Linear Algebraic Groups: a Crash Course

Dave Anderson

January 24, 2011

This is a collection of notes for three lectures, designed to introduce linear algebraic groups quickly in a course on Geometric Invariant Theory. There are several good introductory textbooks; in particular, the books by Humphreys [H], Springer [S], and Borel [B]. Here I merely distill some of the material from Humphreys and Springer.

1 Definitions

We'll work over a fixed algebraically closed base field k.

Definition 1.1 An **algebraic group** G is a group object in the category of varieties over k. That is, G is a group and a variety, and the maps

$$\begin{aligned} G \times G &\to G \\ (g,h) &\mapsto gh \end{aligned} \qquad \text{and} \qquad \begin{aligned} G &\to G \\ g &\mapsto g^{-1} \end{aligned}$$

are morphisms of varieties. (And there is a distinguished k-point $e \in G$, the identity.)

A **homomorphism** of algebraic groups is a group homomorphism that is also a map of varieties.

In schemey language, another way to say this is that the functor h_G : **Schemes** $\to$ **Sets** factors through **Groups**.

Definition 1.2 A **linear algebraic group** is an affine variety that is an algebraic group.

Example 1.3 The multiplicative group $\mathbb{G}_m = k^* = \operatorname{Spec} k[x, x^{-1}]$ is an algebraic group.

The coordinate ring $k[G]$ of a linear algebraic group is a (commutative) Hopf algebra: it comes with maps

$$\delta : k[G] \to k[G] \otimes k[G] = k[G \times G] \qquad \text{(comultiplication)},$$
$$c : k[G] \to k[G] \text{(antipode)},$$
$$\epsilon : k[G] \to k \text{(counit)},$$

corresponding to the multiplication, inverse, and unit maps, respectively.

Example 1.4 For $\mathbb{G}_m$, we have $\delta(x) = x \otimes x$, $c(x) = x^{-1}$, and $\epsilon(x) = 1$.

Exercise 1.5 Work out the maps for the additive group $\mathbb{G}_a = \operatorname{Spec} k[x]$.

A pleasant feature of the theory is that the most important examples (for now) are also the most familiar ones.

Example 1.6 The general linear group is a LAG, with $GL_n = \operatorname{Spec} k[x_{ij}]_{\det}$, as is any Zariski-closed subgroup of GL_n.

Example 1.7 Particular examples of closed subgroups that come up:

$\mathbb{B} \subset GL_n$, upper-triangular matrices ("Borel").

$\mathbb{U} \subset \mathbb{B} \subset GL_n$, strictly upper-triangular matrices, with 1's on diagonal (unipotent).

$\mathbb{D} \subset \mathbb{B} \subset GL_n$, diagonal matrices (maximal torus).

In fact, although we've defined "linear" to mean "affine", it turns out that *all* such groups are closed subgroups of GL_n. (This justifies the terminology.)

Proposition 1.8 *Every linear algebraic group can be embedded as a closed subgroup in some GL_n.*

To prove this, we'll need a couple more basic notions.

Definition 1.9 A **(rational) representation** of G on a k-vector space V is a homomorphism $G \to GL(V)$.

A representation is **irreducible** if there is no nontrivial proper G-stable subspace; that is, no W such that $0 \neq W \subsetneq V$ with $G \cdot W \subseteq W$.

One can talk about representations for infinite-dimensional V, but we'll always assume they're **locally finite**: for all $v \in W$, there is a G-stable, finite-dimensional subspace W with $v \in W \subseteq V$.

Example 1.10 The main (and for us, essentially only) example of this is the action of G on $k[G]$. (Or $k[X]$, when G acts on a variety X.) Here, for $f \in k[G]$, we have $g \cdot f$ defined by $(g \cdot f)(x) = f(g^{-1}x)$ for all $x \in G$. This is sometimes called the action by **left translation** on functions, and it works whenever G acts (on the left) on a variety.

There is also an action by **right translation** which is sometimes useful. (But note that it is still a left action!) Denoting this by r_g, we have $r_g \cdot f$ given by $(r_g \cdot f)(x) = f(xg)$.

