GILLET DESCENT FOR CONNECTIVE K-THEORY

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Abstract. Using Gillet’s technique of projective envelopes, we prove a homological descent theorem for the connective K-homology of schemes.

The aim of this note is to describe an exact descent sequence in connective K-theory. This is a variation on a technique of Gillet, who used it to construct proper pushforwards for K-theory of general schemes, by bootstrapping from Quillen’s construction for quasi-projective schemes. Likewise, by using the present descent sequence, one can extend results stated in [Cai08, DL14] for quasi-projective schemes to general schemes.

Our presentation will be terse: the arguments are all based on ones in the literature, and we refer especially to [Gi] for the details. See also [AP15, Appendix A] for a digest.

All schemes are separated and of finite type over some field. Following Cai [Cai08] (see also [And19, Appendix A]), let $M_i(X) \subseteq \text{Coh}(X)$ be the full subcategory of sheaves whose support has dimension $\leq i$. The connective K-theory groups of $X$ are defined as

$$CK_i(X) := \text{im} \left( \text{K}(M_i(X)) \to \text{K}(M_{i+1}(X)) \right).$$

More generally, Cai defines

$$CK_{i,q-i}(X) := \text{im} \left( \text{K}_q(M_i(X)) \to \text{K}_q(M_{i+1}(X)) \right),$$

although for $q > 0$ his higher K-groups diverge from those of Dai and Levine [DL14]. Our applications focus on the case $q = 0$.

A proper morphism $f: Z \to X$ is an envelope if every subvariety of $X$ is the birational image of some subvariety of $Z$. If $X \to Y$ and $Z \to Y$ are defined relative to some base scheme, and $f$ is a morphism of $Y$-schemes, then one says $f$ is a projective envelope if $Z \to Y$ is projective. A basic fact is that any scheme $X$ admits a projective envelope.

Theorem 1. Let $X \to Y$ be a proper morphism, and let $f: Z \to X$ be a projective envelope (relative to $Y$). Then the sequence

$$CK_i(Z \times_X Z) \xrightarrow{pr_1^* - pr_2^*} CK_i(Z) \xrightarrow{f_*} CK_i(X) \to 0$$

is exact, for all $i$.

This is proved by following Gillet’s argument closely. In brief, using the terminology of [Gi], the theorem follows by applying the same arguments to a projective
hyperenvelope \( Z_\bullet \to X \), using functoriality of the Bousfield-Kan spectral sequence for \( K_*(M_i(Z_\bullet)) \). We will give an outline here, stressing the novel points for our situation; see [AP15, Appendix A] for a similar outline, including most of the terminology and further references.

We will work with an augmented simplicial scheme \( Z_\bullet \to X \). For each non-increasing map of ordinals \( \tau : m \to n \) there is structure map \( \tau : Z_n \to Z_m \). We require the following condition on sheaves of \( O_{Z_\bullet} \)-modules:

\[(*) \text{ For all } \tau : m \to n \text{ and all } p > 0, \text{ we have } R^p \tau_* \mathcal{F} = 0 \text{ as sheaves on } Z_m.\]

Let \( A_i(Z_n) \subseteq M_i(Z_n) \) be the full subcategory of sheaves satisfying \((*)\). These are exact categories, and they fit together to form a simplicial category \( A_i(Z_\bullet) \). We define

\[E^q_{q,i}(Z_\bullet) := K_q(A_i(Z_\bullet)),\]

where for any category \( C \), the \( K \)-group is defined by Quillen’s construction, so \( K_q(C) = \pi_q(\Omega|NQ|C) \). The natural inclusions of categories \( A_i(Z_\bullet) \to A_{i+1}(Z_\bullet) \) induce homomorphisms \( E_{q,i}(Z_\bullet) \to E_{q,i+1}(Z_\bullet) \).

