2021.3.8

# LECTURE 20: LIE THEORY I (Root systems)

Goal: understand enough to work with general $G/P$.

Plan: (1) Root systems

(2) Weyl groups

(3) semisimple LAGs and Lie algebras

## Root systems

Ref: · Bourbaki Lie gps + Lie algs, VI §1

· Humphreys, Intro to Lie alg's

$V$ = finite-dimensional v.s. $/\mathbb{R}$.

$\langle \ , \ \rangle : V \times V^* \longrightarrow \mathbb{R}$, the natural pairing.

Defn: A root system is a subset $R \subset V$ satisfying:

RS1 $R$ is finite, $0 \notin R$, and $R$ spans $V$.

**RS 2** For $\alpha \in R$, $\exists! \, \alpha^\vee \in V^*$ such that $\langle \alpha, \alpha^\vee \rangle = 2$,
and the linear reflection
$s_\alpha : \lambda \longmapsto \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$ preserves $R$.

reflect in the hyperplane $H_\alpha = \{\alpha^\vee = 0\} \subset V$

**RS 3** For $\alpha, \beta \in R$, $\langle \alpha, \beta^\vee \rangle \in \mathbb{Z}$.

Elements of $R$ are called roots.

The elements $\alpha^\vee$ (specified in RS2) are co-roots.

(The set $R^\vee = \{\alpha^\vee \mid \alpha \in R\} \subset V^*$ is a root system - check!)

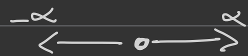
Note: $s_\alpha(\alpha) = -\alpha$. So $R = -R$.

$R$ is reduced if, for each $\alpha \in R$, the only multiples $n\alpha \in R$ are $n = \pm 1$.

$\longrightarrow$ We'll always assume this.

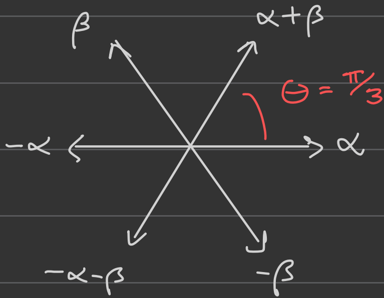
The **rank** of $R$ is $\dim V$.

**Ex:** There's (essentially) only one root system of rk 1




$V \simeq \mathbb{R}$
$R = \{\alpha, -\alpha\}$ $(A_1)$

**Ex:**




$(A_2)$

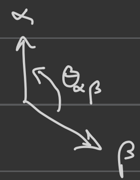
Rmk: in pictures, implicitly $V \simeq \mathbb{R}^n$
w/ standard Euclidean metric,
(Choice of basis + metric is not an issue...)

For now, $(\cdot,\cdot)$ is this standard metric,
so under $V^* \simeq V$ have $\alpha^\vee = \dfrac{2\alpha}{(\alpha,\alpha)}$.

We'll use this a little now, avoid it later.

## Two roots

For $\alpha, \beta \in R$, $\theta_{\alpha\beta}$ is the angle w.r.t. $(\ ,\ )$.




And $\|\alpha\| = \sqrt{(\alpha,\alpha)}$ is the length.

By RS3,

$$\mathbb{Z} \Rightarrow \langle \alpha, \beta^\vee \rangle = \left( \alpha, \frac{2\beta}{(\beta,\beta)} \right) = \frac{2(\alpha,\beta)}{(\beta,\beta)} = 2\frac{\|\alpha\|}{\|\beta\|}\cos\theta_{\alpha\beta}$$

is an integer, as is $\langle \beta, \alpha^\vee \rangle = 2\frac{\|\beta\|}{\|\alpha\|}\cos\theta_{\alpha\beta}$.

So $\langle \alpha, \beta^\vee \rangle \langle \beta, \alpha^\vee \rangle = 4\cos^2\theta_{\alpha\beta} \in \mathbb{Z}$.

