DOUBLE SCHUBERT POLYNOMIALS AND DOUBLE SCHUBERT VARIETIES

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The purpose of this note is to explain the geometry underlying a certain identity of Schubert polynomials, namely,

$$\mathfrak{S}_w(x;y) = \sum_{v^{-1}u=w} \mathfrak{S}_u(x)\,\mathfrak{S}_v(-y), \tag{1}$$

where the sum is over those $u, v \in S_n$ such that $v^{-1}u = w$ and $\ell(u) + \ell(v) = \ell(w)$. (See [LS],[M, (6.3)].) This implies an identity in the cohomology ring of a product of two flag varieties, when the variables are specialized to appropriate Chern classes of universal bundles on $Fl \times Fl$:

$$[\mathbf{\Omega}_w] = \sum_{v^{-1}u=w} [\Omega_u] \times [\Omega_{w_0\,v\,w_0}]. \tag{2}$$

Here $\mathbf{\Omega}_w$ is a certain degeneracy locus in $Fl \times Fl$, which we will call a *double Schubert variety*, since it describes pairs of flags in special position with respect to one another; the degeneracy locus formula of [F1] gives $[\mathbf{\Omega}_w] = \mathfrak{S}_w(x;y)$.

The geometric formula (2) is *a priori* a weaker statement than (1), since it takes place in $H^*(Fl \times Fl)$, a quotient of the polynomial ring $\mathbb{Z}[x,y]$. However, the strong stability property of Schubert polynomials [M, (6.5)] allows one to deduce (1) from (2). See Corollary 3.3 below for a precise formulation.

Our main result is a geometric proof that (2) holds in all Lie types, when the left-hand side is suitably defined. The key ingredient is a simple description of the tangent space to a double Schubert variety at a smooth point (Proposition 2.1). We apply this in Propostion 3.1 to show that $\mathbf{\Omega}_w$ intersects $\widetilde{X}_{w_0\,u\,w_0} \times X_v$ transversally in a single point, where $\widetilde{X}_{w_0\,u\,w_0} \times X_v$ is a product of Schubert varieties whose class is Poincaré dual to $[\Omega_u] \times [\Omega_{w_0\,v\,w_0}]$. The identity (2) is an immediate consequence.

The formulas we prove here are certainly known to experts. In fact, they are implicit in [G, Prop. 4.2], [B, §3.1], and [K, Lemma 1]. (Each of these discusses only the diagonal case, where $w = w_0$; one can deduce (2) by applying divided difference operators.) However, to the best of my knowledge, they have not appeared in the form we present them. In any case, one may regard this note as giving a Lie-theoretic proof of (2). We also discuss a generalization to flag bundles.

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For the most part, we will leave aside questions concerning polynomial representatives for Schubert classes in types other than A, focusing instead on the geometric classes; we note one application, though. In other types, one cannot expect to find polynomials which satisfy all the remarkable properties of the type A Schubert polynomials: for example, Fomin and Kirillov have shown that it is impossible to find polynomials which (1) represent type B Schubert varieties, (2) multiply according to the structure constants of the cohomology ring, and (3) have non-negative integer coefficients [FK]. (Several choices of representatives have been proposed, each with its own merits and disadvantages; see [F2, F3, FK, KT].) However, once candidate polynomials have been chosen, one could in principle use (1) to define double Schubert polynomials. The identity established in (2) can be taken as confirmation that this approach makes good geometric sense. In particular, for any choice of polynomials $P_w(x)$ representing Schubert varieties, the corresponding double polynomials $P_w(x; y)$ will represent degeneracy loci.

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1. Setup

In this section, we set up notation, fix conventions, and recall basic facts about Schubert varieties. All of this is standard, but worth reviewing, as there are many choices for conventions in common use.

