EQUIVARIANT COHOMOLOGY IN ALGEBRAIC GEOMETRY
LECTURE NINE: FLAG VARIETIES

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Remark 0.1. We saw that the study of cohomology and equivariant cohomology of Grassmannians leads to interesting symmetric polynomials, namely, the Schur polynomials $s_\lambda(x)$ and $s_\lambda(x|t)$. These arise in contexts other than intersection theory and representation theory. For example, Griffiths asked which polynomials P in $c_1(E), \dots, c_n(E)$ are positive whenever E is an ample vector bundle on an n-dimensional variety. (That is, $\int_X P(c_1(E), \dots, e_n(E)) > 0$; P should be homogeneous of degree n.)

Bloch showed that the Chern classes $c_1, \dots, c_n$ are positive in this sense; Griffiths gave other examples, such as $c_1^2 - c_2$, $\begin{vmatrix} c_1 & c_2 & c_3 \\ 1 & c_1 & c_2 \\ 0 & 1 & c_1 \end{vmatrix}$, $c_1^2 - 2c_2$, etc.

The complete answer was given by Fulton-Lazarsfeld [Ful-Laz83]. Write $P = \sum a_\lambda s_\lambda$, where s_λ is the Schur polynomial $\det(c_{\lambda_i+j-i})$ and $a_\lambda \in \mathbb{Z}$. Then P is positive if and only if $a_\lambda \geq 0$ for all λ (and at least one is > 0). (So for example $c_1^2 - 2c_2 = (c_1^2 - c_2) - c_2$ is not!)

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Let V be an n-dimensional vector space, let $G = GL(V)$, and let $X = Fl(V)$ be the variety of complete flags in V (so $\dim X = \binom{n}{2}$). For approximation spaces, take $EG_m = \mathrm{Hom}^o(V, \mathbb{C}^m)$ and $BG_m = Gr(n, \mathbb{C}^m)$, with tautological bundle $E \subset \mathbb{C}^m_{Gr}$. We have a diagram

$$\begin{tikzcd} EG_m \times^G X \arrow[r, "\sim"] \arrow[d] & \mathbf{Fl}(E) \arrow[d] \\ EG_m \times^G pt \arrow[r, "\sim"] & Gr(n, \mathbb{C}^m), \end{tikzcd}$$

where the top map is given map $(\varphi, L_\bullet) \mapsto (\mathrm{im}(\varphi), \varphi(L_\bullet))$. (So if $L_\bullet$ is $(L_1 \subset \cdots \subset L_n = V)$, then $\varphi(L_\bullet)$ is $(\varphi(L_1) \subset \cdots \subset \varphi(L_n) = \varphi(V))$.) On $\mathbf{Fl}(E)$, there are tautological flags of subbundles $\mathbb{S}_1 \subset \cdots \subset \mathbb{S}_n = E$ and quotient bundles $E = \mathbb{Q}_n \to \cdots \to \mathbb{Q}_1$, corresponding to the tautological bundles $S_1 \subset \cdots \subset S_n = V_X = Q_n \to \cdots \to Q_1$ on X. (So $Q_i = V/S_{n-i}$.)

Let $x_i = c_1^G(\ker(Q_i \to Q_{i-1})) = c_1(\ker(\mathbb{Q}_i \to \mathbb{Q}_{i-1}))$.

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Proposition 1.1. *We have*

$$H^*_G X = \Lambda_G[x_1, \ldots, x_n]/(e_i(x) - c_i)_{1 \le i \le n}.$$

More generally, for any flag bundle $\mathbf{Fl}(E) \to B$, *we have*

$$H^*\mathbf{Fl}(E) = H^*B[x_1, \ldots, x_n]/(e_i(x) - c_i(E))_{1 \le i \le n}.$$

Proof. To see this, realize $\mathbf{Fl}(E)$ as a sequence of projective bundles over B: Start by forming $\mathbb{P}(E) \to B$, with universal subbundle $U_1 \subset E$; then form $\mathbb{P}(E/U_1) \to \mathbb{P}(E)$, with universal subbundle U_2/U_1; continue until reaching $\mathbf{Fl}(E) = \mathbb{P}(E/U_{n-2}) \to \cdots \to B$. It is clear from this construction that the monomials $x_1^{i_1} \cdots x_n^{i_n}$ with $i_j \le n - j$ form a basis for $H^*\mathbf{Fl}(E)$ over H^*B, and that the relations $e_i(x) = c_i(E)$ hold. Thus $H^*\mathbf{Fl}(E)$ has rank $n!$ over H^*B, and the proposition will follow if the ring on the RHS has the same rank. □