The key thing here is that $k[G]$ is locally finite:

Lemma 1.11 *If $V \subset k[G]$ is a finite-dimensional subspace, then there is a finite-dimensional G-stable subspace W with $V \subseteq W \subseteq k[G]$. (In particular, $k[G]$ is locally finite.)*

Proof. It clearly suffices to treat the case where V is one-dimensional, say spanned by f. Write

$$\widetilde{\delta}(f) = \sum_i m_i \otimes f_i$$

in $k[G \times G]$, corresponding to the map $(g, h) \mapsto g^{-1}h$. Only finitely many terms appear, say $i = 1, \ldots, n$. Then

$$(g \cdot f)(x) = f(g^{-1}x) = \sum_i m_i(g) f_i(x),$$

so

$$g \cdot f = \sum_i m_i(g) f_i$$

lies in the span of $f_1, \ldots, f_n$. Therefore the space W spanned by

$$\{g \cdot f \mid g \in G\},$$

which is manifestly G-stable, is also finite-dimensional. ∎

We now prove the Proposition.

Proof. Take generators $f_1, \ldots, f_n$ for $k[G]$. By the Lemma, we may assume they're a basis for a G-stable subspace. We'll produce an embedding $G \hookrightarrow GL_n$.

In fact, we have a map $k[GL_n] = k[x_{ij}]_{\text{det}} \to k[G]$, as follows. Consider the right translation action. As in the lemma, there are elements $m_{ij} \in k[G]$ with

$$r_g \cdot f_i = \sum_j f_j \, m_{ij}(g).$$

Define the map by $x_{ij} \mapsto m_{ij}$.

Since

$$f_i(g) = f_i(eg) = \sum_j f_j(e) m_{ij}(g),$$

we see that $f_i = \sum_j f_j(e) m_{ij}$, and therefore the m_{ij} also generate $k[G]$. It follows that the map we defined is surjective, so it corresponds to a closed embedding of varieties. ■

This is all good culture, but many of the groups you encounter come automatically linearized. A major example is that of *diagonalizable groups*.

2 Diagonalizable groups and characters

The group of diagonal matrices $\mathbb{D}_n \subset GL_n$ is special in several ways. First, observe that

$$k[\mathbb{D}_n] \cong k[x_1^{\pm 1}, \cdots, x_n^{\pm 1}] \cong k[\mathbb{Z}^n].$$

Definition 2.1 A **character** of an algebraic group G is a homomorphism $\chi : G \to \mathbb{G}_m = k^*$. The set of all characters forms an abelian group under pointwise multiplication, the **character group** of G, denoted $\mathrm{X}(G) = \mathrm{Hom}_{\text{alg. gp.}}(G, \mathbb{G}_m)$. (Warning: the group operation in $\mathrm{X}(G)$ is often written additively, so you may see the character $g \mapsto \chi_1(g)\chi_2(g)$ written as either $\chi_1 \cdot \chi_2$ or $\chi_1 + \chi_2$.)

Example 2.2 For $\mathbb{G}_m$, we have $\mathrm{X}(\mathbb{G}_m) = \mathbb{Z}$ canonically (up to choice of generator $1 \in \mathbb{Z}$), by sending the identity in $\mathrm{X}(G) = \mathrm{Hom}_{\text{alg. gp.}}(\mathbb{G}_m, \mathbb{G}_m)$ to $1 \in \mathbb{Z}$.

The character $\chi : z \mapsto z^n$ then corresponds to the integer n.

From the example, we see $\mathrm{X}(\mathbb{D}_n) \cong \mathbb{Z}^n$, and $k[\mathbb{D}_n] \cong k[\mathrm{X}(\mathbb{D}_n)]$. In other words, the group of diagonal matrices has lots of characters, enough to form a linear basis for all functions. Contrast with this with the case of a simple group like PGL_n, which has no nontrivial characters (since $\ker(\chi)$ would be a nontrivial normal subgroup).

Definition 2.3 A linear algebraic group is **diagonalizable** if it is isomorphic to a closed subgroup of some $\mathbb{D}_n$. A connected diagonalizable group is called a **torus**.

The key fact about diagonalizable groups is the following structure theorem:

Proposition 2.4 *For a linear algebraic group D, the following are equivalent:*

(1). D is diagonalizable

(2). $\mathrm{X}(D)$ is finitely generated, and $k[D] \cong k[\mathrm{X}(D)] := \bigoplus_{\chi \in \mathrm{X}(D)} k \cdot \chi$.