1.1. **Lie groups and root systems.** Let G be a complex simple Lie group, with Lie algebra $\mathfrak{g}$. Fix a maximal torus T and a Borel subgroup $B \supset T$; let $\mathfrak{t}$ and $\mathfrak{b}$ be the Lie algebras of T and B, respectively. Write $B^- = w_0\, B\, w_0$ for the opposite Borel subgroup. Let $W = N(T)/T$ be the Weyl group of G, and choose representatives for W in G. For our purposes, nothing depends on this choice of representatives, so we will use the same notation for elements of W and their lifts, following a common abuse.

Write $\Phi \subset \mathfrak{t}^*$ for the set of roots for $\mathfrak{g}$, and let Φ^+ be the set of positive roots corresponding to B. The Weyl group acts by reflections on the set of Weyl chambers, and hence on the sets of corresponding positive roots; write $\Phi^+_w = w \cdot \Phi^+$. Thus $\Phi^+_{w_0} = w_0 \cdot \Phi^+ = \Phi^-$. For $\alpha \in \Phi$, write E_α for the corresponding root vector in $\mathfrak{g}$, and $\lambda_\alpha : \mathbb{C}^* \to G$ for the one-parameter subgroup of G obtained by exponentiating the root space $\mathfrak{g}_\alpha$.

1.2. **Schubert varieties and fixed points.** Let $X = G/B$ be the flag variety of G. The torus T acts on X with fixed points $p_w = wB/B$ for $w \in W$. The *Schubert cells* are the B-orbits $X^o_w = BwB/B$; the *Schubert varieties* $X_w = \overline{X^o_w}$ are the orbit closures. Write $\widetilde{X}_w = w_0 \cdot X_w$ for the *opposite Schubert variety*; this is the B^--orbit of the T-fixed point $p_{w_0\, w}$. We will also use the notation $\Omega_w = X_{w\, w_0}$. Thus if $\ell(w)$ is the length of $w \in W$, $\ell(w) = \dim X_w = \operatorname{codim} \Omega_w$.

The set of T-fixed points in X_w is $\{p_u \,|\, u \le w\}$, where "$\le$" denotes Bruhat order on W; the fixed points in Ω_w are $\{p_{u\,w_0} \,|\, u \ge w\}$, and the fixed points in $\widetilde{X}_w$ are $\{p_{w_0\,u} \,|\, u \le w\}$.

Each Borel subgroup of G can be regarded as the stabilizer of some flag in X; write $B_w = w\,B\,w^{-1}$ for the stabilizer of the T-fixed point p_w. Note that Φ^+_w is exactly the set of roots such that $\lambda_\alpha \subset B_w$, i.e., λ_α fixes p_w.

1.3. **Cohomology of flag varieties.** The cohomology ring $H^*(X;\mathbb{Z})$ has an additive basis of *Schubert classes*, the classes of Schubert varieties. This basis is self-dual under Poincaré duality: if $w, v \in W$ are such that $\ell(w) = \ell(v)$, then $[\Omega_w]\cdot[\Omega_{w_0\,v}] = [\Omega_w]\cdot[X_{w_0\,v\,w_0}] = \delta_{w,v}[pt]$. This fact can be easily seen by looking at intersections of Schubert cells: Ω^o_w and $\widetilde{X}^o_{w_0\,v\,w_0}$ are disjoint unless $v = w$, in which case they intersect transversally in the point $p_{w\,w_0}$.

1.4. **Double Schubert varieties.** We define the *double Schubert cells* and *double Schubert varieties* in $X \times X$ by

$$\mathbf{\Omega}^o_w = \{(\overline{g}, \overline{h}) \in X \times X \,|\, \overline{h^{-1}g} \in \Omega^o_w\};$$

and

$$\mathbf{\Omega}_w = \{(\overline{g}, \overline{h}) \in X \times X \,|\, \overline{h^{-1}g} \in \Omega_w\},$$

where $\overline{g}$ denotes the image of g in $X = G/B$. Note that the "double Schubert cells" are not topological cells: they are affine bundles over X via either projection. In classical types, $\mathbf{\Omega}_w$ is the set of pairs of flags which meet each other according to conditions determined by w. It is also the degeneracy locus corresponding to the sequence $\pi_2^* S_\bullet \to \pi_1^* Q_\bullet$, where $S_\bullet$ and $Q_\bullet$ are the tautological subbundles and quotient bundles on X, and π_1, π_2 are the projections:

$$\mathbf{\Omega}_w = \{x \,|\, \mathrm{rk}(\pi_2^* S_p \to \pi_1^* Q_q) \le \#(i \le q \,|\, w(i) \le p)\}.$$

(This description depends on an embedding of the Weyl group in a symmetric group; see [FP, §6] for details.)

