Positivity in the cohomology of flag bundles (after Graham)

Dave Anderson

December 15, 2007

In [Gr], Graham proves that the structure constants of the equivariant cohomology ring of a flag variety are positive combinations of monomials in the simple roots:

**Theorem 1 ([Gr, Cor. 4.1])** Let \( X = G/B \) be the flag variety for a complex semisimple group \( G \) with maximal torus \( T \subset B \), and let \( \{ \sigma_w \in H^*_T X \mid w \in W \} \) be the basis of \((B\text{-invariant})\) Schubert classes. Let \( \{ \alpha_i \} \) be the simple roots which are negative on \( B \). Then in the expansion

\[
\sigma_u \cdot \sigma_v = \sum_w c_{uv}^w \sigma_w,
\]

the coefficients \( c_{uv}^w \) are in \( \mathbb{Z}_{\geq 0}[\alpha] \).

Graham deduces this from a more general result about varieties with finitely many unipotent orbits, which is proved using induction and a calculation in the rank-one case. (In fact, A. Knutson points out that Graham’s proof yields a stronger result: as a polynomial in all the negative roots, \( c_{uv}^w \) is a nonnegative combination of squarefree monomials.)

The goal of this note is to give a short, geometric proof of Graham’s positivity theorem, based on a transversality argument. Here I only discuss type \( A \), but other types work as well. (For a type-uniform version, a change of language is needed: one should replace vector bundles with corresponding principal \( G \)-bundles.)

Throughout, \( Fl \) denotes the variety of (complete) flags in \( \mathbb{C}^n \), and if \( V \to X \) is a vector bundle, \( Fl(V) \to X \) is the bundle of flags in \( V \).

Recall that for \( T' \cong (\mathbb{C}^*)^n \), we have \( BT' = (\mathbb{P}^\infty)^{\times n} \) and \( H^*_T Fl = H^*(ET' \times^{T'} Fl) = H^*Fl(E') \), where \( E' \) is the sum of the \( n \) tautological line bundles on \( BT' \). The **effective** action on \( Fl \) is by \( T \cong (\mathbb{C}^*)^n/\mathbb{C}^* \), and
the classifying space for this torus is $BT = (\mathbb{P}^\infty)^{\times n - 1}$. We will usually deal with the effective torus.

Let $\mathbb{P} = \mathbb{P}^m \times \cdots \times \mathbb{P}^m$ $(n - 1$ factors), with $m \gg 0$, and write $H^*\mathbb{P} = \mathbb{Z}[\alpha_1, \ldots, \alpha_{n-1}]$. (We always assume that $m$ is large enough so that there are no relations in the relevant degrees.) Let $M_i = p_i^* (\mathcal{O}(-1))$ be the tautological bundle on the $i$th factor, and let $\alpha_i = -c_1(M_i)$. Note that the class of any effective cycle in $H^*\mathbb{P}$ is a positive polynomial in the $\alpha$'s.

Let

$$L_i = M_1 \otimes \cdots \otimes M_{i-1}$$

for $1 \leq i \leq n$ (so $L_1 = \mathcal{O}$ is the trivial line bundle), and let $E_i = L_1 \oplus \cdots \oplus L_i$. Thus we have a flag $E_\bullet$ in $E = E_n$. Let $\bar{E}_\bullet$ be the opposite flag, with $\bar{E}_i = L_n \oplus \cdots \oplus L_{n+1-i}$. In the flag bundle $p : \text{Fl}(E) \to \mathbb{P}$, with universal quotient flags $Q_\bullet$, we have Schubert loci

$$\Omega_w = \{ x \in \text{Fl}(E) \mid \text{rk}(E_p \to Q_q) \leq \#(i \leq q \mid w(i) \leq p) \}.$$  \hspace{1cm} (1)

Opposite Schubert loci $\bar{\Omega}_w = \Omega_w(\bar{E}_\bullet \to Q_\bullet)$ are defined similarly. We also have “Schubert cell bundles” $\Omega_w^c$: these are affine bundles over $\mathbb{P}$ which are open in the corresponding loci $\Omega_w$, and are defined by replacing the inequality in (1) with an equality.

The classes $[\Omega_w]$ form a basis for $H^*\text{Fl}(E)$ over $H^*\mathbb{P}$, as $w$ ranges over $S_n$. Writing

$$[\Omega_u] : [\Omega_v] = \sum_w c_{uw}^v [\Omega_w]$$

with $c_{uw}^v \in H^*\mathbb{P}$, our main result is the following:

**Proposition 2** The polynomials $c_{uw}^v$ are positive, that is, $c_{uw}^v \in \mathbb{Z}_{\geq 0}[\alpha_1, \ldots, \alpha_{n-1}]$.