(3). Every rational representation of D is isomorphic to a direct sum of one-dimensional representations.

(4). D is isomorphic to $(k^*)^r \times A$, for some finite abelian group A.

Remark 2.5 In (3), the claim is that a representation V breaks up as $V = \bigoplus_\chi V_\chi$, where $V_\chi = \{v \in V \mid g \cdot v = \chi(g)v \text{ for all } g \in D\}$. Characters with $V_\chi \neq 0$ are called **weights**, and V_χ are called **weight spaces**.

Example 2.6 Take $T = (k^*)^2 = \mathbb{D}_2 \subset GL_2$, acting on 2×2 matrices $M_{2,2}$ by conjugation:

$$g \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} z_1 & 0 \\ 0 & z_2 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z_1^{-1} & 0 \\ 0 & z_2 \end{pmatrix} = \begin{pmatrix} a & z_1 z_2^{-1} b \\ z_1^{-1} z_2 c & d \end{pmatrix}.$$

The group $\mathbb{D}_2$ has a basis of characters χ_1, χ_2, with $\chi_i(g) = z_i$, and the four-dimensional vector space $M_{2,2}$ breaks up as

$$M_{2,2} = \underbrace{k \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{\text{weight } \chi_1 \chi_2^{-1}} \oplus \underbrace{k \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{\text{weight } \chi_1^{-1} \chi_2} \oplus \underbrace{\left\{ \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \right\}}_{\text{weight } 0}$$

Remark 2.7 Actually, there's some delicacy about which finite abelian groups A can occur in (4). The condition is that A should have no p-torsion if $\mathrm{char}(k) = p$.

Example 2.8 The diagonalizable group D with character group $\mathrm{X}(D) = \mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$ has $k[D] \cong k[x, x^{-1}, y]/(y^2 - 1)$. So $D \cong \mathbb{G}_m \times \mu_2$. Note that if $\mathrm{char}(k) = 2$, this is a non-reduced group scheme (so not an algebraic group).

We now prove the proposition (see [S, §3]).

Proof. The implications (2)⇒(4)⇒(1)⇒(2) are easily verified. We'll show (2)⇒(3).

Let V be a finite-dimensional (rational) representation of D, corresponding to a homomorphism $\varphi : D \to GL(V)$. Choosing a basis for V, the map φ is given by

$$\varphi(g) = (a_{ij}(g)),$$

for some functions $a_{ij} \in k[D]$. By (2), we can write $a_{ij} = \sum_\chi c_{ij}^\chi \chi$ (with finitely many nonzero terms). Grouping these by characters, we can define matrices $A_\chi = (c_{ij}^\chi)$, and then we have

$$\varphi(g) = \sum_\chi \chi(g) A_\chi.$$

It's easy to see that the endomorphisms A_χ do not depend on the choice of basis.

We claim that A_χ is actually the projection on the weight space V_χ. To see this, we first show that $A_\chi \cdot A_\psi = \delta_{\chi,\psi} A_\chi$. Using $\varphi(gh) = \varphi(g)\varphi(h)$, we obtain

$$\sum_\eta \eta(gh) A_\eta = \sum_B \left(\sum_{A_\chi A_\psi = B} \chi(g)\psi(h) \right) B.$$

We'll write this entrywise, for $\varphi(gh) = \varphi(g)\varphi(h)$, and using $\eta(gh) = \eta(g)\eta(h)$: this is an equality of coefficients

$$\sum_\eta c_{ij}^\eta \eta(g)\eta(h) = \sum_{\chi,\psi} b_{ij}^{\chi,\psi} \chi(g)\psi(h).$$

The maps $(g,h) \mapsto \eta(g)\eta(h)$ and $(g,h) \mapsto \chi(g)\psi(h)$ are characters of $D \times D$. By linear independence of characters (Dedekind's theorem), the coefficients on both sides of the equality must be equal, i.e., $b_{ij}^{\chi,\psi} = \delta_{\chi,\psi} c_{ij}^\chi$. This proves that the A_χ are orthogonal idempotents.