The torus T acts diagonally on $X \times X$, with fixed points (p_u, p_v), for $u, v \in W$. The T-fixed points in the double Schubert cells and varieties are

$$\begin{aligned} \{\text{fixed points in } \mathbf{\Omega}^o_w\} &= \{(p_{u\,w_0}, p_v) \,|\, v^{-1}u = w\}; \\ \{\text{fixed points in } \mathbf{\Omega}_w\} &= \{(p_{u\,w_0}, p_v) \,|\, v^{-1}u \ge w\}. \end{aligned}$$

2. Tangent spaces at fixed points

The tangent space to X at p_{w_0} is $w_0 \cdot \mathfrak{g}/\mathfrak{b}$, and thus has a basis indexed naturally by Φ^+. For any $p \in X$, we will abuse notation by writing E_α for the vector in T_pX which is the image of $E_\alpha \in \mathfrak{g}$ under the quotient and translation maps; thus $T_{p_{w_0}}X$ has basis $\{E_\alpha \,|\, \alpha \in \Phi^+\}$.

More generally, the tangent space to X_w at $p_w \in X^o_w$ has a basis indexed by $\Phi^+ \cap \Phi^-_w$; in particular, $\#(\Phi^+ \cap \Phi^-_w) = \dim X_w = \ell(w)$. Moreover, the map $t \mapsto \lambda_\alpha(t) \cdot p_w$ parametrizes a curve in X_w exactly when $\lambda_\alpha \subset B$ and λ_α does not fix p_w. (See, e.g., [BL], §4-5.)

We now obtain a similar description of the tangent space to a T-fixed point of a double Schubert cell.

Proposition 2.1. *Let $p = (p_{u\,w_0}, p_v) \in X \times X$, and let $w = v^{-1}u$. The tangent space $T_p\mathbf{\Omega}^o_w = T_p\mathbf{\Omega}_w \subseteq T_p(X \times X)$ has basis*

$$\begin{aligned}\mathscr{B} \quad = \quad & \{(E_\alpha, 0) \,|\, \alpha \in \Phi^+_v \cap \Phi^-_{u\,w_0}\} \cup \{(0, E_\beta) \,|\, \beta \in \Phi^+_{u\,w_0} \cap \Phi^-_v\} \\ & \cup \{(E_\gamma, E_\gamma) \,|\, \gamma \in \Phi^-_{u\,w_0} \cap \Phi^-_v\}.\end{aligned}$$

Proof. First we check that the curves obtained by exponentiating the vectors in $\mathscr{B}$ lie in $\mathbf{\Omega}_w$, so $\mathscr{B} \subset T_p\mathbf{\Omega}_w$. Now, $\{\lambda_\alpha(t) \cdot p_{u\,w_0}\} \times \{p_v\}$ is a curve in $\mathbf{\Omega}_w$ exactly when

(1) $\lambda_\alpha(t)$ does not fix $p_{u\,w_0}$ (so the curve is nontrivial), and
(2) $(v\,B)^{-1}\lambda_\alpha(t)\,u\,w_0\,B \subset B\,w\,w_0\,B$.

Since $v^{-1}u = w$, these two conditions are equivalent to

(1) $\lambda_\alpha \not\subset B_{u\,w_0}$, and
(2) $\lambda_\alpha \subset B_{u\,w_0} \cup B_v$.