In our context, the Bousfield-Kan spectral sequence relates \( E_{q,i}(Z_\bullet) \) with the \( K \)-groups \( K_q(M_i(Z_p)) \). (Here it is important that each \( Z_i \) is quasi-projective over the base \( Y \).) Specifically, it gives a convergent spectral sequence

\[E^{1}_{p,q}(i) = K_q(M_i(Z_p)) \Rightarrow E^{0}_{0,0}(i) = K_0(M_i(X)).\]

The differential is given by the alternating sum of face homomorphisms.

The key ingredient in proving Theorem 1 is an analogue of another result of Gillet.

**Lemma 2.** Let \( f : Z_\bullet \to X \) be a projective hyperenvelope. Then

\[f_* : E_{q,i} \to K_q(M_i(X))\]

is an isomorphism, natural with respect to the inclusions from \( i \) to \( i + 1 \).

The proof is exactly the same as in [Gi], everywhere replacing the category of coherent sheaves by the subcategory \( M_i \).

To deduce Theorem 1, one examines the edge homomorphism of the Bousfield-Kan spectral sequence, just as in [Gi]. Let \( Z_1 = Z \times_X Z \) and \( Z_0 = Z \). We have a diagram

\[
\begin{array}{cccc}
K_0(M_i(Z_1)) & \longrightarrow & K_0(M_i(Z_0)) & \longrightarrow & E^2_{0,0}(i) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \\
K_0(M_{i+1}(Z_1)) & \longrightarrow & K_0(M_{i+1}(Z_0)) & \longrightarrow & E^2_{0,0}(i) & \longrightarrow & 0
\end{array}
\]

with exact rows. From the convergence of the spectral sequence we know \( E^2_{0,0}(i) = E_{0,i}(Z_\bullet) \), and from Lemma 2, we have \( E_{0,i}(Z_\bullet) = K_0(M_i(X)) \). Putting all this together,
the images of the vertical arrows form an exact sequence

\[ \text{CK}_i(Z_1) \to \text{CK}_i(Z_0) \to \text{CK}_i(X) \to 0, \]

as claimed. \(\square\)

As noted above, the descent sequence allows one to extend results from the quasi-projective to the general case. For instance, suppose \(V\) is vector bundle of rank \(n\) on a scheme \(X\). The \textit{projective bundle formula} asserts that there is an isomorphism of \(\mathbb{Z}[\beta]\)-modules

\[ \text{CK}_i(X) \otimes_{\mathbb{Z}[\beta]} \text{CK}_i(\mathbb{P}^{n-1}) \to \text{CK}_i(\mathbb{P}(V)). \]

In [Cai08, DL14], this is proved for quasi-projective schemes. For arbitrary \(X\), one can choose a projective envelope \(f: Z \to X\) (relative to an appropriate base scheme \(Y\)), and one has a diagram

\[
\begin{array}{c}
\text{CK}_i(Z_1) \otimes \text{CK}_i(\mathbb{P}^{n-1}) \to \text{CK}_i(Z_0) \otimes \text{CK}_i(\mathbb{P}^{n-1}) \to \text{CK}_i(X) \otimes \text{CK}_i(\mathbb{P}^{n-1}) \to 0 \\
\downarrow \quad \downarrow \quad \downarrow
\end{array}
\]

\[
\begin{array}{c}
\text{CK}_i(\mathbb{P}(V_1)) \to \text{CK}_i(\mathbb{P}(V_0)) \to \text{CK}_i(\mathbb{P}(V)) \to 0,
\end{array}
\]

where \(V_0\) and \(V_1\) are the pullbacks of \(V\) to \(Z_0 = Z\) and \(Z_1 = Z \times_X Z\), respectively. The rows are exact, by Theorem 1, and the left two vertical arrows are isomorphisms, by the quasi-projective case, so the right vertical arrow is also an isomorphism, by the five lemma.

In particular, this allows one to extend the theory of Chern classes developed in [Cai08, §6.7] to all schemes, as sketched in [And19, Appendix A].

References