This implies Graham’s positivity theorem (in this context), since $\mathbb{P}$ approximates $BT$ for $m$ sufficiently large, and $\text{Fl}(E)$ approximates $ET \times^T \text{Fl}$, with $[\Omega_w]$ corresponding to the equivariant class $\sigma_w$. (See [Ed-Gr, §9].)

**Remark 3** Since the equivariant Chow ring of [Ed-Gr] is defined via approximation spaces, one can view Proposition 2 as proving positivity in $A^*_T(G/B)$ (which is isomorphic to $H^*_T(G/B)$).

Proposition 2 is a consequence of a transversality statement:

**Proposition 4** For any $u, v, w \in S_n$, there is a translate $\Omega_w'$ of $\Omega_w$ by the action of a connected algebraic group such that $\Omega_w'$ intersects $\Omega_u$ and $\bar{\Omega}_w$ properly and generically transversally.
To deduce Proposition 2, first note that the intersection $\Omega_u \cap \tilde{\Omega}_{w_0 w}$ is always proper and generically transverse. Thus Proposition 4 says that $\Omega'_{u} \cap (\Omega_u \cap \tilde{\Omega}_{w_0 w})$ is proper and generically transverse. By [F 11, Ex. (8.1.11)], this says that
\[
[\Omega_v] \cdot [\Omega_u] \cdot [\tilde{\Omega}_{w_0 w}] = [\Omega'_{u} \cap \Omega_u \cap \tilde{\Omega}_{w_0 w}].
\]
(Since $\Omega'_{u} = g \cdot \Omega_{v}$ for some $g$ in a connected algebraic group, $[\Omega'_{u}] = [\Omega_{v}]$.) Using relative Poincaré duality (see e.g. [Fu 2 §A.6]), we have
\[
c^w_{uv} = p_*([\Omega_u] \cdot [\Omega_v] \cdot [\tilde{\Omega}_{w_0 w}]) = p_*([\Omega_u \cap \Omega'_{u} \cap \tilde{\Omega}_{w_0 w}]).
\]
This is an effective class in $H^* \mathbb{P}$, so Proposition 2 follows.

Proof of Proposition 4. This is essentially an application of Kleiman’s theorem. The endomorphism bundle
\[
\text{End}(E) = \bigoplus_{i,j} L_i^{-1} \otimes L_j
\]
\[
= \left( \bigoplus_{i<j} M_i \otimes \cdots \otimes M_{j-1} \right) \oplus \mathcal{O}^\oplus_{\mathbb{P}^n} \oplus \left( \bigoplus_{i>j} M_i^{-1} \otimes \cdots \otimes M_{i-1}^{-1} \right)
\]
has global sections in lower-triangular matrices, so the group $B$ of (invertible) lower-triangular matrices acts on $\text{Fl}(E)$, fixing the flag $\tilde{E}_\bullet$ and stabilizing $\tilde{\Omega}_{w_0 w}$. (Note that the entries of a matrix in $B$ are global sections of the line bundles $M_i^{-1} \otimes \cdots \otimes M_{i-1}^{-1}$, i.e., multi-homogeneous polynomials. This is a connected group over $\mathbb{C}$, acting on a fiber $p^{-1}(x) \subset \text{Fl}(E)$ by first evaluating the sections at $x$.)

Now let $H = (GL_{m+1})^{(n-1)}$, and for $b \in B$, let $b_x$ be the evaluation at $x \in \mathbb{P}$ (so the action of $b$ on $p^{-1}(x)$ is by $b_x$). Consider the semidirect product $\Gamma = B \rtimes H$, given by $(h \cdot b \cdot h^{-1})_x = b_{h^{-1}x}$. (This action of $H$ on $B$ is just the usual action of $H$ on global sections of the equivariant vector bundle $\text{End}(E)$. Alternatively, one could take $\Gamma$ to be the subgroup of $\text{Aut}(\text{Fl}(E))$ generated by the images of $B$ and $G$ via the homomorphisms corresponding to their respective actions.) As a semidirect product of connected groups, $\Gamma$ is a connected algebraic group. We claim that the locus $\Omega'_{w_0 w}$ is homogeneous for the action of $\Gamma$. Indeed, $B$ acts transitively on each fiber of $\tilde{\Omega}_{w_0 w}$, and the action of $H$ on $\text{Fl}(E)$ induces a transitive action on the set of fibers of $\tilde{\Omega}_{w_0 w}$. (The line bundles $L_i$ are equivariant for $H$, so $H$ preserves the flag $\tilde{E}_\bullet$, and therefore acts on $\tilde{\Omega}_{w_0 w}$.)
Finally, note that $Ω_u$ and $\tilde{Ω}_{w_0 w}$ intersect transversally, as do $Ω_v$ and $\tilde{Ω}_{w_0 w}$. The proposition follows from Lemma 5 below, taking $U = Ω_u$, $V = Ω_v$, and $W = \tilde{Ω}_{w_0 w}$, with their stratifications by Schubert loci. q.e.d.