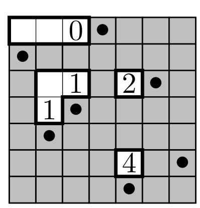
Exercise 1.2. Show that the ring $H^*B[x_1, \ldots, x_n]/(e_i(x) - c_i(E))$ also has a basis of monomials $x_1^{i_1} \cdots x_n^{i_n}$ with $i_j \le n - j$. (Hint: use the relation $\sum(-1)^i e_i \cdot h_{n-i} = 0$.)

Now assume $V = \mathbb{C}^n$, so we have $F_1 \subset \cdots \subset F_n = V$, with $F_i = \mathrm{Span}\{e_1, \ldots, e_i\}$. (Also write $\widetilde{F}_i = \mathrm{Span}\{e_n, \ldots, e_{n+1-i}\}$ for the **opposite flag**.) Note that the subgroup B of upper-triangular matrices preserves $F_\bullet$, and B^{opp} (lower-triangular matrices) preserves $\widetilde{F}_\bullet$. For $w \in S_n$, we have B-invariant **Schubert varieties** in X, defined by

$$\Omega_w(F_\bullet) = \{x \mid \mathrm{rk}(F_p(x) \to Q_q(x)) \le r_w(q,p) \text{ for all } 1 \le q, p \le n\},$$

where $r_w(q,p) = \#\{i \le q \mid w(i) \le p\}$.

As in the Grassmannian case, a subset of these rank conditions suffices to define Ω_w. To see which, form the **diagram** of w, the collection of boxes in the $n \times n$ defined as follows. Place a dot (or a 1) in row i and column $w(i)$, and cross out all boxes which are to the right or below a dot, including the boxes containing dots. The diagram $D(w)$ is the collection of boxes which remain. Here is an example, for $w = 4\,1\,6\,3\,2\,7\,5$. (We use "one-line" notation and write $w = w(1)\,w(2) \cdots w(n)$.)



Note that $r_w(q,p)$ is the number of dots in the upper-left $q \times p$ rectangle. (Some of these have been labelled in the diagram.) The conditions needed to define $\Omega_w(F_\bullet)$ are those coming from the boxes in the southwest corners of $D(w)$ – so in the above example, only five conditions are needed. In fact, this defines $\Omega_w(F_\bullet)$ as a (reduced, irreducible, normal, Cohen-Macaulay) subscheme. The codimension is

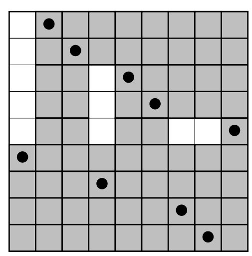
$$\begin{aligned}\ell(w) &= \#\{j < i \,|\, w(j) > w(i)\} \\ &= \#(\text{boxes in } D(w)).\end{aligned}$$

For details, see [Ful92].

Example 1.3. For a partition λ in the $k \times \ell$ rectangle, we have $I(\lambda) = \{i_1 < \cdots < i_k\}$ and $J(\lambda) = \{j_1 < \cdots < j_\ell\}$, with $I(\lambda) \cup J(\lambda) = \{1, \ldots, n\}$. Define $w(\lambda) \in S_n$ by

$$w(\lambda) = j_1\, j_2 \cdots j_\ell\, i_1 \cdots i_k.$$

Note that $\ell(w(\lambda)) = |\lambda| = \#\{i \in I(\lambda),\, j \in J(\lambda) \,|\, j > i\}$. For example, with $k = 4$, $\ell = 5$, and $\lambda = (5, 3, 1, 1)$, we have $w(\lambda) = 2\,3\,5\,6\,9\,1\,4\,7\,8$. The diagram $D(w(\lambda))$ is easy to describe in terms of λ; the pattern is suggested by this example.



