \( a_\lambda \in \mathbb{Z} \). Then \( P \) is positive if and only if \( a_\lambda \geq 0 \) for all \( \lambda \) (and at least one is \( > 0 \)). (So for example \( c_2^2 - 2c_2 = (c_2^2 - c_2) - c_2 \) is not!)

Let \( V \) be an \( n \)-dimensional vector space, let \( G = GL(V) \), and let \( X = Fl(V) \) be the variety of complete flags in \( V \) (so \( \dim X = \binom{n}{2} \)). For approximation spaces, take \( EG_m = \text{Hom}^0(V, \mathbb{C}^m) \) and \( BG_m = Gr(n, \mathbb{C}^m) \), with tautological bundle \( E \subset \mathbb{C}^m_{Gr} \). We have a diagram

\[
\begin{array}{ccc}
EG_m \times^G X & \sim & Fl(E) \\
\downarrow & & \downarrow \\
EG_m \times^G pt & \sim & Gr(n, \mathbb{C}^m),
\end{array}
\]

where the top map is given map \((\varphi, L_\bullet) \mapsto (\text{im}(\varphi), \varphi(L_\bullet))\). (So if \( L_\bullet \) is \( (L_1 \subset \cdots \subset L_n = V) \), then \( \varphi(L_\bullet) \) is \( (\varphi(L_1) \subset \cdots \subset \varphi(L_n) = \varphi(V)) \).) On \( Fl(E) \), there are tautological flags of subbundles \( S_1 \subset \cdots \subset S_n = E \) and quotient bundles \( E = Q_n \rightarrow \cdots \rightarrow Q_1 \), corresponding to the tautological bundles \( S_1 \subset \cdots \subset S_n = V_X = Q_n \rightarrow \cdots \rightarrow Q_1 \) on \( X \). (So \( Q_i = V/S_{n-i} \).)

Let \( x_i = c_1^G(\ker(Q_i \rightarrow Q_{i-1})) = c_1(\ker(Q_i \rightarrow Q_{i-1})) \).

\( Date: \ April 5, 2007. \)
Proposition 1.1. We have

$$H^*_G X = \Lambda_G[x_1, \ldots, x_n]/(e_i(x) - c_i)_{1 \leq i \leq n}.$$ 

More generally, for any flag bundle $\text{Fl}(E) \to B$, we have

$$H^*\text{Fl}(E) = H^* B[x_1, \ldots, x_n]/(e_i(x) - c_i(\text{E}))_{1 \leq i \leq n}.$$ 

Proof. To see this, realize $\text{Fl}(E)$ as a sequence of projective bundles over $B$: Start by forming $\mathbb{P}(E) \to B$, with universal subbundle $U_1 \subset E$; then form $\mathbb{P}(E/U_1) \to \mathbb{P}(E)$, with universal subbundle $U_2/U_1$; continue until reaching $\text{Fl}(E) = \mathbb{P}(E/U_{n-2}) \to \cdots \to B$. It is clear from this construction that the monomials $x_1^{i_1} \cdots x_n^{i_n}$ with $i_j \leq n - j$ form a basis for $H^*\text{Fl}(E)$ over $H^*B$, and that the relations $e_i(x) = c_i(\text{E})$ hold. Thus $H^*\text{Fl}(E)$ has rank $n$ over $H^*B$, and the proposition will follow if the ring on the RHS has the same rank.

Exercise 1.2. Show that the ring $H^* B[x_1, \ldots, x_n]/(e_i(x) - c_i(\text{E}))$ also has a basis of monomials $x_1^{i_1} \cdots x_n^{i_n}$ with $i_j \leq n - j$. (Hint: use the relation $\sum(-1)^i e_i : h_{n-i} = 0$.)

Now assume $V = \mathbb{C}^n$, so we have $F_1 \subset \cdots \subset F_n = V$, with $F_i = \text{Span}\{e_1, \ldots, e_i\}$. (Also write $\overline{F}_i = \text{Span}\{e_n, \ldots, e_{n+1-i}\}$ for the opposite flag.) Note that the subgroup $B$ of upper-triangular matrices preserves $F_\bullet$, and $B^{\text{opp}}$ (lower-triangular matrices) preserves $\overline{F}_\bullet$. For $w \in S_n$, we have $B$-invariant Schubert varieties in $X$, defined by

$$\Omega_w(F_\bullet) = \{x \mid \text{rk}(F_p(x) \to Q_q(x)) \leq r_w(q,p) \text{ for all } 1 \leq q, p \leq n\},$$

where $r_w(q,p) = \#\{i \leq q \mid w(i) \leq p\}$.

