2021. 3. 12

Ref: [Humphreys, LAG]
Springer, LAG

LECTURE 22: LIE THEORY III (Semisimple groups and Lie algebras)

A linear algebraic group ("LAG") is a Zariski-closed subgroup $G \subseteq GL(V)$, for some V.

Equivalently (!) G is an affine algebraic group, i.e., an affine variety with a group structure.

A torus is $T \simeq (\mathbb{C}^*)^n = \operatorname{Spec} \mathbb{C}[x_1^{\pm}, \dots, x_n^{\pm}]$.
$(\mathbb{G}_m)^n$

A maximal torus $T \subseteq G$ is what it seems.

Every LAG G has a Lie algebra $\mathfrak{g} = \operatorname{Lie}(G)$.
This can be defined intrinsically, but a quick way is this: for $G \subseteq GL(V)$,

$$\begin{array}{ccc} T_e G & \subseteq & T_e GL(V) = \operatorname{End}(V) \\ \| & & \| \\ \mathfrak{g} & \subseteq & \mathfrak{gl}(V) \end{array},$$

$[X, Y] = XY - YX$

with bracket $[\cdot, \cdot]$ on $\mathfrak{g}$ induced by commutator on $\mathfrak{gl}(V)$.

G acts on $\mathfrak{g}$ by the adjoint representation.
Again, this may be defined intrinsically, but using $\mathfrak{g} \subseteq \mathfrak{gl}(V) = \mathrm{End}(V)$, its

$$\mathrm{Ad}(g) \cdot X = g X g^{-1} \qquad \text{for } g \in G \text{ and } X \in \mathfrak{g}.$$

In particular, a (maximal) torus $T \subseteq G$ acts on $\mathfrak{g}$ by the adjoint action.

So one has a weight decomposition

$$\mathfrak{g} = \bigoplus_{\chi \in M} \mathfrak{g}_\chi \qquad \longleftarrow \mathfrak{g}_\chi \subseteq \mathfrak{g} \text{ is where } T \text{ acts by character } \chi$$

Defn: The roots of G with respect to T are the nonzero weights for the adjoint action of T on $\mathfrak{g}$:

$$R(G,T) := \{ \chi \in M \mid \chi \neq 0 \text{ and } \mathfrak{g}_\chi \neq 0 \}.$$

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So $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha \in R(G,T)} \mathfrak{g}_\alpha$ as T-modules.

$\left(\begin{bmatrix} z & \\ & z^{-1} \end{bmatrix} \longmapsto z \right) \leftrightarrow$ "1"

Ex: $G = SL_2$, $T = \left\{ \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix} \right\} \simeq \mathbb{C}^*$, $M \cong \mathbb{Z}$.

trace-zero 2x2 matrices $=$ $\mathfrak{sl}_2$

$\mathfrak{sl}_2$ has basis $H = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, $E = E_{12} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, $F = E_{21} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$.

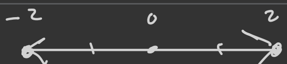
$$z \cdot E = \begin{bmatrix} z & \\ & z^{-1} \end{bmatrix} \begin{bmatrix} & 1 \\ & \end{bmatrix} \begin{bmatrix} z^{-1} & \\ & z \end{bmatrix} = \begin{bmatrix} & z^2 \\ & \end{bmatrix} = z^2 \cdot E \quad \Rightarrow \text{weight } 2$$

$$z \cdot F = \quad \dots\dots \quad = \begin{bmatrix} & \\ z^{-2} & \end{bmatrix} = z^{-2} F \quad \Rightarrow \text{weight } -2$$

$$z \cdot H = H \quad \Rightarrow \text{weight } 0$$

So $R(G,T) = \{2, -2\}$, $\mathfrak{sl}_2 = \underset{\substack{\| \\ \mathbb{C}\cdot H \\ \| \\ \mathfrak{t}}}{\mathfrak{g}_0} \oplus \underset{\substack{\| \\ \mathbb{C}\cdot E}}{\mathfrak{g}_2} \oplus \underset{\substack{\| \\ \mathbb{C}\cdot F}}{\mathfrak{g}_{-2}}$

(A_1)



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Ex: $G = GL_n$, $T =$ diagonal torus $\cong (\mathbb{C}^*)^n$.