That is, $\lambda_\alpha \subset B_v \smallsetminus B_{u\,w_0}$, so $\alpha \in \Phi^+_v \cap \Phi^-_{u\,w_0}$. The case of curves of the form $\{p_{u\,w_0}\} \times \{\lambda_\beta(t) \cdot p_v\}$ is analogous. For curves of the form $\{\lambda_\gamma(t) \cdot p_{u\,w_0}\} \times \{\lambda_\gamma(t) \cdot p_v\}$, the conditions are

(1) $\lambda_\gamma(t)$ fixes neither $p_{u\,w_0}$ nor p_v, and
(2) $(\lambda_\gamma(t)\,v\,B)^{-1}\lambda_\gamma(t)\,u\,w_0\,B \subset B\,w\,w_0\,B$.

The first condition is just the condition $\gamma \in \Phi^-_{u\,w_0} \cap \Phi^-_v$, and the second condition is always satisfied, since

$$\begin{aligned}B^{-1}\,v^{-1}\,\lambda_\gamma(-t)\,\lambda_\gamma(t)\,u\,w_0\,B \quad &= \quad B\,v^{-1}\,u\,w_0\,B \\ &= \quad B\,w\,w_0\,B.\end{aligned}$$

The elements of $\mathscr{B}$ are clearly independent, so to prove the proposition it suffices to show $\#\mathscr{B} = \dim \mathbf{\Omega}_w = 2\dim X - \ell(w)$. Let

$$S = (\Phi^+_{u\,w_0} \cap \Phi^-_v) \,\dot\cup\, (\Phi^-_{u\,w_0} \cap \Phi^+_v) \,\dot\cup\, (\Phi^-_{u\,w_0} \cap \Phi^-_v)$$

be the set of roots indexing $\mathscr{B}$. Note that $\#\Phi = 2\dim X$, and Φ can be written as a disjoint union $S \,\dot\cup\, (\Phi^+_{u\,w_0} \cap \Phi^+_v)$. Therefore

$$\#\mathscr{B} = \#S = 2\dim X - \#(\Phi^+_{u\,w_0} \cap \Phi^+_v).$$

Multiplying by v^{-1}, we see

$$\begin{aligned}\#(\Phi^+_{u\,w_0} \cap \Phi^+_v) = \#(\Phi^+_{w\,w_0} \cap \Phi^+) \quad &= \quad \#(\Phi^-_{w_0\,w\,w_0} \cap \Phi^+) \\ &= \quad \ell(w_0\,w\,w_0) = \ell(w),\end{aligned}$$

and the proposition follows. $\square$

3. The class of $\mathbf{\Omega}_w$

Proposition 3.1. *Let $u, v, w \in W$ be such that $\ell(u) + \ell(v) = \ell(w)$. Then $\mathbf{\Omega}_w$ and $\widetilde{X}_{w_0\,u\,w_0} \times X_v$ are disjoint unless $v^{-1}u = w$, in which case they intersect transversally in the point $p = (p_{u\,w_0}, p_v)$.*

Proof. The Borel fixed point theorem implies that when a torus acts on a (nonempty) projective variety, it must have fixed points. We have noted that the T-fixed points in $\mathbf{\Omega}_w$ are

$$\{(p_{a\,w_0}, p_b) \,|\, b^{-1}a \geq w\},$$

and the fixed points in $\widetilde{X}_{w_0\,u\,w_0} \times X_v$ are

$$\{(p_{a\,w_0}, p_b) \,|\, a \leq u,\ b \leq v\}.$$

We use the following basic properties of Bruhat order (see [Hu]): $b \leq v$ if and only if $b^{-1} \leq v^{-1}$; $\ell(uv) \leq \ell(u) + \ell(v)$; and $a \leq u$, $b \leq v$ implies $ab \leq uv$. If $(p_{a\,w_0}, p_b)$ lies in both $\mathbf{\Omega}_w$ and $\widetilde{X}_{w_0\,u\,w_0} \times X_v$, then, we must have

$$\ell(w) \leq \ell(b^{-1}\,a) \leq \ell(v^{-1}\,u) \leq \ell(v^{-1}) + \ell(u) = \ell(v) + \ell(u) = \ell(w).$$

Thus there are equalities $\ell(a) = \ell(u)$, $\ell(b) = \ell(v)$, and $\ell(v^{-1}\,u) = \ell(w)$. It follows that $a = u$, $b = v$, and $v^{-1}\,u = w$; the only fixed point in the intersection is therefore $(p_{u\,w_0}, p_v)$, and the intersection is empty unless $v^{-1}u = w$.