This restricts the possibilities. For $\alpha \neq \pm\beta$, $\|\alpha\| \leq \|\beta\|$:

| $4\cos^2\theta_{\alpha\beta}$ | $\theta_{\alpha\beta}$ | $(\beta,\beta)/(\alpha,\alpha)$ | |
|---|---|---|---|
| 0 | $\pi/2$ | anything | |
| 1 | $\pi/3$, $2\pi/3$ | 1 | |
| 2 | $\pi/4$, $3\pi/4$ | 2 | |
| 3 | $\pi/6$, $5\pi/6$ | 3 | |
| 4 | $0$, $\pi$ | 1 | $\leftarrow \alpha = \pm\beta$ |

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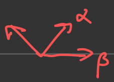
**Exercise:** If $\|\alpha\| \leq \|\beta\|$ and $\alpha \neq \pm\beta$, then $\langle \alpha, \beta^\vee \rangle = \pm 1$ or $0$.

Show this implies:

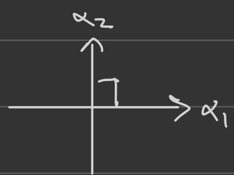

(1) if $(\alpha, \beta) < 0$ then $\alpha + \beta$ is a root.




(2) if $(\alpha, \beta) > 0$ then $\alpha - \beta$ is a root

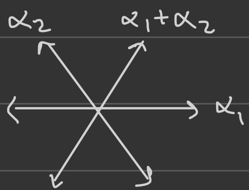
**Exs:** (Rank 2)

$\theta = \frac{\pi}{2}$



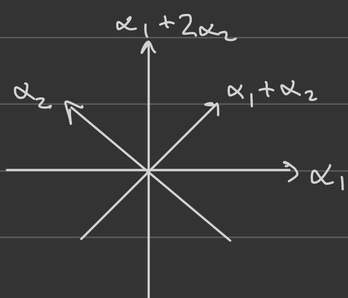

$(A_1 \times A_1)$

$\theta = \frac{2\pi}{3}$



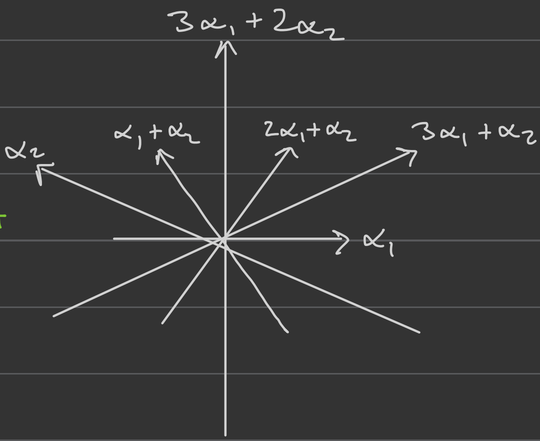

$(A_2)$

$\theta = \frac{3\pi}{4}$




$(B_2)$

20.6




$\theta = \frac{5\pi}{6}$

$(G_2)$

Def$^{\underline{n}}$: $R$ is reducible if $R = R_1 \amalg R_2$ (nontrivially) such that $(R_1, R_2) \equiv 0$, i.e., the decomposition is orthogonal.

$A_1, A_2, B_2, G_2$ — irreducible


$A_1 \times A_1$ — reducible.

# Simple roots, positive roots, Weyl chambers

**Def^n:** A subset $\Delta = \{\alpha_1, \dots, \alpha_n\} \subset R$ is a set of simple roots if

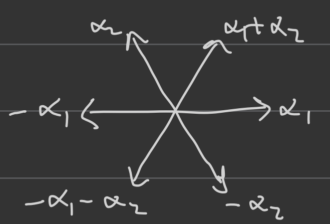
(1) $\Delta$ is a basis of $V$

and

(2) Every $\beta \in R$ is $\beta = \sum c_i \alpha_i$, with all $c_i \in \mathbb{Z}_{\geq 0}$ or all $c_i \in \mathbb{Z}_{\leq 0}$.