Basis for $\mathfrak{gl}_n = (n \times n$ matrices$)$ is E_{ij} 1 in pos (i,j), 0 elsewhere

$M \simeq \mathbb{Z}^n$, basis $t_1, \dots, t_n$. $t_i\left(\begin{bmatrix} z_1 & & 0 \\ & \ddots & \\ 0 & & z_n \end{bmatrix}\right) = z_i$

$z \cdot E_{ij} = z E_{ij} z^{-1} = \frac{z_i}{z_j} E_{ij}$ $\Rightarrow$ weight $t_i - t_j$

$\Rightarrow R(GL_n, T) = \{ t_i - t_j \mid i \neq j \}$ (A_{n-1})

The roots <u>don't</u> span $M \otimes \mathbb{R} \cong \mathbb{R}^n$!

Ex: $G = B =$ upper-triangular $\subseteq GL_n$.
$\cup\!\!|$
$T =$ diagonal $\simeq (\mathbb{C}^*)^n$.

So $\mathfrak{b} \subseteq \mathfrak{gl}_n$ is upper-Δ matrices, basis E_{ij} for $i \leq j$.

Then $R(B, T) = \{ t_i - t_j \mid i < j \}$.

Not a root system!   But $= R^+ \subseteq R(A_{n-1})$!

Ex: $G = Sp_{2n} \subseteq GL_{2n}$, preserving our std form ω.

$$\left[\begin{array}{c|c} & \begin{smallmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{smallmatrix} \\ \hline \begin{smallmatrix} & & -1 \\ & \iddots & \\ -1 & & \end{smallmatrix} & \end{array}\right]$$

Need some basic facts about representations of Lie groups + algebras.

A representation of a LAG is a homomorphism (of alg. gps) $G \to GL(V)$, some v.s. V.

A representation of $\mathfrak{g}$ is a homomorphism (of Lie algs) $\mathfrak{g} \to \mathfrak{gl}(V)$.

① If $G \circlearrowright V$, fixing a vector $v \in V$, then the corresponding rep'n of $\mathfrak{g}$ kills v.

"Think: $\mathfrak{g}$ is derivative of G"

$(g \cdot v = v \ \forall g \in G \Rightarrow X \cdot v = 0 \ \forall X \in \mathfrak{g}.)$

② V, W rep'ns of $G \Rightarrow g \cdot (v \otimes w) = (g \cdot v) \otimes (g \cdot w)$ $g \in G$

$X \in \mathfrak{g}$ $\quad X \cdot (v \otimes w) = (X \cdot v) \otimes w + v \otimes (X \cdot w)$

"Leibniz rule"

③ V^* = dual rep $\Rightarrow (g \cdot \varphi)(v) = \varphi(g^{-1} v)$, $(X \cdot \varphi)(v) = -\varphi(X \cdot v)$.

Ref: [Fulton-Harris, §8]

22.6

Now take $V = \mathbb{C}^{2n}$, $\omega \in \Lambda^2 V^* \subset V^* \otimes V^*$. So

$Sp_{2n} \subseteq GL_{2n}$ is the stabilizer of ω.

$\Rightarrow \mathfrak{sp}_{2n} \subseteq \mathfrak{gl}_{2n}$ is subalgebra that kills ω:

$$= \{ X \mid \omega(X\cdot v, w) + \omega(v, X\cdot w) \equiv 0 \}$$

$$= \{ X \mid X^t \left[\begin{array}{c|c} 0 & I' \\ \hline -I' & 0 \end{array}\right] + \left[\begin{array}{c|c} 0 & I' \\ \hline -I' & 0 \end{array}\right] X = 0 \}.$$

$X = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$

Exercise: Work out the eqns in block matrices!