As in the Grassmannian case, a subset of these rank conditions suffices to define $\Omega_w$. To see which, form the diagram of $w$, the collection of boxes in the $n \times n$ defined as follows. Place a dot (or a 1) in row $i$ and column $w(i)$, and cross out all boxes which are to the right or below a dot, including the boxes containing dots. The diagram $D(w)$ is the collection of boxes which remain. Here is an example, for $w = 4163275$. (We use “one-line” notation and write $w = w(1) w(2) \cdots w(n)$.)

Note that $r_w(q,p)$ is the number of dots in the upper-left $q \times p$ rectangle. (Some of these have been labelled in the diagram.) The conditions needed to define $\Omega_w(F_\bullet)$ are those coming from the boxes in the southwest corners of $D(w)$ — so in the above example, only five conditions are needed. In fact,
EQUIVARIANT COHOMOLOGY IN ALGEBRAIC GEOMETRY

this defines $\Omega_w(F_\bullet)$ as a (reduced, irreducible, normal, Cohen-Macaulay) subscheme. The codimension is

$$\ell(w) = \# \{ j < i \mid w(j) > w(i) \} = \# \text{(boxes in } D(w)).$$

For details, see [Ful92].

**Example 1.3.** For a partition $\lambda$ in the $k \times \ell$ rectangle, we have $I(\lambda) = \{ i_1 < \cdots < i_k \}$ and $J(\lambda) = \{ j_1 < \cdots < j_\ell \}$, with $I(\lambda) \cup J(\lambda) = \{ 1, \ldots, n \}$. Define $w(\lambda) \in S_n$ by

$$w(\lambda) = j_1 j_2 \cdots j_\ell i_1 \cdots i_k.$$ 

Note that $\ell(w(\lambda)) = |\lambda| = \# \{ i \in I(\lambda), j \in J(\lambda) \mid j > i \}$. For example, with $k = 4$, $\ell = 5$, and $\lambda = (5, 3, 1, 1)$, we have $w(\lambda) = 235691478$. The diagram $D(w(\lambda))$ is easy to describe in terms of $\lambda$; the pattern is suggested by this example.

![Diagram of $D(w(\lambda))$]

The Schubert variety $\Omega_{w(\lambda)}(F_\bullet)$ comes from a Schubert variety on a Grassmanian: there is a projection $f : Fl(V) \to Gr(k, V)$, with $f^{-1} \Omega_\lambda(F_\bullet) = \Omega_{w(\lambda)}(F_\bullet)$.

All of this works for flag bundles, without change. On $X = Fl(E) \to B$, we have a flag $F_1 \subset \cdots \subset F_n = E_X$, and Schubert loci $\Omega_w(F_\bullet) \subset X$. When $B = BT$, we have $[\Omega_w(F_\bullet)] = [\Omega_w(F_\bullet)]^T \in H^*_T X$.

Write $\sigma_w = [\Omega_w(F_\bullet)]^T$ and $\tilde{\sigma_w} = [\Omega_w(F_\bullet)]^T$. For $w \in S_n$, these classes give two bases for $H^*_T X$ over $\Lambda_T$.

**Proposition 1.4.** The bases $\{ \sigma_w \}$ and $\{ \tau_w = \tilde{\sigma_{w_0 w}} \}$ are Poincaré dual; that is,

$$\rho_*(\sigma_w \cdot \tau_v) = \delta_{wv} \in \Lambda,$$

where $\rho$ is the map $X \to \text{pt}$.

**Proof.** We show that if $\ell(w) + \ell(v) \geq \binom{n}{2} = \dim X$, then $\Omega_w(F_\bullet)$ meets $\Omega_v(F_\bullet)$ only if $v = w_0 w$, and in this case they meet transversally at the point

$$p_w = \langle e_{w(n)} \rangle \subset \langle e_{w(n)}, e_{w(n-1)} \rangle \subset \cdots \subset \langle e_{w(n)}, \ldots, e_{w(1)} \rangle.$$
(Note that at \( p_w \), the map \( F_p(p_w) = \langle e_1, \ldots, e_p \rangle \to Q_q(p_w) = \mathbb{C}^n / \langle e_w(n), \ldots, e_w(n+1-q) \rangle = \langle e_w(1), \ldots, e_w(q) \rangle \) has rank \( r_w(q,p) \).