The Schubert variety $\Omega_{w(\lambda)}(F_\bullet)$ comes from a Schubert variety on a Grassmannian: there is a projection $f : Fl(V) \to Gr(k, V)$, with $f^{-1}\Omega_\lambda(F_\bullet) = \Omega_{w(\lambda)}(F_\bullet)$.

All of this works for flag bundles, without change. On $\mathbf{X} = \mathbf{Fl}(E) \to B$, we have a flag $\mathbb{F}_1 \subset \cdots \subset \mathbb{F}_n = E_{\mathbf{X}}$, and Schubert loci $\boldsymbol{\Omega}_w(\mathbb{F}_\bullet) \subset \mathbf{X}$. When $B = BT$, we have $[\boldsymbol{\Omega}_w(\mathbb{F}_\bullet)] = [\Omega_w(F_\bullet)]^T \in H_T^* X$.

Write $\sigma_w = [\Omega_w(F_\bullet)]^T$ and $\widetilde{\sigma}_w = [\Omega_w(\widetilde{F}_\bullet)]^T$. For $w \in S_n$, these classes give two bases for $H_T^* X$ over Λ_T.

Proposition 1.4. *The bases $\{\sigma_w\}$ and $\{\tau_w = \widetilde{\sigma}_{w_0\, w}\}$ are Poincaré dual; that is,*

$$\rho_*(\sigma_w \cdot \tau_v) = \delta_{w\,v} \in \Lambda,$$

where ρ is the map $X \to pt$.

Proof. We show that if $\ell(w) + \ell(v) \geq \binom{n}{2} = \dim X$, then $\Omega_w(F_\bullet)$ meets $\Omega_v(\widetilde{F}_\bullet)$ only if $v = w_0\, w$, and in this case they meet transversally at the point

$$p_w = \langle e_{w(n)} \rangle \subset \langle e_{w(n)}, e_{w(n-1)} \rangle \subset \cdots \subset \langle e_{w(n)}, \ldots, e_{w(1)} \rangle.$$

(Note that at p_w, the map

$$F_p(p_w) = \langle e_1, \dots, e_p\rangle \to Q_q(p_w) = \mathbb{C}^n/\langle e_{w(n)}, \dots, e_{w(n+1-q)}\rangle = \langle e_{w(1)}, \dots, e_{w(q)}\rangle$$

has rank $r_w(q,p)$.)

A neighborhood $U_w \cong \mathbb{A}^{\binom{n}{2}}$ of p_w in X is given by the set of matrices with 1's in positions $(i, w(i))$ and arbitrary entries below. The flag associated to such a matrix has parts spanned by the rows, reading from the bottom up. For example, if $w = 4\,1\,6\,3\,2\,7\,5$, then

$$U_w = \begin{pmatrix} & & & 1 & & & \\ 1 & & & * & & & \\ * & & & * & & 1 & \\ * & & 1 & * & & * & \\ * & 1 & * & * & & * & \\ * & * & * & * & & * & 1 \\ * & * & * & * & 1 & * & * \end{pmatrix}.$$

The intersection of U_w with the Schubert variety

$$\Omega_w(F_\bullet) = \{L_\bullet \mid \dim(F_p \cap L_q) \geq p - \#\{i \leq q \mid w(i) \leq p\}\}$$

is given by setting entries to the right of 1's to 0:

$$\Omega^o_w(F_\bullet) = \begin{pmatrix} & & & 1 & & & \\ 1 & & & 0 & & & \\ * & & & * & & 1 & \\ * & & 1 & 0 & & 0 & \\ * & 1 & 0 & 0 & & 0 & \\ * & * & * & * & & * & 1 \\ * & * & * & * & 1 & 0 & 0 \end{pmatrix}.$$

(Note that there are $\ell(w)$ such entries.)