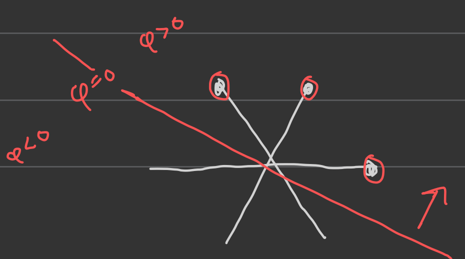
**Thm:** simple roots exist!

**Ex:**




$\Rightarrow \{\alpha_1, \alpha_2\} = \Delta$ for $A_2$.

**Def^n:** A subset $R^+ \subset R$ is a set of positive roots if $\exists\, \varphi \in V^*$ such that $\varphi(\alpha) \neq 0 \ \forall \alpha \in R$ and $\varphi(\alpha) > 0 \ \forall \alpha \in R^+$.

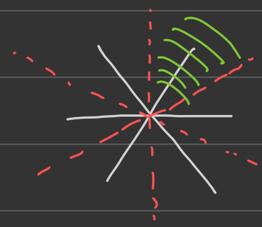


(Positive roots obviously exist.)

Negative roots are $R^- = -R^+$, so $R = R^+ \amalg R^-$.

Defn: Let $H_\alpha \subset V$ be the hyperplane defined by $\{\alpha^\vee = 0\}$.

A Weyl chamber is a connected component of $V \setminus \bigcup_\alpha H_\alpha$.



Propn: There are canonical bijections

$$\{\text{sets of simple roots } \Delta\} \longleftrightarrow \{\text{sets of positive roots } R^+\}$$

$$\{\text{Weyl chambers } \mathring{\sigma}\}$$

Ref: [B, VI, §1.5]

$$\Delta = \{\alpha_1, \dots, \alpha_n\} \rightsquigarrow R^+ = \{\textstyle\sum c_i \alpha_i \mid c_i \geq 0\} \subseteq R,$$

$$\mathring{\sigma} = \{\lambda \in V \mid \langle \lambda, \alpha_i^\vee \rangle > 0 \ \forall\ i = 1, \dots, n\}$$

**Propn:** If $\alpha, \beta \in \Delta$, $\alpha \neq \beta$, then $\langle \alpha, \beta^\vee \rangle \leq 0$.

**Pf:** Assume $\|\alpha\| \leq \|\beta\|$, and for contradiction $(\alpha, \beta) > 0$. Then by exercise, $\alpha - \beta$ is a root $\times$.

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## Weights

**Defn:** $\lambda \in V$ is a weight if $\langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}$ $\forall \alpha \in R$.

- Weights form a lattice in $V$, the weight lattice $M_{wt} \subseteq V$.
- Roots are weights (by RS 3), span the root lattice $M_{rt} \subset M_{wt} \subset V$

Fixing simple roots $\Delta = \{\alpha_1, \dots, \alpha_n\}$, the dominant Weyl chamber is $\mathring{\sigma} = \{\lambda \mid \langle \lambda, \alpha_i^\vee \rangle > 0 \text{ for } \alpha_i \in \Delta\}$

A weight $\lambda \in M_{wt}$ is dominant if it's in the closure $\sigma = \overline{\mathring{\sigma}} = \{\langle \lambda, \alpha_i^\vee \rangle \geq 0\}$.

20.10

The **fundamental weights** $\varpi_1, \dots, \varpi_n \in M_{wt}$ are the dual basis to $\alpha_1^\vee, \dots, \alpha_n^\vee$ $(\in V^*)$

I.e., $\langle \varpi_i, \alpha_j^\vee \rangle = \delta_{ij}$.

$\rightsquigarrow$ These are the generators for cone $\sigma$ in $M_{wt}$.