Ex: $$\mathfrak{sp}_4 = \left[\begin{array}{cc|cc} a & b & e & f \\ c & d & g & e \\ \hline h & i & -d & -b \\ j & h & -c & -a \end{array}\right]$$

For $$T = \left\{ \left[\begin{array}{ccc|ccc} z_n^{-1} & & & & & \\ & \ddots & & & 0 & \\ & & z_1^{-1} & & & \\ \hline & & & z_1 & & \\ & 0 & & & \ddots & \\ & & & & & z_n \end{array}\right] \right\} \cong (\mathbb{C}^*)^n \subseteq Sp_{2n},$$

[clarify]

$$R(Sp_{2n}, T) = \{ t_i - t_j \mid i \neq j \} \cup \{ \pm t_i \pm t_j \mid i \leq j \}$$

(C_n)

E.g., for $n=2$:

$$\begin{array}{c c}
 & \begin{array}{cccc} \bar{2} & \bar{1} & 1 & 2 \end{array} \\
\begin{array}{c} \bar{2} \\ \bar{1} \\ 1 \\ 2 \end{array} &
\left[\begin{array}{cc|cc}
a & \frac{z_1}{z_2} b & \frac{1}{z_1 z_2} e & \frac{1}{z_2^2} f \\
\frac{z_2}{z_1} c & d & \frac{1}{z_1^2} g & \frac{1}{z_1 z_2} e \\
\hline
z_1 z_2 h & z_1^2 i & -d & -\frac{z_1}{z_2} b \\
z_2^2 j & z_1 z_2 h & -\frac{z_2}{z_1} c & -a
\end{array}\right]
\end{array}$$

(Work out some $n>2$!)

Weyl group

Now G is a connected LAG and $T \subseteq G$ max'l torus.

$$N_G(T) = \{ g \in G \mid g z g^{-1} \in T \ \forall z \in T \} \quad \text{(normalizer)}$$

$$\triangledown$$

$$C_G(T) = \{ g \in G \mid g z g^{-1} = z \ \forall z \in T \} \quad \text{(centralizer)}$$

$$W = W(G,T) := N_G(T) / C_G(T).$$

For most of our examples, $C_G(T) = T$, so $W = N_G(T)/T$.

22.8

Ex: $G = \left[\begin{array}{cc|cc} * & * & \multicolumn{2}{c}{0} \\ & * & & \\ \hline \multicolumn{2}{c|}{0} & 1 & * \\ & & & 1 \end{array}\right] \supseteq T = \left[\begin{array}{cc|cc} * & & & \\ & * & & \\ \hline & & 1 & \\ & & & 1 \end{array}\right]$

has $C_G(T) = \left[\begin{array}{cc|cc} * & & & \\ & * & & \\ \hline & & 1 & * \\ & & & 1 \end{array}\right] \supsetneq T$. [In fact, $W(G,T) = \{e\}$ in this example...]

Prop: $C_G(T) = N_G(T)^\circ$ (identity component),
and $[N_G(T) : C_G(T)] < \infty$, so $W(G,T)$ is finite.

[Humphreys §22]

Rmk: All maximal tori $T \subset G$ are conjugate,
so the choice of T doesn't matter:
as an abstract group, W depends only on G.

The Weyl group acts on $M = M(T) := \mathrm{Hom}(T, \mathbb{C}^*)$.
Given $w \in W$, choose a lift $n_w \in N_G(T)$.
For $\lambda \in M$, $z \in T$,

$$(w \cdot \lambda)(z) = \lambda(n_w^{-1} z n_w).$$

Check indep't of choice of n_w! (And that this a gp action.)

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Prop: This W action preserves the roots $R = R(G,T) \subseteq M$.

Pf: Take $\alpha \in R$, $X \in \mathfrak{g}_\alpha$ nonzero, $z \in T$, so $Ad(z) \cdot X = \alpha(z) X$.

Let $n = n_w$ be a lift of $w \in W$.

The claim is that $Ad(n) \cdot X \in \mathfrak{g}_{w(\alpha)}$.

(using a rep'n of $\mathfrak{g}$)

Compute: $$\begin{aligned} Ad(z)\,Ad(n) \cdot X &= z\,(n X n^{-1})\,z^{-1} \\ &= n\,(n^{-1} z n X n^{-1} z^{-1} n)\,n^{-1} \\ &= n\,(\alpha(n^{-1} z n) \cdot X)\,n^{-1} \\ &= w(\alpha)(z) \cdot (Ad(n) \cdot X). \end{aligned}$$ $\boxtimes$

22.10

Solvable + Unipotent groups

Defn: An element $x \in G$ of a LAG is:

- semisimple if $\exists$ a faithful rep'n $\rho : G \hookrightarrow GL_n$ so that $\rho(x)$ is diagonal $\begin{bmatrix} * & & 0 \\ & \ddots & \\ 0 & & * \end{bmatrix}$

- unipotent if ......
  s.t. $\rho(x)$ is strictly upper-Δ $\begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

Thm (Jordan decomposition):

(1) For any $x \in G$, $\exists!$ semisimple x_s, unipotent x_u s.t. $x = x_s x_u$ $= x_u x_s$.