It remains to check that the intersection is transversal. For this, we compare the tangent spaces of the two varieties at $p = (p_{u\,w_0}, p_v)$. The tangent space to $\widetilde{X}_{w_0\,u\,w_0} \times X_v$ at p has basis

$$\{(E_\gamma, 0) \,|\, \gamma \in \Phi^- \cap \Phi^-_{u\,w_0}\} \cup \{(0, E_\gamma) \,|\, \gamma \in \Phi^+ \cap \Phi^-_v\}. \tag{3}$$

The roots γ appearing in (3) are exactly those appearing in

$$\{(E_\gamma, E_\gamma) \,|\, \gamma \in \Phi^-_{u\,w_0} \cap \Phi^-_v\} \subseteq \mathscr{B};$$

that is, $\Phi^-_{u\,w_0} \cap \Phi^-_v = (\Phi^- \cap \Phi^-_{u\,w_0}) \,\dot\cup\, (\Phi^+ \cap \Phi^-_v)$. To see this, write

$$\begin{aligned} U &= \Phi^- \cap \Phi^-_{u\,w_0}, \\ V &= \Phi^+ \cap \Phi^-_v, \\ A^- &= (\Phi^-_{u\,w_0} \cap \Phi^-_v) \cap \Phi^-, \text{ and} \\ A^+ &= (\Phi^-_{u\,w_0} \cap \Phi^-_v) \cap \Phi^+, \end{aligned}$$

so $\Phi^-_{u\,w_0} \cap \Phi^-_v = A^- \,\dot\cup\, A^+$. Note $A^- \subseteq U$ and $A^+ \subseteq V$, so $\#A^- \leq \#U$ and $\#A^+ \leq \#V$. The proof of Proposition 2.1 shows that $\#(\Phi^-_{u\,w_0} \cap \Phi^-_v) = \ell(w)$, so

$$\#A^- + \#A^+ = \ell(w) = \ell(u) + \ell(v) = \#U + \#V.$$

It follows that $A^- = U$ and $A^+ = V$. $\square$

The formula given in (2) for the class of a double Schubert variety is an immediate consequence:

Theorem 3.2. *In $H^*(X \times X)$, we have*

$$[\mathbf{\Omega}_w] = \sum [\Omega_u] \times [\Omega_{w_0\, v\, w_0}],$$

where the sum is over u, v such that $v^{-1}u = w$ and $\ell(u) + \ell(v) = \ell(w)$.

Proof. Indeed, by Proposition 3.1,

$$\int [\mathbf{\Omega}_w] \cdot ([\widetilde{X}_{w_0\, u\, w_0}] \times [X_v]) = \begin{cases} 1 & \text{if } v^{-1}u = w \text{ and } \ell(u) + \ell(v) = \ell(w); \\ 0 & \text{otherwise,} \end{cases}$$

and $[\widetilde{X}_{w_0\, u\, w_0}] \times [X_v] = [\Omega_{w_0\, u}] \times [\Omega_{v\, w_0}]$ is Poincaré dual to $[\Omega_u] \times [\Omega_{w_0\, v\, w_0}]$. □

When $X = Fl(V)$ is the variety of (type A) flags in an n-dimensional vector space V, the identity (1) can be deduced from Theorem 3.2:

Corollary 3.3. *For $w \in S_n$, there is an identity of polynomials*

$$\mathfrak{S}_w(x; y) = \sum_{v^{-1}u = w} \mathfrak{S}_u(x)\, \mathfrak{S}_v(-y),$$

summing over those $u, v \in S_n$ such that $v^{-1}u = w$ and $\ell(u) + \ell(v) = \ell(w)$.