The situation for $\Omega_v(\widetilde{F}_\bullet)$ is the same, but with the matrices reflected from left to right. Thus $p_{w_0\,v} \in \Omega_v(\widetilde{F}_\bullet)$, and the intersection of $\Omega_v(\widetilde{F}_\bullet)$ with $U_{w_0\,v}$ is given by setting entries to the left of 1's equal to 0. For $v = w_0\,w$, we see that $\Omega_w(F_\bullet)$ and $\Omega_{w_0\,w}(\widetilde{F}_\bullet)$ intersect transversally at the origin (as coordinate planes in U_w). Also, we see that if $\Omega_w(F_\bullet) \cap \Omega_v(\widetilde{F}_\bullet)$ is nonempty, then $w(i) \leq w_0 v(i)$ for all i, so $\ell(w) \leq \ell(w_0 v) = \binom{n}{2} - \ell(v)$, i.e., $\ell(w) + \ell(v) \leq \binom{n}{2}$. $\square$

Note that p_w is a smooth point of $\Omega_w(F_\bullet)$, and since Ω^o_w is an affine space, it is easy to compute $T_{p_w}\Omega_w$. The torus $T = (\mathbb{C}^*)^n$ acts with weight $t_{w(i)} - t_{w(j)}$ on an entry in position $(j, w(i))$ with $i < j$ and $w(i) < w(j)$.

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We now consider the equivariant Giambelli formula for Schubert varieties in X. This is given by the *double Schubert polynomials* $\mathfrak{S}_w(x|y)$ of Lascoux and Schützenberger [Las-Sch82]. Here $w \in S_n$ is a permutation, and $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_n)$ are two sets of variables.

These are defined as follows. For $F \in \mathbb{Z}[x, y]$ and $1 \leq i \leq n-1$, define the **divided difference operator** $\partial_i = \partial_i^x$ by

$$\partial_i F = \frac{F(x;y) - F(\dots, x_{i+1}, x_i, \dots; y)}{x_i - x_{i+1}} = \frac{F - s_i(F)}{x_i - x_{i+1}}.$$

(Here $s_i = (i, i+1)$ is the simple transposition exchanging i and $i+1$.) Note that ∂_i ignores the y variables, and $\partial_i F = 0$ iff F is symmetric in x_i and x_{i+1}. For $w \in S_n$, write $w = w_0\, s_{i_1} \cdots s_{i_\ell}$ with ℓ minimal, so $\ell = \binom{n}{2} - \ell(w)$. (To do this, successively swap adjacent entries of w to reach w_0. For example,

$$\begin{aligned} w = 3\,1\,5\,2\,4 &\xrightarrow{s_2} 3\,5\,1\,2\,4 \xrightarrow{s_3} 3\,5\,2\,1\,4 \xrightarrow{s_4} 3\,5\,2\,4\,1 \\ &\xrightarrow{s_3} 3\,5\,4\,2\,1 \xrightarrow{s_1} 5\,3\,4\,2\,1 \xrightarrow{s_2} 5\,4\,3\,2\,1 = w_0 \end{aligned}$$

shows $w = w_0\, s_2\, s_1\, s_3\, s_4\, s_3\, s_2$.)

Definition 2.1. With notation as above, the **double Schubert polynomial** is defined by

$$\mathfrak{S}_w(x|y) = \partial_{i_\ell} \circ \cdots \circ \partial_{i_1} \left(\prod_{i+j \leq n} (x_i - y_j) \right).$$

This is independent of the choice of the expression for w; as for many such assertions, there are algebraic proofs (see [Mac91]) and geometric proofs. Note that the y variables act as "scalars" here. In fact, there is a "Leibniz rule"

$$\partial_i(F \cdot G) = (\partial_i F)\, G + (s_i F)(\partial_i G),$$

so any function F which is symmetric in the x variables is a scalar for the action of the divided difference operators.

Example 2.2. The Schubert polynomials for $n = 3$ are as follows:

$$\begin{tikzcd}
 & \mathfrak{S}_{321} = (x_1 - y_1)(x_1 - y_2)(x_2 - y_1) \arrow[dl, "\partial_1"'] \arrow[dr, "\partial_2"] & \\
\mathfrak{S}_{231} = (x_1 - y_1)(x_2 - y_1) \arrow[d, "\partial_2"'] & & \mathfrak{S}_{312} = (x_1 - y_1)(x_1 - y_2) \arrow[d, "\partial_1"'] \\
\mathfrak{S}_{213} = x_1 - y_1 \arrow[dr, "\partial_1"'] & & \mathfrak{S}_{132} = x_1 + x_2 - y_1 - y_2 \arrow[dl, "\partial_2"] \\
 & \mathfrak{S}_{123} = 1. &
\end{tikzcd}$$

