GILLET DESCENT FOR CONNECTIVE K-THEORY

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Abstract. Using Gillet's technique of projective envelopes, we prove a homological descent theorem for the connective K-homology of schemes.

The aim of this note is to describe an exact descent sequence in connective K-theory. This is a variation on a technique of Gillet, who used it to construct proper pushforwards for K-theory of general schemes, by bootstrapping from Quillen's construction for quasi-projective schemes. Likewise, by using the present descent sequence, one can extend results stated in [Cai08, DL14] for quasi-projective schemes to general schemes.

Our presentation will be terse: the arguments are all based on ones in the literature, and we refer especially to [Gi] for the details. See also [AP15, Appendix A] for a digest.

All schemes are separated and of finite type over some field. Following Cai [Cai08] (see also [And19, Appendix A]), let $\mathcal{M}_i(X) \subseteq \mathbf{Coh}(X)$ be the full subcategory of sheaves whose support has dimension $\leq i$. The *connective K-theory* groups of X are defined as

$$CK_i(X) := \mathrm{im}\left(K_\circ(\mathcal{M}_i(X)) \to K_\circ(\mathcal{M}_{i+1}(X))\right).$$

More generally, Cai defines

$$CK_{i,q-i}(X) := \mathrm{im}\left((K_q(\mathcal{M}_i(X)) \to K_q(\mathcal{M}_{i+1}(X))\right),$$

although for $q > 0$ his higher K-groups diverge from those of Dai and Levine [DL14]. Our applications focus on the case $q = 0$.

A proper morphism $f\colon Z \to X$ is an *envelope* if every subvariety of X is the birational image of some subvariety of Z. If $X \to Y$ and $Z \to Y$ are defined relative to some base scheme, and f is a morphism of Y-schemes, then one says f is a *projective envelope* if $Z \to Y$ is projective. A basic fact is that any scheme X admits a projective envelope.

Theorem 1. *Let $X \to Y$ be a proper morphism, and let $f\colon Z \to X$ be a projective envelope (relative to Y). Then the sequence*

$$CK_i(Z \times_X Z) \xrightarrow{pr_{1*} - pr_{2*}} CK_i(Z) \xrightarrow{f_*} CK_i(X) \to 0$$

is exact, for all i.

This is proved by following Gillet's argument closely. In brief, using the terminology of [Gi], the theorem follows by applying the same arguments to a projective

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hyperenvelope $Z_\bullet \to X$, using functoriality of the Bousfield-Kan spectral sequence for $K_*(\mathcal{M}_i(Z_\bullet))$. We will give an outline here, stressing the novel points for our situtation; see [AP15, Appendix A] for a similar outline, including most of the terminology and further references.

We will work with an augmented simplicial scheme $Z_\bullet \to X$. For each non-increasing map of ordinals $\tau\colon \mathbf{m} \to \mathbf{n}$ there is structure map $\tau\colon Z_n \to Z_m$. We require the following condition on sheaves of $\mathcal{O}_{Z_n}$-modules:

(*) For all $\tau\colon \mathbf{m} \to \mathbf{n}$ and all $p > 0$, we have $R^p\tau_*\mathscr{F} = 0$ as sheaves on Z_m.

Let $\mathcal{A}_i(Z_n) \subseteq \mathcal{M}_i(Z_n)$ be the full subcategory of sheaves satisfying ((*)). These are exact categories, and they fit together to form a simplicial category $\mathcal{A}_i(Z_\bullet)$. We define

$$\mathcal{E}_{q,i}(Z_\bullet) := K_q(\mathcal{A}_i(Z_\bullet)),$$

where for any category C, the K-group is defined by Quillen's construction, so $K_q(C) = \pi_q(\Omega|NQC|)$. The natural inclusions of categories $\mathcal{A}_i(Z_\bullet) \to \mathcal{A}_{i+1}(Z_\bullet)$ induce homomorphisms $\mathcal{E}_{q,i}(Z_\bullet) \to \mathcal{E}_{q,i+1}(Z_\bullet)$.