**Ex: ($A_2$)** $\mathbb{R}^3$, with std basis $e_1, e_2, e_3$,
std pairing $(e_i, e_j) = \delta_{ij}$.

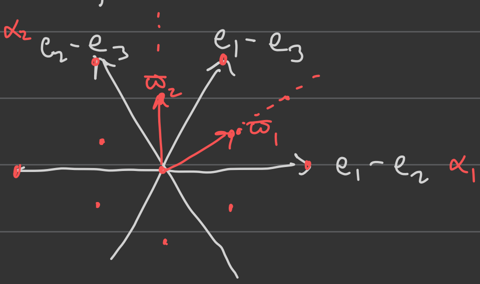
$V \subset \mathbb{R}^3$ subspace $\{\lambda = a_1 e_1 + a_2 e_2 + a_3 e_3 \mid a_1 + a_2 + a_3 = 0\}$.

$\Delta = \{\alpha_1, \alpha_2\} = \{e_1 - e_2, e_2 - e_3\}$.

Using $(\ ,\ )$, so $\alpha_i^\vee = \dfrac{2\alpha_i}{(\alpha_i, \alpha_i)} = \alpha_i$.

$\Rightarrow \varpi_1 = e_1 - \frac{1}{3}(e_1 + e_2 + e_3)$

$\varpi_2 = e_1 + e_2 - \frac{2}{3}(e_1 + e_2 + e_3)$




(Etc. ...) See $M_{wt}$ = hexagonal lattice generated by $\varpi_1, \varpi_2$.

20.11

## Classification [Bourbaki §4, Humphreys §11]

**Defn:** Fix simple roots $\Delta = \{\alpha_1, \dots, \alpha_n\}$.
The matrix $\left(\langle \alpha_i, \alpha_j^\vee \rangle\right)_{1 \le i, j \le n}$ is the Cartan matrix of $R$ (wrt $\Delta$).

**Propn:** The Cartan matrix determines $R$.
Precisely, let $\Delta \subset R$, $\Delta' \subset R'$ be root systems with simple roots. Assume the Cartan matrices are the same: $\langle \alpha_i, \alpha_j^\vee \rangle = \langle \alpha_i', \alpha_j'^\vee \rangle$.

Then the isomorphism $\phi : V \to V'$, $\alpha_i \mapsto \alpha_i'$, sends $R$ to $R'$.

**Rmk:** From defn, $\alpha_j = \sum_{i=1}^{n} \langle \alpha_j, \alpha_i^\vee \rangle \varpi_i$.
So to express fundamental wts in terms of roots, invert the Cartan mtx.

**Cor:** $\left| {}^{M_{wt}}/_{M_{rt}} \right| = \det\left(\langle \alpha_i, \alpha_j^\vee \rangle\right)$.

${}^{M_{wt}}/_{M_{rt}}$ is the fundamental group of $R$...

20.12

The Dynkin diagram of $R$ (wrt $\Delta$) is a decorated graph on nodes $\{1, \dots, n\}$ $\longleftrightarrow \{\alpha_1, \dots, \alpha_n\}$

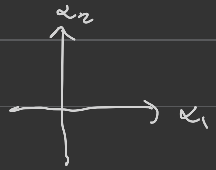
with $i$ connected to $j$ by $\langle \alpha_i, \alpha_j^\vee \rangle \langle \alpha_j, \alpha_i^\vee \rangle$ edges

$= 4\cos^2\theta_{ij}$

Draw arrow from $i$ to $j$ if $\|\alpha_i\| > \|\alpha_j\|$ (and $\overset{i}{\circ}\!\!\rightarrow\!\!\overset{j}{\circ}$)

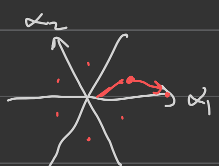
( long $\to$ short )

Ex:

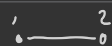



$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$


$(A_1 \times A_1)$

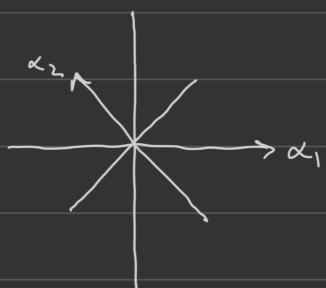



$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$




$(A_2)$

index $|M_{wt}/M_{rt}| = 3$




$\begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}$


$(B_2)$

index $|M_{wt}/M_{rt}| = 2$

if $\|\alpha_i\| < \|\alpha_j\|$ then $\langle \alpha_i, \alpha_j^\vee \rangle = -1$ or $0$.

