2021.3.10

LECTURE 21: LIE THEORY II (Weyl group)

Ref: [Humphreys, Reflection groups + Coxeter groups]

A root system $R \subset V$ comes with reflections s_α for each $\alpha \in R$, preserving R. RS 2

Defn: The Weyl group $W(R)$ is the subgroup of $GL(V)$ generated by the reflections

$$s_\alpha : \lambda \longmapsto \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$$

for $\alpha \in R$.

Fixing a set of simple roots Δ, we write $s_i = s_{\alpha_i}$ for the simple reflections; $S = \{s_1, \dots, s_n\}$.

In fact, W is generated by S. More precisely, recall the possible angles θ_{ij} between α_i, α_j, are $\theta = \frac{\pi}{2}, \frac{2\pi}{3}, \frac{3\pi}{4}, \frac{5\pi}{6}$ (for $i \neq j$).

θ_{ij} are w.r.t. $(\, , \,)$, chosen to be W-invariant!

Write $m(i,j) =$ denominator of θ_{ij} $(= 2, 3, 4, \text{ or } 6)$.

Prop: (W, S) is a Coxeter group:

$$W = \langle s_1, \dots, s_n \mid (s_i s_j)^{m(i,j)} = s_i^2 = 1 \rangle$$

[Bourbaki §1], [Humphreys §1.9]

The exponent $m(i,j)$ is determined by the Dynkin diagram:

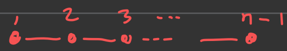
# edges $i - j$	$m(i,j)$	
0	2	← s_i, s_j commute
1	3	← $s_i s_j s_i = s_j s_i s_j$
2	4	
3	6	

So in fact, $W = W(R)$ depends only on the underlying graph, not the arrows. So $W(B_n) = W(C_n)$.

(Clear from defn: reflection gp only depends on directions of α's, not their lengths.)

Ex: $W(A_{n-1}) = \langle s_1, \dots, s_{n-1} \mid s_i^2 = 1,\ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1},\ s_i s_j = s_j s_i \ |i-j|>1 \rangle$

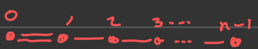
$\cong S_n$,



by $s_i \longmapsto (i \leftrightarrow i+1)$.

The representation on V is induced from the usual permutation representation of S_n on $\mathbb{R}^n$.

Ex: $W(B_n) = W(C_n) = \left\langle s_0, \dots, s_{n-1} \;\middle|\; \begin{array}{ll} s_i^2 = 1 & s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \ (i>0) \\ s_0 s_1 s_0 s_1 = s_1 s_0 s_1 s_0 & \\ & s_i s_j = s_j s_i \ \ |i-j|>1 \end{array} \right\rangle$

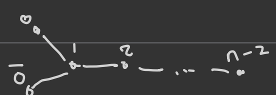


$\cong W_n$ (signed permutations) $\cong (\mathbb{Z}/2\mathbb{Z})^n \rtimes S_n$

by $s_i \longmapsto i \leftrightarrow i+1$ for $i>0$

$s_0 \longmapsto 1 \leftrightarrow \bar{1}$. (Acts on V by $s_0(e_1) = -e_1$.)

Ex: $W(D_n) = \left\langle s_0, s_{\bar{0}}, s_1, \dots, s_{n-2} \;\middle|\; \begin{array}{ll} s_i^2 = 1 & s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \ (i>0) \\ s_0 s_1 s_0 = s_1 s_0 s_1, & s_{\bar{0}} s_1 s_{\bar{0}} = s_1 s_{\bar{0}} s_1 \\ & s_i s_j = s_j s_i \text{ otherwise} \end{array} \right\rangle$



$\hookrightarrow W_n$

Represent $W(D_n)$ on $V = \mathbb{R}^n$ with std. basis $e_0, e_1, \dots, e_{n-1}$.

by $s_i \longmapsto (i+1) \leftrightarrow (i+2)$ $i = 0, \dots, n-2$

$s_{\bar{0}} \longmapsto 12 \leftrightarrow \bar{2}\bar{1}$.

Image is $W_n' = \{ w \in W_n \mid w \text{ has an } \underline{\text{even}} \ \# \text{ of signs} \}$

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By the Prop^n, every $w \in W$ is $w = s_{i_1} \cdots s_{i_\ell}$, for some minimal ℓ.

Def^n: Such a minimal $\ell = \ell(w)$ is the length of w. An expression of minimal length is reduced.

Ex: In W_3, $w = 2\,\bar{3}\,1 = $

$$\begin{array}{ccc} 1 & 2 & 3 \\ \hline 2 & 1 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \\ \bar{3} & 2 & 1 \\ 2 & \bar{3} & 1 \end{array} = s_1 s_2 s_1 s_0 s_1$$

so $\ell(w) = 5$.

Check this agrees with our earlier description of $\ell(w)$!

Prop^n: $\ell(w) = \#\left(w^{-1}(R^-) \cap R^+\right) = \#\left(\text{positive roots made negative by } w\right)$

$$= \#\left(w(R^+) \cap R^-\right)$$

[Humphreys §1.7]

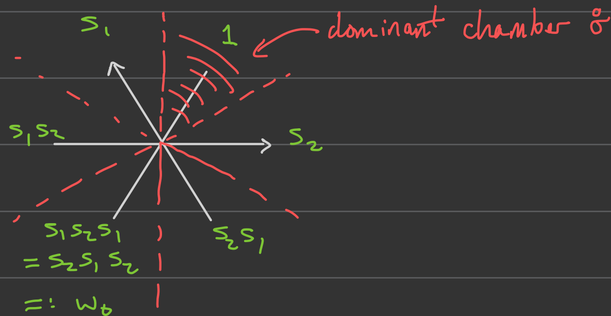
Exercise: For $\alpha \in \Delta$, $s_\alpha\left(R^+ \setminus \{\alpha\}\right) = R^+ \setminus \{\alpha\}$. a case of the Prop^n!

Exercise: Show $\displaystyle\sum_{i=1}^{n} \varpi_i = \frac{1}{2} \sum_{\alpha \in R^+} \alpha$. use previous exercise!

Prop^n: $W = W(R)$ acts simply transitively on the set of Weyl chambers.

[Bourbaki §1], [Humphreys §1.8]

Ex:



Cor: There's a unique longest element w_0, taking the dominant chamber $\mathring{\sigma}$ to the antidominant chamber $-\mathring{\sigma}$.

WARNING: usually w_0 is not just multiplication by -1!

Cor: Up to re-labelling, everything so far is independent of choice of Δ (Dynkin diag., Cartan matrix, etc.)