A neighborhood \( U_w \cong A^{\binom{n}{2}} \) of \( p_w \) in \( X \) is given by the set of matrices with 1’s in positions \((i, w(i))\) and arbitrary entries below. The flag associated to such a matrix has parts spanned by the rows, reading from the bottom up. For example, if \( w = 4163275 \), then

\[
U_w = \begin{pmatrix}
1 \\
1 & * \\
* & * & 1 \\
* & 1 & * & * \\
* & 1 & * & * & * \\
* & * & * & * & * & 1 \\
* & * & * & 1 & * & *
\end{pmatrix}.
\]

The intersection of \( U_w \) with the Schubert variety

\[
\Omega_w(F_\bullet) = \{ L_\bullet \mid \dim(F_p \cap L_q) \geq p - \# \{ i \leq q \mid w(i) \leq p \} \}
\]

is given by setting entries to the right of 1’s to 0:

\[
\Omega^o_w(F_\bullet) = \begin{pmatrix}
1 \\
1 & 0 \\
* & * & 1 \\
* & 1 & 0 & 0 \\
* & 1 & 0 & 0 & 0 \\
* & * & * & * & * & 1 \\
* & * & * & 1 & 0 & 0
\end{pmatrix}.
\]

(Note that there are \( \ell(w) \) such entries.)

The situation for \( \Omega_w(\bar{F}_\bullet) \) is the same, but with the matrices reflected from left to right. Thus \( p_{w_0 v} \in \Omega_w(\bar{F}_\bullet) \), and the intersection of \( \Omega_w(\bar{F}_\bullet) \) with \( U_{w_0 v} \) is given by setting entries to the left of 1’s equal to 0. For \( v = w_0 w \), we see that \( \Omega_w(F_\bullet) \) and \( \Omega_{w_0 w}(\bar{F}_\bullet) \) intersect transversally at the origin (as coordinate planes in \( U_w \)). Also, we see that if \( \Omega_w(F_\bullet) \cap \Omega_v(\bar{F}_\bullet) \) is nonempty, then \( w(i) \leq w_0 v(i) \) for all \( i \), so \( \ell(w) \leq \ell(w_0 v) = \binom{n}{2} - \ell(v) \), i.e., \( \ell(w) + \ell(v) \leq \binom{n}{2} \).

\( \square \)

Note that \( p_w \) is a smooth point of \( \Omega_w(F_\bullet) \), and since \( \Omega^o_w \) is an affine space, it is easy to compute \( T_{p_w} \Omega_w \). The torus \( T = (\mathbb{C}^*)^n \) acts with weight \( t_{w(i)} - t_{w(j)} \) on an entry in position \((j, w(i))\) with \( i < j \) and \( w(i) < w(j) \).

We now consider the equivariant Giambelli formula for Schubert varieties in \( X \). This is given by the double Schubert polynomials \( \mathcal{G}_w(x|y) \) of Lascoux
and Schützenberger \cite{Las-Sch82}. Here \( w \in S_n \) is a permutation, and \( x = (x_1, \ldots, x_n) \) and \( y = (y_1, \ldots, y_n) \) are two sets of variables.

These are defined as follows. For \( F \in \mathbb{Z}[x, y] \) and \( 1 \leq i \leq n - 1 \), define the \textbf{divided difference operator} \( \partial_i = \partial_i^F \) by
\[
\partial_i F = \frac{F(x; y) - F(\ldots, x_{i+1}, x_i, \ldots; y)}{x_i - x_{i+1}} = \frac{F - s_i(F)}{x_i - x_{i+1}}.
\]
(Here \( s_i = (i, i+1) \) is the simple transposition exchanging \( i \) and \( i+1 \).) Note that \( \partial_i \) ignores the \( y \) variables, and \( \partial_i F = 0 \) iff \( F \) is symmetric in \( x_i \) and \( x_{i+1} \).

For \( w \in S_n \), write \( w = w_0 s_1 \cdots s_\ell \) with \( \ell \) minimal, so \( \ell = \left( \frac{n}{2} \right) - \ell(w) \). (To do this, successively swap adjacent entries of \( w \) to reach \( w_0 \). For example,
\[
w = 3 \ 1 \ 5 \ 2 \ 4 \xrightarrow{s_2} 3 \ 5 \ 1 \ 2 \ 4 \xrightarrow{s_3} 3 \ 5 \ 2 \ 1 \ 4 \xrightarrow{s_4} 3 \ 5 \ 4 \ 2 \ 1 \xrightarrow{s_3} 5 \ 3 \ 4 \ 2 \ 1 \xrightarrow{s_2} 5 \ 4 \ 3 \ 2 \ 1 = w_0
\]
shows \( w = w_0 s_2 s_1 s_4 s_3 s_2 \).)