(2) For any homom. $G \xrightarrow{\varphi} H$, have
$$\varphi(x)_s = \varphi(x_s) \quad \text{and} \quad \varphi(x)_u = \varphi(x_u).$$

[H, §15.3] (Use familiar Jordan normal form for GL_n)

Defn: G is unipotent if all its elements are, i.e., $x = x_u$ for all $x \in G$.

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Defn: G is solvable if the series

$$G \supseteq (G,G) \supseteq ((G,G),(G,G)) \supseteq \cdots$$

terminates in $\{e\}$, where (G,G) is the commutator subgroup.

Note: Any subgp of a solvable (resp., unipotent) group is again solvable (resp., unipotent).

Main Exs:

$$\left\{\begin{bmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{bmatrix}\right\} = B \subseteq GL_n \quad \text{solvable}$$

$(B,B) = \mathcal{U}$

$(\mathcal{U},\mathcal{U}) = \left\{\begin{bmatrix} 1 & 0 & * \\ & \ddots & 0 \\ 0 & & 1 \end{bmatrix}\right\}$

$$\left\{\begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}\right\} = \mathcal{U} \subseteq GL_n \quad \text{unipotent}$$

Thm: G is unipotent iff any representation $\rho: G \to GL(V)$ can be "strictly upper-triangularized" i.e., there's a basis of V so that

$$\rho(G) \subseteq \mathcal{U} = \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}.$$

[H, § 17.5]

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Thm ("Lie-Kolchin"): A (connected) LAG G is solvable iff any representation $\rho: G \to GL(V)$ can be upper-triangularized: there's a basis of V so that
$$\rho(G) \subseteq \left\{ \begin{bmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{bmatrix} \right\}.$$
[H, §17.6]

So: unipotent groups $\longleftrightarrow$ closed subgroups of $\mathcal{U} = \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$

conn. solvable groups $\longleftrightarrow$ conn. closed subgps of $B = \begin{bmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{bmatrix}$.

Defn: A **Borel subgroup** $B \subset G$ is maximal (closed) connected solvable subgroup.

- A torus is connected + solvable, so contained in some Borel.
- Likewise, any (connected) unipotent gp is contained in a Borel.

Thm: All Borel subgroups are conjugate: if $B, B' \subseteq G$ are Borels, then $B' = xBx^{-1}$ for some $x \in G$.
[Humphreys §21.3] [• For $G = GL_n$: $Fl(\mathbb{C}^n)$ is homogeneous!]

Cor: All maximal tori are conjugate (as are max'l unipotents).

Cor: Let $T, T' \subset G$ be maximal tori. There are isomorphisms $M(T) \xrightarrow{\sim} M(T')$ and $W(G,T) \xrightarrow{\sim} W(G,T')$ inducing an isomorphism $R(G,T) \xrightarrow{\sim} R(G,T')$, compatible with W-actions.

(All induced by $T' \longrightarrow T$, $t' \longmapsto g t' g^{-1}$, where $T = gT'g^{-1}$.)

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Semisimple + Reductive Groups

Now: G is connected + nontrivial

Defn: The radical of G is

$R(G)$ = max'l connected normal solvable subgp

unique! [H, §19.5]

The unipotent radical is

$R_u(G)$ = max'l connected normal unipotent subgp

(also unique)

Ex: $\cdot$ $R(GL_n)$ = scalar matrices $\simeq \mathbb{C}^*$

$R_u(GL_n) = \{e\}$ (trivial)

$\cdot$ For $B = B_n^+ = \begin{bmatrix} * & & * \\ & \ddots & \\ 0 & & * \end{bmatrix}$, $R(B) = B$.

And $R_u(B) = \mathcal{U} = \begin{bmatrix} 1 & & * \\ & \ddots & \\ 0 & & 1 \end{bmatrix}$. $\leftarrow$ $B/R_u(B) \cong (\mathbb{C}^*)^n$.