Specializing the y variables to 0, we obtain the **(single) Schubert polynomials** $\mathfrak{S}_w(x) = \mathfrak{S}_w(x|0)$. These also be defined similarly as $\partial_{i_\ell} \circ \cdots \circ \partial_{i_1}(x_1^{n-1} \cdots x_{n-1})$. In fact, one can write $\mathfrak{S}_w(x) = \sum a_I x_1^{i_1} \cdots x_n^{i_n}$, with

$i_j \leq n - j$ and $a_I \geq 0$. There are nice combinatorial formulas for the coefficients a_I; see [Bil-Joc-Sta93], [Win02]. One can read the top monomial (with respect to a certain term order) of $\mathfrak{S}_w(x)$ from the diagram $D(w)$: this is $\prod x_i^{D(w)_i}$, where $D(w)_i$ is the number of boxes in the ith row of $D(w)$.

As w varies over $S_\infty = \bigcup_n S_n$, the Schubert polynomials $\mathfrak{S}_w(x)$ form a linear basis for $\mathbb{Z}[x_1, x_2, \ldots]$. In fact, $\mathfrak{S}_w(x)$ is symmetric in x_k and x_{k+1} iff $w(k) < w(k+1)$ iff $\partial_k \mathfrak{S}_w = 0$. Thus the polynomials $\mathfrak{S}_w$ with $w(n+1) < w(n+2) < \cdots$ form a basis for $\mathbb{Z}[x_1, \ldots, x_n]$.

Theorem 2.3. *With $x_i = c_1^T(\ker(Q_i \to Q_{i-1}))$ as above, we have*

$$\sigma_w = \mathfrak{S}_w(x|t).$$

Equivalently, given bundles $F_1 \subset \cdots F_n = E$ on a base B, and $\mathbf{X} = \mathbf{Fl}(E)$ with universal quotient bundles $E \to Q_{n-1} \to \cdots \to Q_1$, we have

$$[\mathbf{\Omega}_w] = \mathfrak{S}_w(x|y),$$

where $\mathbf{\Omega}_w$ is the locus defined by $\mathrm{rk}(F_p \to Q_q) \leq r_w(q,p)$, $x_i = c_1(\ker(Q_i \to Q_{i-1}))$, and $y_i = c_1(F_i/F_{i-1})$.

Proof. First consider the case $w = w_0$. Then $\mathbf{\Omega}_{w_0}$ is the locus where $F_p \to Q_{n-p}$ vanishes for all p, i.e., $S_p = F_p$, where $S_\bullet$ is the tautological subbundle. (It is also the image of the canonical section $B \to \mathbf{X}$ corresponding to the flag $F_\bullet$ on B.) One way to compute its class is as follows. The locus where $F_1 \to Q_{n-1}$ vanishes has class $(x_1 - y_1) \cdots (x_{n-1} - y_1)$. On this locus, the vanishing of (the restriction of) $F_2/F_1 \to Q_{n-2}$ has class $(x_1 - y_2) \cdots (x_{n-2} - y_2)$. Continuing in this way and using the projection formula, we see $[\mathbf{\Omega}_{w_0}] = \prod_{i+j \leq n} (x_i - y_j)$.

Now suppose we know the formula for some w. If $w(k) > w(k+1)$, let $v = w \cdot s_k$. The theorem will follow from the following:

Claim . $[\mathbf{\Omega}_v] = \partial_k [\mathbf{\Omega}_w]$.

In fact, we will also show that $\partial_k [\mathbf{\Omega}_w] = 0$ when $w(k) < w(k+1)$. (Note that ∂_k is well-defined on $\Lambda[x_1, \ldots, x_n]/(e_i(x) - e_i(t))_{1 \leq i \leq n}$; this follows from the Leibniz formula.)

To prove the claim, let $Y_k = \mathbf{Fl}(1, 2, \ldots, \widehat{n-k}, \ldots, n; E)$, so $p : \mathbf{X} \to Y_k$ is a $\mathbb{P}^1$-bundle: $\mathbf{X} = \mathbb{P}(S_{n-k+1}/S_{n-k-1})$. Form the fiber product

$$\begin{tikzcd} & Z_k \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ \mathbf{X} \arrow[dr, "p"'] & & \mathbf{X} \arrow[dl, "p"] \\ & Y_k, & \end{tikzcd}$$

so $Z_k = \{(L_\bullet, L'_\bullet \,|\, L_i = L'_i \text{ for } i \neq n-k\}$.

