SCHUBERT VARIETIES ARE LOG FANO OVER THE INTEGERS

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Abstract. Given a Schubert variety X_w, we exhibit a divisor Δ, defined over $\mathbb{Z}$, such that the pair (X_w, Δ) is log Fano in all characteristics.

Let $X_w = \overline{BwB/B}$ be a Schubert variety in a Kac-Moody flag variety G/B over an algebraically closed field of arbitrary characteristic. Fix a reduced word for the Weyl group element w, and let $\varphi : \widetilde{X}_w \to X_w$ be the corresponding Bott-Samelson resolution. Let $\partial = \partial_1 + \cdots + \partial_k$ be the complement of the open B-orbit in X_w, written as a sum of prime divisors, and define $\widetilde{\partial} = \widetilde{\partial}_1 + \cdots + \widetilde{\partial}_\ell$ similarly; the divisor $\widetilde{\partial}$ is a simple normal crossings divisor. Here $\ell = \ell(w)$ is equal to the length of w, which in turn is equal to the dimension of X_w.

Denote by ρ the sum of all fundamental weights of (the simply connected form of) G, and let $\mathcal{L}(\rho)$ be the corresponding ample line bundle on X_w. Choosing B-invariant sections of $\mathcal{L}(\rho)$ and $\varphi^*\mathcal{L}(\rho)$, these line bundles correspond to divisors

$$a_1\partial_1 + \cdots + a_k\partial_k, \qquad \text{and} \qquad b_1\widetilde{\partial}_1 + \cdots + b_\ell\widetilde{\partial}_\ell,$$

for some nonnegative integers a_i and b_i. In fact, they are all positive, by the Lemma below.

Recall that for a normal irreducible variety Y and an effective $\mathbb{Q}$-divisor D, the pair (Y, D) is **Kawamata log terminal (klt)** if $K_Y + D$ is $\mathbb{Q}$-Cartier, and for all proper birational maps $\pi : Y' \to Y$, the pullback $\pi^*(K_Y + D) = K_{Y'} + D'$ has $\pi_* K_{Y'} = K_Y$ and $\lfloor D' \rfloor \leq 0$. The pair is **log Fano** if it is klt and $-(K_Y + D)$ is ample.

In [6], Schwede and Smith prove that globally F-regular varieties are log Fano, for some choice of boundary divisor that may depend on the characteristic. Motivated by the fact that Schubert varieties are known to be globally F-regular [3], they ask if there exists a single boundary divisor that works uniformly, in all characteristics. The purpose of this note is to answer their question affirmatively:

Theorem. *The pair (X_w, Δ) is log Fano, where $\Delta = c_1\partial_1 + \cdots + c_k\partial_k$, and $c_i = 1 - a_i/M$ for some integer M greater than all a_i.*

We will need a lemma.

Lemma. *For all i, we have $b_i > 0$.*

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Proof. Let $(\alpha_1, \dots, \alpha_\ell)$ be the sequence of simple roots corresponding to the reduced word defining $\widetilde{X}_w$, and let $s_1, \dots, s_\ell$ be the corresponding simple reflections. For each i, set

$$v_i = s_1 \cdots \widehat{s}_i \cdots s_\ell = w s_{\gamma_i}, \tag{1}$$

where

$$\gamma_i = s_\ell s_{\ell-1} \cdots s_{i+1}(\alpha_i). \tag{2}$$

We claim that

$$b_i = \langle \rho, \gamma_i^\vee \rangle = \mathrm{ht}(\gamma_i), \tag{3}$$

where $\mathrm{ht}(\beta) = |\sum n_i|$ if the root $\beta = \sum n_i \alpha_i$ as a sum of simple roots. (Since the expression $w = s_1 \cdots s_\ell$ is *reduced*, each γ_i is a positive root.)

The claimed expression follows from Chevalley's formula for multiplying a Schubert class by a divisor [1], but it is easy enough to prove directly. We first review some basic geometry of the Bott-Samelson variety. We have $\widetilde{X}_w = (P_{\alpha_1} \times \cdots \times P_{\alpha_\ell})/B^\ell$, where P_α is the minimal parabolic subgroup, and B^ℓ acts by

$$(p_1, p_2, \dots, p_\ell) \cdot (b_1, b_2, \dots, b_\ell) = (p_1 b_1, b_1^{-1} p_2 b_2, \dots, b_{\ell-1}^{-1} p_\ell b_\ell).$$

For each i, fix a representative $\dot{s}_i \in P_{\alpha_i}$. The torus T acts on $\widetilde{X}_w$ by left multiplication, and its fixed points correspond to subsets $I \subseteq [\ell] := \{1, \dots, \ell\}$. Specifically, the point $p(I)$ is defined by $p_i = \dot{s}_i$ if $i \in I$ and $p_i = e$ otherwise. The divisor $\widetilde{\partial}_i$ is defined by the equation $p_i = e$.