Proof. Let X be as above, and consider the sequence

$$\pi_2^* S_1 \subset \cdots \subset \pi_2^* S_n = V = \pi_1^* Q_n \to \cdots \to \pi_1^* Q_1$$

of universal sub- and quotient bundles on $X \times X$. Letting

$$x_i = c_1(\ker(Q_i \to Q_{i-1}))$$

and

$$y_i = c_1(S_i / S_{i-1}),$$

the degeneracy locus formula of [F1] gives $[\mathbf{\Omega}_w] = \mathfrak{S}_w(x; y)$, so the left-hand sides of (1) and (2) are equal in the quotient ring $H^*(X \times X)$. On the other hand, $[\Omega_u] \times 1 = \mathfrak{S}_u(x)$, and one can show that $1 \times [\Omega_{w_0\, v\, w_0}] = \mathfrak{S}_v(-y)$. (For the latter, use the dualizing map $D : X \to X$ given by

$$D : (F_1 \subset F_2 \subset \cdots \subset F_n = V) \mapsto ((V/F_{n-1})^\vee \subset (V/F_{n-2})^\vee \subset \cdots \subset V^\vee).$$

One checks that $D^*(y_i) = -y_{n+1-i}$, and $D^{-1}(\Omega_v) = \Omega_{w_0\, v\, w_0}$. With the conventions as described above, we have $1 \times [\Omega_v] = \mathfrak{S}_v(y_n, \ldots, y_1)$, so

$$1 \times [\Omega_{w_0\, v\, w_0}] = D^*(1 \times [\Omega_v]) = D^* \mathfrak{S}_v(y_n, \ldots, y_1) = \mathfrak{S}_v(-y).)$$

Thus we have equality of the right-hand sides of (1) and (2), again modulo the ideal of relations defining $H^*(X \times X)$. Since this is true for any sufficiently large n, the stability property of Schubert polynomials (see [M]) implies that (1) holds as an equality of polynomials. □

4. Schubert bundles in classical types

Flag varieties for classical groups generalize easily to flag bundles over an arbitrary base Z. In this section, we will show how to rephrase the identity (2) in this globalized setup, when Z is a nonsingular variety.

Let $V \to Z$ be a vector bundle of rank m, equipped with a bilinear form $\langle\ ,\ \rangle : V \otimes V \to \mathbb{C}$. (The form should be zero for type A, symplectic for type C, or nondegenerate symmetric for types B and D.) The *flag bundle* $\mathbf{Fl} = \mathbf{Fl}_{\langle\ ,\ \rangle}(V) \to Z$ parametrizes all isotropic flags in V; the fiber over $z \in Z$ is the classical isotropic flag variety for $V(z)$. The flag bundle comes with tautological flags of subbundles $S_\bullet$ and quotient bundles $Q_\bullet$.

Assume for simplicity that V splits as a sum of line bundles, $V = L_1 \oplus \cdots \oplus L_m$, such that the flag $E_\bullet$ given by $E_k = L_1 \oplus \cdots \oplus L_k$ is isotropic. (That is, $\langle E_i, E_{m-i}\rangle = 0$ for all i.) For each $w \in W$, the flag

$$E^w_\bullet : (L_{w(1)} \subset L_{w(1)} \oplus L_{w(2)} \subset \cdots \subset L_{w(1)} \oplus \cdots \oplus L_{w(m)} = V)$$

is also isotropic. Write $\widetilde{E}_\bullet = E^{w_0}_\bullet$.