The Dynkin diagram and Cartan mtx encode equivalent info.

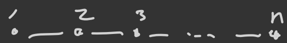
Rmk: $R$ is irreducible iff its Dynkin diagram is connected

(wrt some/any $\Delta \subset R$ )

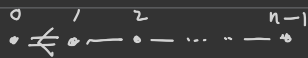
20.13

**Thm** ("Cartan-Killing"): The irreducible root systems are:

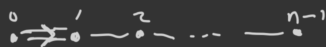
($A_n$)



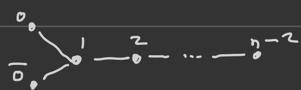

($B_n$)




($C_n$)

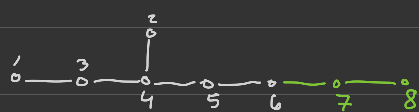



($D_n$)




"classical series"

($E_{6,7,8}$)




($F_4$)


($G_2$)


[van der Waerden]
Idea: classify possible Cartan matrices, by excluding subgraphs.

E.g.  X

**Exs**:

($A_{n-1}$) Take $\mathbb{R}^n$ with orthonormal basis $e_1, \dots, e_n$.

"$PGL_n$"

Two common choices: $V \subset \mathbb{R}^n$ subspace $\{\sum a_i e_i \mid \sum a_i = 0\}$

or: $V = \mathbb{R}^n / \mathbb{R}\cdot(e_1 + \dots + e_n)$. "$SL_n$"

These two are related by $V \leftrightarrow V^*$

20.14

This time we'll the the second one, the quotient.

$\Delta = \{\alpha_1, \dots, \alpha_{n-1}\}, \quad \alpha_i = \bar{e}_i - \bar{e}_{i+1}.$

$R^+ = \{\bar{e}_i - \bar{e}_j \mid i<j\} \qquad R^- = \{\bar{e}_i - \bar{e}_j \mid i>j\}.$

$\alpha_i^\vee = e_i^* - e_{i+1}^* \in V^* \subseteq \mathbb{R}^n$ (coordinates sum to 0)

$\varpi_i = \bar{e}_1 + \cdots + \bar{e}_i \in V$

$s_{\alpha_i} = s_i$ reflects $\bar{e}_i \leftrightarrow \bar{e}_{i+1}$, fixes other $\bar{e}_j$'s.

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$(C_n)$ $V = \mathbb{R}^n$, same basis $e_1, \dots, e_n$.

Rmk: $B_2 \simeq C_2$

$$R = \{\underbrace{\pm e_i \pm e_j \mid i<j}_{\text{short}}\} \cup \{\underbrace{\pm 2e_i}_{\text{long}}\}$$

$$\Delta = \{\overset{\alpha_0}{-2e_1}, \underbrace{\overset{\alpha_1}{e_1 - e_2}, \overset{\alpha_2}{e_2 - e_3}, \dots, \overset{\alpha_{n-1}}{e_{n-1} - e_n}}_{\text{sub-system of type } A_{n-1}}\}$$

(type $B_n$)

Coroots: $\overset{\alpha_0^\vee}{-e_1^*}, \ e_1^* - e_2^*, \dots, e_{n-1}^* - e_n^*$

Weights: $\varpi_0 = \frac{-1}{2}(e_1 + \cdots + e_n)$

$\varpi_i = e_n + \cdots + e_{n+1-i} \quad (i = 1, \dots, n-1)$

- Exercise: Work out others!!
- Also write out $R^+$, $R^-$ for type C. Compare with $Fl_\omega(\mathbb{C}^{2n})$.