Exercise 2.4. (i) If $w(k) > w(k+1)$, p_1 maps $p_2^{-1}\mathbf{\Omega}_w$ birationally onto $\mathbf{\Omega}_v$, with $v = w\,s_k$.
(ii) If $w(k) < w(k+1)$, $p_1(p_2^{-1}\mathbf{\Omega}_w) \subset \mathbf{\Omega}_w$.
(iii) $(p_1)_* \circ p_2^* = p^* \circ p_* = \partial_k$.

Note that the assertions in (i) and (ii) are local, so they can be reduced to the case of a point. The third statement is a general fact about $\mathbb{P}^1$-bundles: If $\mathbb{P}(W) \to Y$ is a $\mathbb{P}^1$-bundle, with tautological quotient $W \to Q \to 0$, and $x = c_1(Q)$, then $p_*(x) = 1 \in H^*Y$.

Therefore we have

$$[\mathbf{\Omega}_v] = (p_1)_* p_2^*[\mathbf{\Omega}_w] \quad = \quad (p_1)_* p_2^* \mathfrak{S}_w = \partial_k \mathfrak{S}_w = \mathfrak{S}_v$$

when $w(k) > w(k+1)$. On the other hand, if $w(k) < w(k+1)$, we have

$$0 = (p_1)_* p_2^*[\mathbf{\Omega}_w] \quad = \quad (p_1)_* p_2^* \mathfrak{S}_w = \partial_k \mathfrak{S}_w.$$

□

The fact that the definition of $\mathfrak{S}_w$ is independent of choices follows, since by choosing a suitable base B with E of sufficiently large rank, one can assume the x's and y's are independent up to any given degree.

Remark 2.5. Schubert polynomials are characterized by the fact that for a general map of flagged vector bundles

$$F_1 \subset \cdots \subset F_n \xrightarrow{\varphi} E_n \to \cdots \to E_1,$$

with degeneracy locus

$$\Omega_w(\varphi) = \{x \mid \operatorname{rk}(F_p(x) \to E_q(x)) \le r_w(q,p)\},$$

we have

$$[\Omega_w(\varphi)] = \mathfrak{S}_w(x|y),$$

where $x_i = c_1(\ker(Q_i \to Q_{i-1}))$ and $y_i = c_1(F_i/F_{i-1})$. See [Ful92].

Many other algebraic properties of Schubert polynomials can be proven geometrically.

Proposition 2.6. $\mathfrak{S}_w(y|x) = (-1)^{\ell(w)} \mathfrak{S}_{w^{-1}}(x|y)$.

Proof. Replacing the sequence

$$F_1 \subset \cdots \subset F_{n-1} \subset E \to Q_{n-1} \to \cdots \to Q_1$$

with

$$Q_1^\vee \subset \cdots \subset Q_{n-1}^\vee \subset E^\vee \to F_{n-1}^\vee \to \cdots \to F_1^\vee,$$

interchanges x_i and $-y_i$, and w and w^{-1}. □

Corollary 2.7. *One can compute Schubert polynomials using divided difference operators acting on the y variables. If $w = s_{i_\ell} \cdots s_{i_1} w_0$, with ℓ minimal, then*

$$\mathfrak{S}_w(x|y) = (-1)^{\ell(w)} \partial^y_{i_\ell} \circ \cdots \circ \partial^y_{i_1} \prod_{i+j \leq n} (x_i - y_j).$$

Remark 2.8. Computationally, it is hard to compute the polynomials $\mathfrak{S}_w$ from the definition. For example, $\mathfrak{S}_{s_k}$ is a linear polynomial, but to use the definition, one has to start from the top and apply $\binom{n}{2} - 1$ divided difference operators. However, by the above symmetry it is enough to compute $\mathfrak{S}_{s_k}(x) = \mathfrak{S}_{s_k}(x|0)$, and this is easy: the fact that $\partial_i \mathfrak{S}_{s_k}(x) = \delta_{ik}$ implies

$$\mathfrak{S}_{s_k}(x|y) = x_1 + \cdots + x_k - (y_1 + \cdots + y_k).$$

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