The *Schubert bundle* $\Omega_w \subseteq \mathbf{Fl}$ is defined as the locus

$$\Omega_w = \{x \mid \operatorname{rk}(E_p(x) \to Q_q(x)) \leq \#(i \leq q \mid w(i) \leq p)\}.$$

Replacing $E_\bullet$ with $\widetilde{E}_\bullet$, we write $\widetilde{\Omega}_w$ for the *opposite Schubert bundle*. Finally, we define the *double Schubert bundle* in $\mathbf{Fl} \times_Z \mathbf{Fl}$ to be

$$\mathbf{\Omega}_w = \{x \mid \operatorname{rk}(\pi_2^* S_p(x) \to \pi_1^* Q_q(x)) \leq \#(i \leq q \mid w(i) \leq p)\}.$$

Note that locally on Z, we have

$$\mathbf{\Omega}_w = \{(\overline{g}, \overline{h}) \in \mathbf{Fl} \times_Z \mathbf{Fl} \mid \overline{h^{-1}g} \in \Omega_w\}.$$

The globalization of Theorem 3.2 is the following:

Theorem 4.1. *Let $\mathbf{Fl} \to Z$ be a classical flag bundle on a nonsingular variety Z. Then we have the following identity in $H^*(\mathbf{Fl} \times_Z \mathbf{Fl})$:*

$$[\mathbf{\Omega}_w] = \sum_{v^{-1}u = w} [\Omega_u] \times [\widetilde{\Omega}_{w_0\, v\, w_0}].$$

Proof. By the Leray-Hirsch theorem, the classes $[\Omega_a]$, for $a \in W$, form a basis for $H^*(\mathbf{Fl})$ over $H^*(Z)$; similarly, the classes $[\Omega_a] \times [\widetilde{\Omega}_b]$ form a basis for $H^*(\mathbf{Fl} \times_Z \mathbf{Fl})$ over $H^*(Z)$. If $p : \mathbf{Fl} \to Z$ is the projection, the relative Poincaré duality pairing is given by $(\alpha, \beta) = p_*(\alpha \cdot \beta)$, and the class dual to $[\Omega_a]$ is $[\widetilde{\Omega}_{w_0\, a}]$. Therefore, it will suffice to prove an analogue of Proposition 3.1, i.e., to show that $\mathbf{\Omega}_w$ and $\widetilde{\Omega}_{w_0\, u} \times \Omega_v$ are disjoint unless $v^{-1}u = w$, in which case they intersect transversally in a section of the bundle. In fact, this follows from Proposition 3.1, since it can be done locally. $\square$

5. Equivariant cohomology

As an application of Theorem 4.1, consider the case where $Z = BT$ is the classifying space for a torus acting on **Fl**. Identifying $T = (\mathbb{C}^*)^n$, one can take approximation spaces $Z_m = \mathbb{P}^m \times \cdots \times \mathbb{P}^m$ (n factors). Let L_i be the tautological subbundle $\mathcal{O}(-1)$ on the ith factor, so we have flags $E_\bullet^w$ on Z_m as above. Then the formula gives

$$[\mathbf{\Omega}_w]^T = \sum_{v^{-1}u=w} [\Omega_u]^T \times [\widetilde{\Omega}_{w_0\, v\, w_0}]^T, \tag{4}$$

where $\mathbf{\Omega}_w \subset Fl(n) \times Fl(n)$, and Ω_u and $\widetilde{\Omega}_{w_0\, v\, w_0}$ are the Schubert varieties in $Fl(n)$ defined in Section 1. Since the locus in $\mathbf{Fl} \times_Z \mathbf{Fl}$ whose class is $[\mathbf{\Omega}_w]^T$ is the degeneracy locus for the sequence $\pi_2^* S_\bullet \to \pi_1^* Q_\bullet$, we have $[\mathbf{\Omega}_w]^T = \mathfrak{S}_w(x;y)$, where $x_i = c_1^T(\ker(Q_i \to Q_{i-1}))$ and $y_i = c_1^T(S_i/S_{i-1})$. (That is, the x's and y's are ordinary Chern classes for the quotient and subbundles on **Fl**, or equivalently, equivariant Chern classes for the bundles on Fl.) Then Equation (4) yields the following identity for double Schubert polynomials:

$$\mathfrak{S}_w(x;y) = \sum_{v^{-1}u=w} \mathfrak{S}_u(x;t)\, \mathfrak{S}_v(-y;t). \tag{5}$$

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