Finally, we have $1 = \varphi(e) = \sum \chi(e) A_\chi = \sum A_\chi$. Together with the previous paragraph, this proves the claim. Indeed, for $v \in \operatorname{im}(A_\chi)$, we have $v = A_\chi w$, so $\varphi(g)v = \sum_\psi \psi(g) A_\psi A_\chi w = \chi(g)v$; therefore $\operatorname{im}(A_\chi) \subseteq V_\chi$. On the other hand, $V = \bigoplus \operatorname{im}(A_\chi)$, so we must have $\operatorname{im}(A_\chi) = V_\chi$. ∎

Dual to characters, we have *one-parameter subgroups*—these play a crucial role in GIT.

Definition 2.9 For an algebraic group G, a **one-parameter subgroup (1-psg)** is a homomorphism $\lambda : \mathbb{G}_m \to G$. Write $\mathrm{Y}(G)$ for the group $\mathrm{Hom}_{\text{alg. gp.}}(\mathbb{G}_m, G)$, with pointwise multiplication.

Note that $\mathrm{Y}(G)$ is *not* necessarily commutative. However, there is always a pairing

$$\mathrm{X}(G) \times \mathrm{Y}(G) \to \mathbb{Z}$$

given by $(\chi, \lambda) \mapsto \chi \circ \lambda \in \mathrm{Hom}_{\text{alg. gp.}}(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}$.

Exercise 2.10 When $G = T$ is a torus, show that this is a perfect pairing, i.e., it identifies $\mathrm{Y}(T)$ with $\mathrm{X}(T)^\vee = \mathrm{Hom}_{\mathbb{Z}}(\mathrm{X}(T), \mathbb{Z})$. Is this true more generally when $G = D$ is diagonalizable? *(Answer: Yes, in the sense* $\mathrm{Y}(D) = \mathrm{X}(D)^\vee$*; note that any 1-psg must have image in the connected component* D°*, so there is no torsion in* $\mathrm{Y}(D)$*.)*

3 Reductive groups

The groups with well-behaved invariant theory, for which GIT works best, are the *reductive groups*. Over an algebraically closed field, they're (essentially) classified by Cartan-Killing.

Here I just give definitions and examples, without proof.

3.1 Jordan decomposition

For a linear algebraic group G, an element $x \in G$ is **semisimple** if there is a faithful representation $\rho : G \to GL_n$ such that $\rho(x)$ is diagonal. An element x is **unipotent** if there is a ρ such that $\rho(x) \in \mathbb{U}_n$ is strictly upper-triangular.

Proposition 3.1 *For any* $x \in G$*, there are unique elements* $x_s, x_u \in G$ *such that* $x = x_s x_u = x_u x_s$*, with* x_s *semisimple and* x_u *unipotent.*

Moreover, any homomorphism $\varphi : G \to H$ *preserves semisimple and unipotent parts.*

Reference: [H, §15.3].

3.2 Unipotent and solvable groups

Definition 3.2 A LAG G is **unipotent** if all elements $x \in G$ are unipotent.

The *commutator subgroup* $(G,G) \subseteq G$ is the group generated by all elements $ghg^{-1}h^{-1}$, for $g,h \in G$. It is a closed subgroup [H, ??].

Definition 3.3 The group G is **solvable** if the series

$$G = G_0 \supseteq G_1 \supseteq G_2 \supseteq \cdots ,$$

with $G_i = (G_{i-1}, G_{i-1})$, terminates in the trivial group $\{e\}$.

Example 3.4 The group $\mathbb{U}_n \subset GL_n$ is unipotent, essentially by definition. It is also solvable; the filtration has the first i superdiagonals equal to 0. E.g.,

$$\begin{pmatrix} 1 & * & * \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \supseteq \begin{pmatrix} 1 & 0 & * \\ 0 & 1 & * \\ 0 & 0 & 1 \end{pmatrix} \supseteq \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Example 3.5 The group $\mathbb{B}_n \subset GL_n$ is solvable, since $(\mathbb{B}_n, \mathbb{B}_n) = \mathbb{U}_n$.

Clearly any subgroup of a solvable group is solvable, and similarly for unipotent groups.

Also, every unipotent group is solvable. This follows from the fact that $\mathbb{U}_n$ is, together with:

Proposition 3.6 ("Lie-Kolchin") *If G is unipotent, then for every representation $\rho : G \to GL(V)$, there is a basis of V such that $\rho(G) \subseteq \mathbb{U}_n$.*

If G is solvable, then for every representation $\rho : G \to GL(V)$, there is a basis of V such that $\rho(G) \subseteq \mathbb{B}_n$.