For each $i = 1, \dots, \ell$, one can find a T-invariant curve C_i in $\widetilde{X}_w$, whose fixed points are $p([\ell])$ and $p([\ell] \setminus \{i\})$. By considering T-fixed points, it is easy to see that $C_i \cap \widetilde{\partial}_i = p([\ell] \setminus \{i\})$, and $C_i \cap \widetilde{\partial}_j = \emptyset$ if $i \neq j$. It follows that $b_i = (\varphi^* \mathcal{L}(\rho) \cdot C_i)$.

By the projection formula, this intersection number is equal to $(\mathcal{L}(\rho) \cdot \varphi(C_i))$. The curve $\varphi(C_i)$ is the unique T-invariant curve in $X_w \subseteq G/B$ with fixed points wB and $v_i B$. The line bundle $\mathcal{L}(\rho)$ on G/B is invariant under the action of the Weyl group, so we may apply a translation by w^{-1}; now $w^{-1}\varphi(C_i)$ is the curve with fixed points eB and $w^{-1} v_i B = s_{\gamma_i} B$. The formula (3) follows, since for any dominant weight λ and any positive root β, if C is the T-invariant curve in G/B joining eB and $s_\beta B$, we have $(\mathcal{L}(\lambda) \cdot C) = \langle \lambda, \beta^\vee \rangle$. □

Proof of Theorem. The canonical divisors for X_w and $\widetilde{X}_w$ are well known (see, e.g., [5]). We have

$$K_{X_w} = -(a_1 + 1)\partial_1 - \cdots - (a_k + 1)\partial_k$$

and

$$K_{\widetilde{X}_w} = -(b_1 + 1)\widetilde{\partial}_1 - \cdots - (b_\ell + 1)\widetilde{\partial}_\ell.$$

By the definition of Δ, the (integral) divisor

$$\begin{aligned} MK_{X_w} + M\Delta &= (-Ma_1 - M + M - a_1)\partial_1 + \cdots + (-Ma_1 - M + M - a_1)\partial_k \\ &= -(M+1)a_1\partial_1 - \cdots - (M+1)a_k\partial_k \end{aligned}$$

comes from a section of $\mathcal{L}(-\rho)^{\otimes M+1}$, so $K_{X_w}+\Delta$ is $\mathbb{Q}$-Cartier and anti-ample.

Now set $\widetilde{c}_i = 1 - b_i/M$, and let $\widetilde{\Delta} = \sum \widetilde{c}_i \widetilde{\partial}_i$. From the definition, we have

$$\begin{aligned}\varphi^*(K_{X_w}+\Delta) &= \frac{M+1}{M}(-b_1\widetilde{\partial}_1 - \cdots - b_\ell\widetilde{\partial}_\ell)\\ &= (-b_1 - 1 + \frac{1}{M}(M-b_1))\widetilde{\partial}_1 + \cdots + (-b_\ell - 1 + \frac{1}{M}(M-b_\ell))\widetilde{\partial}_\ell\\ &= K_{\widetilde{X}_w} + \widetilde{\Delta}.\end{aligned}$$

The Lemma implies that $\lfloor \widetilde{\Delta} \rfloor \leq 0$.

Next we show that $\varphi_*\widetilde{\Delta} = \Delta$. Since X_w is normal, for each $j = 1, \ldots, k$, there is a unique $i(j)$ such that $\varphi(\widetilde{\partial}_{i(j)}) = \partial_j$; the remaining $\widetilde{\partial}_i$'s are collapsed by φ. It follows that for any divisor $\widetilde{D} = \sum_i d_i\,\widetilde{\partial}_i$, we have $\varphi_* D = \sum_j d_{i(j)}\,\partial_j$. For the same reason, $\varphi_*\varphi^*\mathcal{L}(\rho) = \mathcal{L}(\rho)$, so $a_j = b_{i(j)}$. The claim follows.

Finally, the map $\varphi : \widetilde{X}_w \to X_w$ is a rational resolution (see [5]), so in particular we have $\varphi_* K_{\widetilde{X}_w} = K_{X_w}$. Applying [2, Lemma 2.30 and Corollary 2.31], it follows that (X_w, Δ) is klt, and the Theorem is proved. □

Remark. The formula

$$-K_{\widetilde{X}_w} = \sum_i (\langle \rho, \gamma_i^\vee \rangle + 1)\,\widetilde{\partial}_i$$

proved in the Lemma can be found in [4, Proof of Proposition 10], and is also valid for Bott-Samelson varieties corresponding to non-reduced words. In this generality the coefficients on the right-hand side may be negative, since some of the γ_i's will be negative roots. Note, however, that the anticanonical divisor is always effective.

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