$\mathcal{U} \trianglelefteq B$

$B = T \cdot \mathcal{U}$

T: torus (not unique)

$\mathcal{U}$: unipotent (unique)

Ex: $P = \left[\begin{array}{cc|cc} * & * & * & * \\ * & * & * & * \\ \hline \multicolumn{2}{c|}{0} & * & * \\ & & * & * \end{array}\right] \subset GL_4$.

$\Rightarrow R_u(P) = \left[\begin{array}{cc|cc} 1 & 0 & * & * \\ 0 & 1 & * & * \\ \hline 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$ $\quad P = \underbrace{\left[\begin{array}{c|c} * & 0 \\ \hline 0 & * \end{array}\right]}_{L \cong GL_2 \times GL_2} \cdot R_u(P)$

$Z(SL_n) = \mathbb{Z}/n\mathbb{Z}$, conn. comp. is $\{e\}$

Defn: G is **semisimple** if $R(G) = \{e\}$ (ex: SL_n)

G is **reductive** if $R_u(G) = \{e\}$ (ex: GL_n)

Any semisimple group is reductive (since $R_u \subseteq R$ always).

If G is semisimple, its center $Z(G)$ is finite.

(Otherwise the component $Z(G)^\circ$ would be conn. norm. solv.)

If G is reductive, then $Z(G)^\circ$ is a torus, $Z(G)^\circ = R(G)$, and $(G, G) \subset G$ is semisimple.

[H, §19.5]

$SL_n = (GL_n, GL_n) \subset GL_n$

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For any connected G, $G/R(G)$ is semisimple,
$G/R_u(G)$ is reductive.

Ex: GL_n = reductive.
$SL_n = (GL_n, GL_n)$ semisimple.
$PGL_n = GL_n / Z(GL_n) = R(GL_n)$ is semisimple.

Prop: Let G be semisimple with max'l torus T.
Then $R = R(G,T)$ is a root system
nonzero weights of T on $\mathfrak{g}$
in $V = \mathfrak{t}^*_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$,
with Weyl group $W = W(G,T)$.

[H, §27.1]

Rmk: Main difference between semisimple + reductive is the requirement that V be spanned by R.
For reductive G, replacing V by $V' = \mathrm{span}(R(G,T))$ produces a root system. (Corresp. to ss quotient $G/Z(G)^\circ$.)
(Think of GL_n, with $M \cong \mathbb{Z}^n$, $V \cong \mathbb{R}^n$, but R span an $(n-1)$-dim'l subsp.)

Defn: G is **simple** if it has no nontrivial closed connected normal subgroup, and is non-commutative.

- non-comm. rules out trivial cases $\mathbb{G}_m = \mathbb{C}^*$, $\mathbb{G}_a = \mathbb{C}$.
- SL_n is simple as an LAG, though not as an abstract group.

Propn: Suppose G is semisimple. Then G is simple iff $R(G,T)$ is an irreducible root system.

rank of semisimple group := dim (max'l torus).

More on roots

For connected semisimple G with maximal torus T,

$\alpha \in R(G,T)$ is a character $\alpha : T \longrightarrow \mathbb{C}^*$.

Thm: $G_\alpha := C_G(\underbrace{\ker(\alpha)^\circ}_{\text{sub-torus of } T})$ is a connected reductive gp, and (G_α, G_α) is semisimple of rank 1.

[SGA3] or [Springer, §6.4.7]

Ex: $\alpha = t_2 - t_3$, $(\mathbb{C}^*)^4 = T \longrightarrow \mathbb{C}^*$, $\ker(\alpha) = \begin{bmatrix} * & & & \\ & a & & \\ & & a & \\ & & & * \end{bmatrix}$, $G_\alpha = \begin{bmatrix} * & & & \\ & * & * & \\ & * & * & \\ & & & * \end{bmatrix}$

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Semisimple rank 1 groups $\longleftrightarrow$ root sys of type (A_1)

$\Rightarrow G = SL_2$ or PGL_2

There's a corresponding map

$$SL_2 \twoheadrightarrow (G_\alpha, G_\alpha) \hookrightarrow G_\alpha \hookrightarrow G.$$