3.3 Borel subgroups

As usual, G is a LAG.

Definition 3.7 A **Borel subgroup** of G is a maximal connected closed solvable subgroup.

For example, $\mathbb{B}_n$ is a Borel subgroup in GL_n.

Theorem 3.8 *All Borel subgroups of G are conjugate in G.*

Reference: [H, §21.3]. The proof uses: (1) $G/B \cong \{\text{Borel subgroups}\}$ is a projective variety, and (2) the *Borel fixed point theorem*, which says that when a solvable group acts on a projective variety, there is always a fixed point.

Corollary 3.9 *All maximal tori in G are conjugate.*

3.4 Semisimple and reductive groups

Assume G is a nontrivial connected LAG.

Definition 3.10 The **radical** of G is the maximal connected *normal* solvable subgroup, $R(G)$. The **unipotent radical** is the maximal connected normal unipotent subgroup, $R_u(G)$.

(These are unique, see [H, §19.5].)

Example 3.11 We have $R(GL_n) = \{\text{scalar matrices}\} \cong k^*$. (For maximality, consider the sequence $1 \to k^* \to GL_n \to PGL_n \to 1$, noting that PGL_n is simple.)

Example 3.12 For

$$P = \left\{ \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \end{pmatrix} \right\} \subseteq GL_4,$$

we have

$$R_u(P) = \left\{ \begin{pmatrix} 1 & 0 & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\}.$$

Definition 3.13 The group G is **semisimple** if $R(G) = \{e\}$. It is **reductive** if $R_u(G) = \{e\}$.

Example 3.14 The groups SL_n, PGL_n, and $SL_n \times SL_m$ are semisimple. The groups GL_n and T are reductive.

Remark 3.15 (1) Semisimple implies reductive, since $R_u \subseteq R$.

(2) If G is semisimple, then its center $Z(G)$ is finite. (Otherwise the connected component $Z(G)^\circ$ would be a nontrivial solvable group.)

(3) If G is reductive, then $Z(G)^\circ = R(G)$ is a torus, and (G, G) is semisimple.

(4) For any (connected) G, the quotient $G/R(G)$ is semisimple, and $G/R_u(G)$ is reductive.

Example 3.16 GL_n is reductive, and $(GL_n, GL_n) = SL_n$ and $GL_n/k^* = PGL_n$ are (semi)simple.

3.5 Classification

I won't be able to describe the Cartan-Killing classification here, but its existence is worth mentioning. Up to finite quotient, semisimple groups are products of *simple* groups, and these are classified (over an algebraically closed field). There are four infinite families—represented by SL_n, SO_{2n+1}, Sp_{2n}, and SO_{2n}—and five exceptional types—G_2, F_4, E_6, E_7, E_8.

4 Actions on varieties: some examples

Here are a couple (counter)examples of algebraic group actions on varieties, worth keeping in mind.

Example 4.1 If a torus T acts on a nonsingular projective variety with isolated fixed points, then at least one fixed point has a T-invariant open affine neighborhood, in fact isomorphic to affine space (Bialynicki-Birula).

On the other hand, if the variety X is singular, this may fail. For example, consider $T = k^*$ acting on a nodal rational curve; there is only one T-fixed point, and any T-invariant neighborhood must be the whole (projective) curve. (Incidentally, this implies that the nodal curve cannot be embedded equivariantly in any smooth projective T-variety.)

Example 4.2 If a unipotent group acts on a nonsingular projective variety with isolated fixed points, there may still not be invariant affine open neighborhoods. For example, consider $\mathbb{U}_2 \subset GL_2$ acting on $\mathbb{P}^1 = \mathbb{P}(k^2)$ by the standard action. The point $[1, 0]$ is fixed, but the only invariant neighborhood is all of $\mathbb{P}^1$.

References

[B] A. Borel, *Linear Algebraic Groups*, Springer, 1991.

[H] J. Humphreys, *Linear Algebraic Groups*, Springer, 1981.

[S] T. A. Springer, *Linear Algebraic Groups*, second edition, Birkhäuser, 1998.