\textbf{Definition 2.1.} With notation as above, the \textbf{double Schubert polynomial} is defined by
\[
S_w(x|y) = \partial_{\ell} \circ \cdots \circ \partial_1 \left( \prod_{i+j \leq n} (x_i - y_j) \right).
\]

This is independent of the choice of the expression for \( w \); as for many such assertions, there are algebraic proofs (see \cite{Mac91}) and geometric proofs. Note that the \( y \) variables act as “scalars” here. In fact, there is a “Leibniz rule”
\[
\partial_i(F \cdot G) = (\partial_i F) G + (s_i F)(\partial_i G),
\]
so any function \( F \) which is symmetric in the \( x \) variables is a scalar for the action of the divided difference operators.

\textbf{Example 2.2.} The Schubert polynomials for \( n = 3 \) are as follows:
\[
\begin{align*}
S_{321} &= (x_1 - y_1)(x_1 - y_2)(x_2 - y_1) \\
S_{231} &= (x_1 - y_1)(x_2 - y_1) \\
S_{312} &= (x_1 - y_1)(x_1 - y_2) \\
S_{213} &= x_1 - y_1 \\
S_{132} &= x_1 + x_2 - y_1 - y_2 \\
S_{123} &= 1.
\end{align*}
\]

Specializing the \( y \) variables to 0, we obtain the \textbf{(single) Schubert polynomials} \( S_w(x) = S_w(x|0) \). These also be defined similarly as \( \partial_{\ell} \circ \cdots \circ \partial_1(x_1^{n-1} \cdots x_{n-1}) \). In fact, one can write \( S_w(x) = \sum a_j x_1^{i_1} \cdots x_n^{i_n} \), with
$i_j \leq n - j$ and $a_I \geq 0$. There are nice combinatorial formulas for the coefficients $a_I$; see [Bil-Joc-Sta93], [Win02]. One can read the top monomial (with respect to a certain term order) of $\mathfrak{S}_w(x)$ from the diagram $D(w)$: this is $\prod x_i^{D(w)_i}$, where $D(w)_i$ is the number of boxes in the $i$th row of $D(w)$.

As $w$ varies over $S_\infty = \bigcup_n S_n$, the Schubert polynomials $\mathfrak{S}_w(x)$ form a linear basis for $\mathbb{Z}[x_1, x_2, \ldots]$. In fact, $\mathfrak{S}_w(x)$ is symmetric in $x_k$ and $x_{k+1}$ iff $w(k) < w(k+1)$ iff $\partial_k \mathfrak{S}_w = 0$. Thus the polynomials $\mathfrak{S}_w$ with $w(n+1) < w(n+2) < \cdots$ form a basis for $\mathbb{Z}[x_1, \ldots, x_n]$.

**Theorem 2.3.** With $x_i = c_i^t (\ker(Q_i \to Q_{i-1}))$ as above, we have

$$\sigma_w = \mathfrak{S}_w(x|y),$$

where $\Omega_w$ is the locus defined by $\text{rk} (F_p \to Q_q) \leq r_w(q, p)$, $x_i = c_1(\ker(Q_i \to Q_{i-1}))$, and $y_i = c_1(F_i/F_{i-1})$.

**Proof.** First consider the case $w = w_0$. Then $\Omega_{w_0}$ is the locus where $F_p \to Q_{n-p}$ vanishes for all $p$, i.e., $S_p = F_p$, where $S_\bullet$ is the tautological subbundle. (It is also the image of the canonical section $B \to X$ corresponding to the flag $F_\bullet$ on $B$.) One way to compute its class is as follows. The locus where $F_1 \to Q_{n-1}$ vanishes has class $(x_1 - y_1) \cdots (x_{n-1} - y_1)$. On this locus, the vanishing of (the restriction of) $F_2/F_1 \to Q_{n-2}$ has class $(x_1 - y_2) \cdots (x_{n-2} - y_2)$. Continuing in this way and using the projection formula, we see $[\Omega_{w_0}] = \prod_{i+j \leq n}(x_i - y_j)$.

Now suppose we know the formula for some $w$. If $w(k) > w(k+1)$, let $v = w \cdot s_k$. The theorem will follow from the following:

**Claim.** $[\Omega_w] = \partial_k [\Omega_w]$.

In fact, we will also show that $\partial_k [\Omega_w] = 0$ when $w(k) < w(k+1)$. (Note that $\partial_k$ is well-defined on $\Lambda[x_1, \ldots, x_n]/(e_i(x) - e_i(t))_{1 \leq i \leq n}$; this follows from the Leibniz formula.)