We'll sometimes write the composition as

$$\begin{array}{ccc} SL_2 & \xrightarrow{\varphi_\alpha} & G \\ \cup & & \cup \\ \mathbb{C}^* \simeq T_\alpha & \longrightarrow & T \end{array} \qquad \begin{array}{ccc} \begin{bmatrix} 0 & * \\ 0 & 0 \end{bmatrix} & \longrightarrow & \mathfrak{g}_\alpha \\ \cap & & \cap \\ \mathfrak{sl}_2 & \longrightarrow & \mathfrak{g} \end{array}$$

$z = \begin{bmatrix} z & 0 \\ 0 & z^{-1} \end{bmatrix}$

The coroot $\alpha^\vee$ is this one-parameter subgroup (1-psg),

$$\alpha^\vee : \mathbb{C}^* = T_\alpha \longrightarrow T.$$

[Springer, §7.1]

These play an important role. Later we'll see how they determine T-invariant curves in G/B.

Classification

In addition to root data, some topological information is needed to classify simple LAG's.

Propn: For semisimple G and max'l torus T, let $R = R(G,T)$ be the root system, with weight and root lattices $M_{wt} \supseteq M_{rt}$, and $M = M(T)$.

Then $M_{wt} \supset M \supset M_{rt}$, and

$$M^\vee / M^\vee_{wt} \xrightarrow{\sim} \pi_1(G, e)$$

$$\begin{array}{c} M^\vee / M^\vee_{wt} \\ \text{S}\| \\ M_{wt}/M \end{array}$$

for $G/\mathbb{C}$

(1-psg $\varphi : \mathbb{C}^* \to T \subset G$ generates a based loop), $S^1 \subset \mathbb{C}^*$

[Fulton-Harris §23.1]

→ [other ref?] (Helgason?)

22.20

Thm (1) (isom.) G, G' = simple LAG's, with max'l tori T, T'.

If $R(G,T) \cong R(G',T')$ and $\pi_1(G) \cong \pi_1(G')$, then* there's an isom $G \xrightarrow{\sim} G'$ taking T to T'.

* One exception: $R(G,T)$ of type (D_n), $n \geq 6$ even, and $\pi_1(G) = \mathbb{Z}/2\mathbb{Z}$. There are two possible (G,T).

[Why??]

(2) (existence) For R = irred. root system with fundamental group M_{wt}/M_{rt}, and any $M_{wt} \supset M \supset M_{rt}$, there's a simple LAG G with max'l torus T such that $R(G,T) = R$ and $\pi_1(G) \cong M_{wt}/M$.

22.21

Exs (most of them!):

(A_{n-1}) SL_n $(\pi_1 = \{e\})$ PGL_n $(\pi_1 = \mathbb{Z}/n\mathbb{Z})$

$n \geq 2$ (B_n) SO_{2n+1} $(\pi_1 = \mathbb{Z}/2\mathbb{Z})$ $Spin_{2n+1}$ $(\pi_1 = \{e\})$

$n \geq 2$ (C_n) Sp_{2n} $(\pi_1 = \{e\})$ PSp_{2n} $(\pi_1 = \mathbb{Z}/2\mathbb{Z})$

$n \geq 4$ (D_n) SO_{2n} $(\pi_1 = \mathbb{Z}/2\mathbb{Z})$ $Spin_{2n}$ $(\pi_1 = \{e\})$

$\pi_1(D_n) = \underbrace{(\mathbb{Z}/4\mathbb{Z})}_{\text{odd } n}$ or $\underbrace{(\mathbb{Z}/2\mathbb{Z})^2}_{\text{even } n}$

... and 5 exceptional types.

There are some coincidences.

Exercise: Show $\mathfrak{sl}_2 \simeq Sym^2 \mathbb{C}^2$ as SL_2-modules.
Show $PGL_2 \subseteq GL(\mathfrak{sl}_2)$ fixes a symmetric nondeg. bilinear form, and conclude $PGL_2 \cong SO_3$.

The extreme cases $\underbrace{\pi_1(G) = \{e\}}_{M = M_{wt}}$ and $\underbrace{\pi_1(G) = M_{wt}/M_{rt}}_{M = M_{rt}}$ are called the simply connected (SL_n) and adjoint (PGL_n) groups, respectively.