In our context, the Bousfield-Kan spectral sequence relates $\mathcal{E}_{i,q}(Z_\bullet)$ with the K-groups $K_q(\mathcal{M}_i(Z_p))$. (Here it is important that each Z_i is quasi-projective over the base Y.) Specificially, it gives a convergent spectral sequence

$$E^1_{p,q}(i) = K_q(\mathcal{M}_i(Z_p)) \Rightarrow \mathcal{E}_{p+q,i}(Z_\bullet).$$

The differential is given by the alternating sum of face homomorphisms.

The key ingredient in proving Theorem 1 is an analogue of another result of Gillet.

Lemma 2. *Let $f\colon Z_\bullet \to X$ be a projective hyperenvelope. Then*

$$f_*\colon \mathcal{E}_{q,i} \to K_q(\mathcal{M}_i(X))$$

is an isomorphism, natural with respect to the inclusions from i to $i+1$.

The proof is exactly the same as in [Gi], everywhere replacing the category of coherent sheaves by the subcategory $\mathcal{M}_i$.

To deduce Theorem 1, one examines the edge homomorphism of the Bousfield-Kan spectral sequence, just as in [Gi]. Let $Z_1 = Z \times_X Z$ and $Z_0 = Z$. We have a diagram

$$\begin{array}{ccccccc}
K_0(\mathcal{M}_i(Z_1)) & \longrightarrow & K_0(\mathcal{M}_i(Z_0)) & \longrightarrow & E^2_{0,0}(i) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
K_0(\mathcal{M}_{i+1}(Z_1)) & \longrightarrow & K_0(\mathcal{M}_{i+1}(Z_0)) & \longrightarrow & E^2_{0,0}(i) & \longrightarrow & 0
\end{array}$$

with exact rows. From the convergence of the spectral sequence we know $E^2_{0,0}(i) = \mathcal{E}_{0,i}(Z_\bullet)$, and from Lemma 2, we have $\mathcal{E}_{0,i}(Z_\bullet) = K_0(\mathcal{M}_i(X))$. Putting all this together, the images of the vertical arrows form an exact sequence

$$CK_i(Z_1) \to CK_i(Z_0) \to CK_i(X) \to 0,$$

as claimed. □

As noted above, the descent sequence allows one to extend results from the quasi-projective to the general case. For instance, suppose V is vector bundle of rank n on a scheme X. The *projective bundle formula* asserts that there is an isomorphism of $\mathbb{Z}[\beta]$-modules

$$CK_*(X) \otimes_{\mathbb{Z}[\beta]} CK_*(\mathbb{P}^{n-1}) \to CK_*(\mathbb{P}(V)).$$

In [Cai08, DL14], this is proved for quasi-projective schemes. For arbitrary X, one can choose a projective envelope $f\colon Z \to X$ (relative to an appropriate base scheme Y), and one has a diagram

$$\begin{array}{ccccccc} CK_*(Z_1) \otimes CK_*(\mathbb{P}^{n-1}) & \longrightarrow & CK_*(Z_0) \otimes CK_*(\mathbb{P}^{n-1}) & \longrightarrow & CK_*(X) \otimes CK_*(\mathbb{P}^{n-1}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow & & \\ CK_*(\mathbb{P}(V_1)) & \longrightarrow & CK_*(\mathbb{P}(V_0)) & \longrightarrow & CK_*(\mathbb{P}(V)) & \longrightarrow & 0, \end{array}$$

where V_0 and V_1 are the pullbacks of V to $Z_0 = Z$ and $Z_1 = Z \times_X Z$, respectively. The rows are exact, by Theorem 1, and the left two vertical arrows are isomorphisms, by the quasi-projective case, so the right vertical arrow is also an isomorphism, by the five lemma.

In particular, this allows one to extend the theory of Chern classes developed in [Cai08, §6.7] to all schemes, as sketched in [And19, Appendix A].

References

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