**Lemma 5** Let $X$ be a nonsingular variety over a field of characteristic 0, with an action of a connected algebraic group $Γ$. Let $U, V, W ⊂ X$ be subvarieties with stratifications

$$U_0 ⊂ \cdots ⊂ U_ℓ = U,$$

$$V_0 ⊂ \cdots ⊂ V_m = V,$$

$$W_0 ⊂ \cdots ⊂ W_n = W,$$

with each stratum $U_i \setminus U_{i-1}$ nonsingular. Assume also that $Γ$ acts on $W$, with each stratum $W_i \setminus W_{i-1}$ a disjoint union of homogeneous spaces.

If $U_i \setminus U_{i-1}$ meets $W_k \setminus W_{k-1}$ transversally for all $i, k$, and similarly for $V_j \setminus V_{j-1}$ and $W_k \setminus W_{k-1}$, then there is an element $g ∈ Γ$ such that $g \cdot V$ meets $U \cap W$ properly and generically transversally.

This can be deduced from results found in [Sp]; see also [Si] for a vast generalization. The proof of this version is quite short, so we give it here.

**Proof.** Applying Kleiman’s theorem (cf. [Ha, III.10.8]) to the pairs $(U_i \setminus U_{i-1} ∩ W_k \setminus W_{k-1})$ and $(V_j \setminus V_{j-1} ∩ W_k \setminus W_{k-1})$ inside the homogeneous space $W_k \setminus W_{k-1}$, we can choose $g ∈ Γ$ such that each intersection

$$(U_i \setminus U_{i-1} ∩ W_k \setminus W_{k-1}) \cap g \cdot (V_j \setminus V_{j-1} ∩ W_k \setminus W_{k-1})$$

$$= (U_i \setminus U_{i-1} ∩ W_k \setminus W_{k-1}) \cap (g \cdot V_j \setminus g \cdot V_{j-1} ∩ W_k \setminus W_{k-1})$$

is transverse, so the intersection $U \cap W \cap g \cdot V$ is proper and generically transverse. q.e.d.

**Remark 6** All that is required in the proof of Proposition 4 are the facts that $P$ is homogeneous for the action of an algebraic group $H$, and $L_i$ are $H$-equivariant line bundles such that $L_i^{-1} \otimes L_j$ is globally generated for $i > j$.

**Remark 7** To recover the result that for (type A) equivariant Schubert calculus, the structure constants $c_{uv}^w$ are in $\mathbb{Z}_{≥0}[t_2 - t_1, \ldots, t_n - t_{n-1}]$, let $P' = (\mathbb{P}^n')^×n$ and choose a map $φ : P' → P$ such that $φ^* M_i = M_i' \otimes (M_i'_{i+1})^{-1}$, where $M_i'$ is the tautological bundle on the $i$th factor of $P'$, with $t_i = c_1(M_i')$. (Note that $φ$ will not be holomorphic!)
The $T'$-equivariant class of a Schubert variety (for $T' = (\mathbb{C}^*)^n$) can be identified with the class of the locus $\Omega_w(E'_\bullet \to Q_\bullet) \subset \text{Fl}(E')$, where $E'_i = M'_1 \oplus \cdots \oplus M'_i$ is a flag of bundles on $\mathbb{P}'$. Since this is $\varphi^{-1}\Omega_w$, the equivariant structure constants are $\varphi^*c^{uv}_w$, which are positive in the variables $\varphi^*\alpha_i = t_{i+1} - t_i$.

**Remark 8** The naive choice of flag, with $F_i = M_1 \oplus \cdots \oplus M_i$, does not work: The bundle $\text{End}(F)$ has only diagonal global sections, so the corresponding loci $\Omega_o^w$ are not homogeneous. This explains why one does not see positivity over $\mathbb{P}'$.

**Acknowledgements.** This note was inspired by William Fulton’s lectures on equivariant cohomology [Fu2], and I thank him, as well as Allen Knutson and Ezra Miller, for comments on the manuscript. Thanks also to Sue Sierra for interesting discussions, and for bringing [Sp] to my attention. Finally, I am grateful for the hospitality of Mathematics Department at Columbia University.

**References**


Department of Mathematics, University of Michigan, Ann Arbor, MI 48109

E-mail address: dandersn@umich.edu