SCHUBERT VARIETIES ARE LOG FANO OVER THE INTEGERS

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Abstract. Given a Schubert variety \( X_w \), we exhibit a divisor \( \Delta \), defined over \( \mathbb{Z} \), such that the pair \((X_w, \Delta)\) is log Fano in all characteristics.

Let \( X_w = \overline{BwB/B} \) be a Schubert variety in a Kac-Moody flag variety \( G/B \) over an algebraically closed field of arbitrary characteristic. Fix a reduced word for the Weyl group element \( w \), and let \( \varphi : \tilde{X}_w \to X_w \) be the corresponding Bott-Samelson resolution. Let \( \partial = \partial_1 + \cdots + \partial_k \) be the complement of the open \( B \)-orbit in \( X_w \), written as a sum of prime divisors, and define \( \tilde{\partial} = \tilde{\partial}_1 + \cdots + \tilde{\partial}_\ell \) similarly; the divisor \( \tilde{\partial} \) is a simple normal crossings divisor. Here \( \ell = \ell(w) \) is equal to the length of \( w \), which in turn is equal to the dimension of \( X_w \).

Denote by \( \rho \) the sum of all fundamental weights of (the simply connected form of) \( G \), and let \( L(\rho) \) be the corresponding ample line bundle on \( X_w \). Choosing \( B \)-invariant sections of \( L(\rho) \) and \( \varphi^* L(\rho) \), these line bundles correspond to divisors
\[
a_1 \partial_1 + \cdots + a_k \partial_k, \quad \text{and} \quad b_1 \tilde{\partial}_1 + \cdots + b_\ell \tilde{\partial}_\ell,
\]
for some nonnegative integers \( a_i \) and \( b_i \). In fact, they are all positive, by the Lemma below.

Recall that for a normal irreducible variety \( Y \) and an effective \( \mathbb{Q} \)-divisor \( D \), the pair \((Y, D)\) is **Kawamata log terminal (klt)** if \( K_Y + D \) is \( \mathbb{Q} \)-Cartier, and for all proper birational maps \( \pi : Y' \to Y \), the pullback \( \pi^*(K_Y + D) = K_{Y'} + D' \) has \( \pi_* K_{Y'} = K_Y \) and \( [D'] \leq 0 \). The pair is **log Fano** if it is klt and \(- (K_Y + D)\) is ample.

In [6], Schwede and Smith prove that globally \( F \)-regular varieties are log Fano, for some choice of boundary divisor that may depend on the characteristic. Motivated by the fact that Schubert varieties are known to be globally \( F \)-regular [3], they ask if there exists a single boundary divisor that works uniformly, in all characteristics. The purpose of this note is to answer their question affirmatively:

**Theorem.** The pair \((X_w, \Delta)\) is log Fano, where \( \Delta = c_1 \partial_1 + \cdots + c_k \partial_k \), and \( c_i = 1 - a_i/M \) for some integer \( M \) greater than all \( a_i \).

We will need a lemma.

**Lemma.** For all \( i \), we have \( b_i > 0 \).

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Proof. Let \((\alpha_1, \ldots, \alpha_\ell)\) be the sequence of simple roots corresponding to the reduced word defining \(X_w\), and let \(s_1, \ldots, s_\ell\) be the corresponding simple reflections. For each \(i\), set
\[
(1) \quad v_i = s_1 \cdots \hat{s}_i \cdots s_\ell = ws_i,
\]
where
\[
(2) \quad \gamma_i = s_\ell s_{\ell-1} \cdots s_{i+1}(\alpha_i).
\]
We claim that
\[
(3) \quad b_i = \langle \rho, \gamma_i \gamma \rangle = \text{ht}(\gamma_i),
\]
where \(\text{ht}(\beta) = |\sum n_i|\) if the root \(\beta = \sum n_i \alpha_i\) as a sum of simple roots. (Since the expression \(w = s_1 \cdots s_\ell\) is reduced, each \(\gamma_i\) is a positive root.)

The claimed expression follows from Chevalley’s formula for multiplying a Schubert class by a divisor \([1]\), but it is easy enough to prove directly. We first review some basic geometry of the Bott-Samelson variety. We have \(\tilde{X}_w = (P_{\alpha_1} \times \cdots \times P_{\alpha_\ell})/B^\ell\), where \(P_\alpha\) is the minimal parabolic subgroup, and \(B^\ell\) acts by
\[
(p_1, p_2, \ldots, p_\ell) \cdot (b_1, b_2, \ldots, b_\ell) = (p_1 b_1, b_1^{-1} p_2 b_2, \ldots, b_\ell^{-1} p_\ell b_\ell).
\]
For each \(i\), fix a representative \(\hat{s}_i \in P_{\alpha_i}\). The torus \(T\) acts on \(\tilde{X}_w\) by left multiplication, and its fixed points correspond to subsets \(I \subseteq [\ell] := \{1, \ldots, \ell\}\). Specifically, the point \(p(I)\) is defined by \(p_i = \hat{s}_i\) if \(i \in I\) and \(p_i = e\) otherwise. The divisor \(\partial I\) is defined by the equation \(p_i = e\).