To prove the claim, let $Y_k = \text{Fl}(1, 2, \ldots, n-k, \ldots, n; E)$, so $p : X \to Y_k$ is a $\mathbb{P}^1$-bundle: $X = \mathbb{P}(S_{n-k+1}/S_{n-k-1})$. Form the fiber product

$$
\begin{array}{ccc}
\text{Z}_k & \xrightarrow{p_1} & \text{X} \\
\downarrow & & \downarrow \\
\text{Y}_k & \xrightarrow{p} & \text{X}
\end{array}
$$

so $Z_k = \{(L_\bullet, L'_\bullet) \mid L_i = L'_i \text{ for } i \neq n-k \}$. 
Exercise 2.4. (i) If \( w(k) > w(k+1) \), \( p_1 \) maps \( p_2^* \Omega_w \) birationally onto \( \Omega_v \), with \( v = w s_k \).

(ii) If \( w(k) < w(k+1) \), \( p_1(p_2^* \Omega_w) \subset \Omega_w \).

(iii) \( (p_1)_* p_2^* \Omega_w = \partial_k \).

Note that the assertions in (i) and (ii) are local, so they can be reduced to the case of a point. The third statement is a general fact about \( \mathbb{P}^1 \)-bundles: If \( \mathbb{P}(W) \to Y \) is a \( \mathbb{P}^1 \)-bundle, with tautological quotient \( W \to Q \to 0 \), and \( x = c_1(Q) \), then \( p_*(x) = 1 \in H^*Y \).

Therefore we have
\[
[\Omega_v] = (p_1)_* p_2^* \Omega_w = \partial_k \mathcal{G}_w = \mathcal{G}_v
\]
when \( w(k) > w(k+1) \). On the other hand, if \( w(k) < w(k+1) \), we have
\[
0 = (p_1)_* p_2^* \Omega_w = (p_1)_* p_2^* \mathcal{G}_w = \partial_k \mathcal{G}_w.
\]

The fact that the definition of \( \mathcal{G}_w \) is independent of choices follows, since by choosing a suitable base \( B \) with \( E \) of sufficiently large rank, one can assume the \( x \)'s and \( y \)'s are independent up to any given degree.

Remark 2.5. Schubert polynomials are characterized by the fact that for a general map of flagged vector bundles
\[
F_1 \subset \cdots \subset F_n \xrightarrow{\varphi} E_n \to \cdots \to E_1,
\]
with degeneracy locus
\[
\Omega_w(\varphi) = \{x \mid \text{rk}(F_p(x) \to E_q(x)) \leq r_w(q,p)\},
\]
we have
\[
[\Omega_w(\varphi)] = \mathcal{G}_w(x|y),
\]
where \( x_i = c_1(\ker(Q_i \to Q_{i-1})) \) and \( y_i = c_1(F_i/F_{i-1}) \). See [Ful92].

Many other algebraic properties of Schubert polynomials can be proven geometrically.

Proposition 2.6. \( \mathcal{G}_w(y|x) = (-1)^{\ell(w)} \mathcal{G}_{w^{-1}}(x|y) \).

Proof. Replacing the sequence
\[
F_1 \subset \cdots \subset F_{n-1} \subset E \to Q_{n-1} \to \cdots \to Q_1
\]
with
\[
Q_1^\vee \subset \cdots \subset Q_{n-1}^\vee \subset E^\vee \to F_{n-1}^\vee \to \cdots \to F_1^\vee,
\]
interchanges \( x_i \) and \( -y_i \), and \( w \) and \( w^{-1} \). \( \square \)
**Corollary 2.7.** One can compute Schubert polynomials using divided difference operators acting on the \( y \) variables. If \( w = s_{i_\ell} \cdots s_{i_1} w_0 \), with \( \ell \) minimal, then

\[
\mathcal{S}_w(x|y) = (-1)^{\ell(w)} \partial_{i_\ell}^y \circ \cdots \circ \partial_{i_1}^y \prod_{i+j \leq n} (x_i - y_j).
\]

**Remark 2.8.** Computationally, it is hard to compute the polynomials \( \mathcal{S}_w \) from the definition. For example, \( \mathcal{S}_{s_k} \) is a linear polynomial, but to use the definition, one has to start from the top and apply \( (n-1) \) divided difference operators. However, by the above symmetry it is enough to compute \( \mathcal{S}_{s_k}(x) = \mathcal{S}_{s_k}(x|0) \), and this is easy: the fact that \( \partial_i \mathcal{S}_{s_k}(x) = \delta_{ik} \) implies

\[
\mathcal{S}_{s_k}(x|y) = x_1 + \cdots + x_k - (y_1 + \cdots + y_k).
\]