For each \(i = 1, \ldots, \ell\), one can find a \(T\)-invariant curve \(C_i\) in \(\tilde{X}_w\), whose fixed points are \(p(\ell)\) and \(p(\ell) \setminus \{i\}\). By considering \(T\)-fixed points, it is easy to see that \(C_i \cap \partial_i = p(\ell) \setminus \{i\}\), and \(C_i \cap \partial_j = \emptyset\) if \(i \neq j\). It follows that \(b_i = (\varphi^* \mathcal{L}(\rho) \cdot C_i)\).

By the projection formula, this intersection number is equal to \(\langle \mathcal{L}(\rho) \cdot \varphi(C_i) \rangle\). The curve \(\varphi(C_i)\) is the unique \(T\)-invariant curve in \(X_w \subseteq G/B\) with fixed points \(wB\) and \(v_iB\). The line bundle \(\mathcal{L}(\rho)\) on \(G/B\) is invariant under the action of the Weyl group, so we may apply a translation by \(w^{-1}\); now \(w^{-1} \varphi(C_i)\) is the curve with fixed points \(eB\) and \(w^{-1} v_i B = s_i B\). The formula (3) follows, since for any dominant weight \(\lambda\) and any positive root \(\beta\), if \(C\) is the \(T\)-invariant curve in \(G/B\) joining \(eB\) and \(s_\beta B\), we have \(\langle \mathcal{L}(\lambda) \cdot C \rangle = \langle \lambda, \beta \rangle\). \(\square\)

Proof of Theorem. The canonical divisors for \(X_w\) and \(\tilde{X}_w\) are well known (see, e.g., [5]). We have
\[
K_{X_w} = -(a_1 + 1) \partial_1 - \cdots - (a_k + 1) \partial_k
\]
and
\[
K_{\tilde{X}_w} = -(b_1 + 1) \tilde{\partial}_1 - \cdots - (b_\ell + 1) \tilde{\partial}_\ell.
\]
By the definition of \(\Delta\), the (integral) divisor
\[
MK_{X_w} + M\Delta = -(M a_1 - M + M - a_1) \partial_1 + \cdots + (M a_1 - M + M - a_1) \partial_k
\]
\[
\quad = -(M + 1) a_1 \partial_1 - \cdots - (M + 1) a_k \partial_k
\]
and
comes from a section of $\mathcal{L}(-\rho)^{\otimes M+1}$, so $K_{X_w} + \Delta$ is $\mathbb{Q}$-Cartier and anti-ample.

Now set $\tilde{c}_i = 1 - b_i/M$, and let $\tilde{\Delta} = \sum \tilde{c}_i \tilde{\partial}_i$. From the definition, we have

$$\varphi^*(K_{X_w} + \Delta) = \frac{M+1}{M} (-b_1 \tilde{\partial}_1 - \cdots - b_\ell \tilde{\partial}_\ell)$$

$$= (-b_1 - 1 + \frac{1}{M}(M - b_1)) \tilde{\partial}_1 + \cdots + (-b_\ell - 1 + \frac{1}{M}(M - b_\ell)) \tilde{\partial}_\ell$$

$$= K_{\tilde{X}_w} + \tilde{\Delta}.$$  

The Lemma implies that $\lfloor \tilde{\Delta} \rfloor \leq 0$.

Next we show that $\varphi_* \tilde{\Delta} = \Delta$. Since $X_w$ is normal, for each $j = 1, \ldots, k$, there is a unique $i(j)$ such that $\varphi(\tilde{\partial}_{i(j)}) = \partial_j$; the remaining $\tilde{\partial}_i$'s are collapsed by $\varphi$. It follows that for any divisor $D = \sum_i d_i \tilde{\partial}_i$, we have $\varphi_* D = \sum_j d_{i(j)} \partial_j$. For the same reason, $\varphi_* \varphi^* \mathcal{L}(\rho) = \mathcal{L}(\rho)$, so $a_j = b_{i(j)}$. The claim follows.

Finally, the map $\varphi : \tilde{X}_w \to X_w$ is a rational resolution (see [5]), so in particular we have $\varphi_* K_{\tilde{X}_w} = K_{X_w}$. Applying [2, Lemma 2.30 and Corollary 2.31], it follows that $(X_w, \Delta)$ is klt, and the Theorem is proved. \hfill \Box

**Remark.** The formula

$$-K_{\tilde{X}_w} = \sum_i ((\rho, \gamma_i^\vee) + 1) \tilde{\partial}_i$$

proved in the Lemma can be found in [4, Proof of Proposition 10], and is also valid for Bott-Samelson varieties corresponding to non-reduced words. In this generality the coefficients on the right-hand side may be negative, since some of the $\gamma_i$'s will be negative roots. Note, however, that the anticanonical divisor is always effective.

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